Princeton University
Spring 2025 MAT425: Measure Theory
HW8
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1. (*Egorov's theorem*) Let $\left(\mathbb{R}^d, \mathfrak{B}\left(\mathbb{R}^d\right), \lambda\right)$ be the usual measure space and $\left\{ f_n : \mathbb{R}^d \to \mathbb{C} \right\}_{n\in\mathbb{N}}$ be a sequence of measurable functions. Show that if there exists some $S \in \mathfrak{B}\left(\mathbb{R}^d\right)$ such that $\lambda(S) < \infty$ such that $\left\{ f_n \right\}_n$ converges λ-almost-everywhere on S to some function $f : S \to \mathbb{C}$, then for any $\varepsilon > 0$ there exists some $M \in \mathfrak{B}\left(\mathbb{R}^d\right)$ with $M \subseteq S$ such that $\lambda(M) < \varepsilon$ and $\left\{ f_n \right\}_n$ converges *uniformly* to f on $S \setminus M$.

2. Find a counter-example of the above theorem that is violated *because* $\lambda(S) < \infty$ *is violated.*

3. (*Luzin's theorem*) Take the same $\left(\mathbb{R}^d, \mathfrak{B}\left(\mathbb{R}^d\right), \lambda\right)$ and let $f : \mathbb{R}^d \to \mathbb{C}$ be a measurable function. Show that

   (a) For any $\varepsilon > 0$ and any $S \in \mathfrak{B}\left(\mathbb{R}^d\right)$ such that $\lambda(S) < \infty$, there exists $F \in \operatorname{Closed}\left(\mathbb{R}^d\right)$ such that $\lambda(S \setminus F) < \varepsilon$ and such that $\left.f\right|_F : F \to \mathbb{C}$ is continuous.

   (b) For any $\varepsilon > 0$ and any $S \in \mathfrak{B}\left(\mathbb{R}^d\right)$ such that $\lambda(S) < \infty$ and such that S is *locally compact*, there exists $F \in \operatorname{Cpt}\left(\mathbb{R}^d\right)$ such that $\lambda(S \setminus F) < \varepsilon$, and such that $\left.f\right|_F : F \to \mathbb{C}$ is continuous. Moreover, there exists a *continuous* function $g : \mathbb{R}^d \to \mathbb{C}$ with compact support such that $\left.f\right|_F = \left.g\right|_F$ and such that
   $$\sup_{x\in\mathbb{R}^d} |g(x)| \leq \sup_{x\in\mathbb{R}^d} |f(x)| \,.$$

4. Let a measure be given by
   $$\mu = \sum_{x\in S} c_x \delta_x$$
   where $S \subseteq X$ is countable and $(X, \operatorname{Msrbl}(X))$ is a measurable space, and $\left\{ c_x \right\}_{x\in S} \subseteq \mathbb{C}$ is some sequence. Calculate $|\mu|$.

5. Let the Hermitian matrices be denoted by
   $$\operatorname{Herm}_N(\mathbb{C}) \equiv \left\{ A \in \operatorname{Mat}_N(\mathbb{C}) \mid A = A^* \right\}$$
   and the unitary matrices
   $$\mathcal{U}(N) \equiv \left\{ U \in \operatorname{Mat}_N(\mathbb{C}) \,\middle|\, U^* = U^{-1} \right\} .$$
   With the notation $\mathbb{T} := \mathbb{S}^1$, we denote by $\mathbb{T}^N$ all $N \times N$ *diagonal* unitary matrices, which is an Abelian subgroup of $\mathcal{U}(N)$. We note that as real vector spaces,
   $$\operatorname{Herm}_N(\mathbb{C}) \cong \mathbb{R}^{N^2} .$$
   Moreover, as real manifolds, $\dim_{\mathbb{R}}(\mathcal{U}(N)) = N^2$. As such, when we unitarily diagonalize a Hermitian matrix $A = A^*$ to factorize it as
   $$A = U^* \Lambda U$$
   with $U \in \mathcal{U}(N)$ the matrix of orthonormal eigenvectors and $\Lambda = \operatorname{diag}(\Lambda_1, \cdots, \Lambda_N) \in \mathbb{R}^N$ the eigenvalues, the matrix U is not fully determined, since U determines the eigenvectors of A, but each of these eigenvectors is still free to have a phase gauge degree of freedom: If $A\psi = a\psi$ then also $A\mathrm{e}^{\mathrm{i}\theta}\psi = a\mathrm{e}^{\mathrm{i}\theta}\psi$. As such, $U\operatorname{diag}\left(\mathrm{e}^{\mathrm{i}\theta_1}, \cdots, \mathrm{e}^{\mathrm{i}\theta_N}\right)$ (for some $\theta_1, \cdots, \theta_N \in \mathbb{R}$) is also a "valid" unitary which diagonalizes A. If we want to work towards a change of variable

formula, we need the diagonalization map to be well-defined. One way to deal with this is to rather work with the quotient space

$$\mathcal{U}(N)/\mathbb{T}^N\,,$$

i.e., equivalence classes of unitary matrices up to diagonal unitary matrices (which are precisely the phases of the eigenvectors). Since both $\mathcal{U}(N)$ and $\mathbb{T}^N$ are Lie groups, we need to establish that the quotient $\mathcal{U}(N)/\mathbb{T}^N$ is also one and consider it as a real manifold of dimension N^2-N. We then need to find a chart for this manifold. Once this is done, we define a map

$$\varphi:\mathrm{Herm}_N(\mathbb{C})\to\mathbb{R}^N\times\left(\mathcal{U}(N)/\mathbb{T}^N\right)$$

by

$$A\mapsto(\Lambda,[U])\equiv\left(\varphi_\Lambda(A),\varphi_{[U]}(A)\right)$$

where $U\in\mathcal{U}(N)$, $\Lambda\in\mathbb{R}^N$ and $A\equiv U^*\Lambda U$.

Work out the change of variable formula in this case for φ, i.e., find some measurable $\delta:\mathbb{R}^N\to\mathbb{C}$ measurable so that the following equation holds for any measurable $f:\mathbb{R}^N\to\mathbb{C}$:

$$\int_{A\in\mathrm{Herm}_N(\mathbb{C})}f\left(\varphi_\Lambda(A)\right)\mathrm{d}\lambda(A)=\int_{\Lambda\in\mathbb{R}^N}f(\Lambda)\,\delta(\Lambda)\,\mathrm{d}\lambda(\Lambda)\,.$$

We identify

$$\delta(\Lambda)=\int_{[U]\in\mathcal{U}(N)/\mathbb{T}^N}\left|\det\left((\mathcal{D}\varphi)(\Lambda,[U])\right)\right|\mathrm{d}H([U])$$

where $H:\mathrm{Msrbl}\left(\mathcal{U}(N)/\mathbb{T}^N\right)\to[0,1]$ is the appropriate measure.

6. Let $(\Omega,\mathrm{Msrbl}(\Omega),\mathbb{P})$ be a probability space. Find a sequence $\{\,E_\alpha\,\}_{\alpha\in A}\subseteq\mathrm{Msrbl}(\Omega)$ which is merely *pairwise independent* yet not fully independent according to the definition.

7. Let $X,Y,Z:\Omega\to[0,\infty)$ be independent identically distributed random variables, all with the distribution $\mu:\mathrm{Msrbl}(\Omega)\to[0,1]$. Define

$$F(t):=\mu((0,t])\qquad(t>0)\;.$$

Show that the probability of the event

$$\left\{\,\omega\in\Omega\;\middle|\;X(\omega)\,t^2+Y(\omega)\,t+Z(\omega)=0\text{ for the unknown }t\text{ has real roots}\,\right\}$$

equals

$$\int_{t=0}^\infty\int_{s=0}^\infty F\left(\frac{t^2}{4s}\right)\mathrm{d}\mu(t)\mathrm{d}\mu(s)\,.$$

8. Let $X:\Omega\to\mathbb{R}$ be a random variable with $\frac{\mathrm{d}\mathbb{P}_X}{\mathrm{d}\lambda}(-x)=\frac{\mathrm{d}\mathbb{P}_X}{\mathrm{d}\lambda}(x)$ for all $x\in\mathbb{R}$. Calculate $\frac{\mathrm{d}\mathbb{P}_{X^2}}{\mathrm{d}\lambda}$ in terms of $\frac{\mathrm{d}\mathbb{P}_X}{\mathrm{d}\lambda}$.

9. (*The Hausdorff moment problem*) Let $\{\,m_n\,\}_{n=1}^\infty\subseteq\mathbb{R}$ be given. We seek necessary and sufficient conditions on this sequence for there to exist a random variable $X:\mathbb{R}\to[0,1]$ such that

$$\mathbb{E}[X^n]=m_n\quad(n\in\mathbb{N})\;.$$

A sequence m is called *completely monotonic* iff

$$(-1)^k\left((L-\mathbb{1})^k\,m\right)_n\geq0\qquad(n,k\in\mathbb{N}_{\geq0})$$

where L is the left shift operator on sequences, $(Lm)_n\equiv m_{n+1}$. Show that m is the moments of a random variable iff m is completely monotonic.

10. One could also ask which functions $f:[0,\infty)\to[0,\infty)$ are the Laplace transform of some positive Borel measure, i.e., so that there exists a positive Borel measure

$$\mu:\mathfrak{B}([0,\infty))\to[0,\infty)$$

so that

$$f(t)=\int_{x=0}^\infty\mathrm{e}^{-tx}\mathrm{d}\mu(x)\qquad(t\in[0,\infty))\;.$$

Define a function $f:[0,\infty)\to[0,\infty)$ to be *completely monotone* iff it is continuous on $[0,\infty)$, smooth on $(0,\infty)$ and satisfies

$$(-1)^n f^{(n)}(t)\geq0\qquad(n\in\mathbb{N},t>0)\;.$$

Show that f is completely monotone iff it is the Laplace transform of some non-negative finite Borel measure on $[0,\infty)$.