Princeton University
Spring 2025 MAT425: Measure Theory
HW7
Mar 30th 2025

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1. Provide an example of a measure $\mu : \mathcal{B}(\mathbb{R}^n) \to \mathbb{C}$ and a point $x \in \mathbb{R}^n$ such that

$$\lim_{\varepsilon \to 0^+} \frac{\mu(B_\varepsilon(x))}{\lambda(B_\varepsilon(x))}$$

does not exist.

2. Let $\mu : \mathcal{B}(\mathbb{S}^1) \to \mathbb{C}$ be a measure (here we parametrize $\mathbb{S}^1 = [0, 2\pi)$ with end points identified) and define

$$\hat{\mu} : \mathbb{Z} \to \mathbb{C}$$

via

$$n \mapsto \int \mathrm{e}^{-int} \mathrm{d}\mu(t) .$$

   (a) Prove that if $\lim_{n\to\infty} \hat{\mu}(n) = 0$ then $\lim_{n\to-\infty} \hat{\mu}(n) = 0$. *Hint*: argue why μ may be replaced by $|\mu|$ by replacing μ with f where f is a trigonometric polynomial, a continuous function, or a bounded Borel function.

   (b) What is a condition on μ so that there exists some $k \in \mathbb{N}$ such that

$$\hat{\mu}(n+k) = \hat{\mu}(n) \qquad (n \in \mathbb{Z})\,?$$

3. Show that if $f \in L^1(\mathbb{R}^n \to \mathbb{C}, \lambda)$ and $x \in \mathbb{R}^n$ is a Lebesgue point of f then $|f(x)| \leq (\mathcal{M}_\lambda f)(x)$.

4. Find a continuous monotone function $f : \mathbb{R} \to \mathbb{R}$ so that f is not constant on *any* interval although $f' = 0$ λ-almost-everywhere.

5. Find a monotone function $f : \mathbb{R} \to \mathbb{R}$ so that f' exists and is finite on $\mathbb{R}$, but f' is not a continuous function.

6. Let $f \in L^1(\mathbb{R}^n \to \mathbb{C}, \lambda)$ such that $\mathcal{M}_\lambda f \in L^1(\mathbb{R}^n \to \mathbb{C}, \lambda)$. Show that $f = 0$ λ-almost-everywhere. *Hint*: For every $f \in L^1(\mathbb{R}^n)$ show there is a constant $c > 0$ such that

$$(\mathcal{M}_\lambda f)(x) \gtrsim c\|x\|^{-n} \qquad (\text{large } x) .$$

That means that $\mathcal{M}_\lambda f$ is *not* in L^1.

7. Define f by

$$x \mapsto \begin{cases} \frac{1}{x[\log(x)]^2} & x \in \left(0, \frac{1}{2}\right) \\ 0 & \text{else} \end{cases} .$$

Show that $f \in L^1$. Show that

$$(\mathcal{M}_\lambda f)(x) \geq \frac{1}{|2x \log(2x)|} \qquad \left(x \in \left(0, \frac{1}{4}\right)\right)$$

and hence

$$\int_0^1 \mathcal{M}_\lambda f \mathrm{d}\lambda = \infty .$$

8. Calculate both the symmetric derivative and the Hardy-Littlewood maximal function for the following measures:

   (a) δ_{x_0} for some $x_0 \in \mathbb{R}^n$.

   (b) c, the counting measure on $\mathbb{R}^n$.

   (c) $\varphi_{\lambda,f}$ for some $f \in L^1(\mathbb{R}^n \to \mathbb{C}, \lambda)$, i.e., $\mathrm{d}\varphi_{\lambda,f} \equiv f\mathrm{d}\lambda$. Make your calculation more explicit assuming that f is continuous.

9. Let

$$\varphi : [0,\infty) \times [0,2\pi) \to \mathbb{R}^2$$

   be given by

$$\varphi(r,\theta) := \begin{bmatrix} r\cos(\theta) \\ r\sin(\theta) \end{bmatrix} .$$

   (a) Show that φ is continuously differentiable and injective.

   (b) Calculate $\mathscr{D}\varphi$ and $|\det(\mathscr{D}\varphi)|$.

   (c) Let $f : [0,\infty) \times [0,2\pi) \to \mathbb{C}$ be given. Calculate the right hand side of

$$\int_{\mathbb{R}^2} f\mathrm{d}\lambda = \int_{[0,\infty)\times[0,2\pi)} (f\circ\varphi)(r,\theta)\,|\det(\mathscr{D}\varphi)|(r,\theta)\,\mathrm{d}\lambda(r,\theta) .$$

10. A map $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is a contraction iff

$$L_\varphi := \sup_{x,y} \frac{\|\varphi(x) - \varphi(y)\|}{\|x-y\|} < 1 .$$

   (a) Let $\mathbb{1} : \mathbb{R}^n \to \mathbb{R}^n$ by the identity mapping $x \mapsto x$. Show that if $\mathbb{1} - \varphi$ is a contraction then φ is injective.

   (b) Show that if $\|\mathbb{1} - \mathscr{D}\varphi\| < 1$ then φ is injective.

11. Prove that if $\mu : \mathfrak{M} \to \mathbb{C}$ is a measure such that $\mu(X) = |\mu|(X)$ then $\mu = |\mu|$.

12. Let $\mu : \mathfrak{M} \to \mathbb{C}$ be a measure. Show that for all $A \in \mathfrak{M}$,

$$\begin{aligned} |\mu|(A) &= \sup\left(\left\{ \sum_{i=1}^{n} |\mu(A_i)| \;\middle|\; n \in \mathbb{N} \wedge A_1, \cdots A_n \text{ is a finite pairwise disjoint partition of } A \right\}\right) \\ &= \sup\left(\left\{ \left|\int_A f\mathrm{d}\mu\right| \;\middle|\; |f| \leq 1 \right\}\right) . \end{aligned}$$