Princeton University
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HW7 Sample Solutions
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Question 1

Construct a measure $\mu : \mathcal{B}(\mathbb{R}^n) \to \mathbb{C}$ and a point $x \in \mathbb{R}^n$ for which the limit below does not exist.

$$\lim_{\epsilon \to 0^+} \frac{\mu(B_\epsilon(x))}{\lambda(B_\epsilon(x)}$$

Solution. We will take $x = 0$ and give a example that works uniformly for all n. Define $E \subseteq \mathbb{R}^n$ by

$$E := \{x \in \mathbb{R}^n \setminus \{0\} \mid \text{the integer } \lfloor -\log_3 \|x\| \rfloor \text{ is even}\}$$

One checks that $E \in \mathcal{B}(\mathbb{R}^n)$.[1] Define μ by $d\mu = \chi_E \, d\lambda$. That is $\mu(S) = \lambda(S \cap E)$ for all $S \in \mathcal{B}(\mathbb{R}^n)$. Consider the limit above with ϵ taking the values 3^{-m} for $m \in \mathbb{N}$. If $\epsilon = 3^{-m}$ with m is odd, we have $B_\epsilon(0) \cap E \subseteq \overline{B}_{\epsilon/3}(0)$.[2] So

$$\mu(B_\epsilon(0)) = \lambda(B_\epsilon(0) \cap E) \leq \lambda(\overline{B}_{\epsilon/3}(0)) = \frac{1}{3^n} \lambda\left(B_\epsilon(0)\right)$$

If $\epsilon = 3^{-m}$ with m even, then $B_\epsilon(0) - \overline{B}_{\epsilon/3}(0) \subseteq E$. Hence

$$\mu(B_\epsilon(0)) = \lambda(B_\epsilon(0) \cap E) \geq \lambda(B_\epsilon(0)) - \lambda(\overline{B}_{\epsilon/3}(0)) = \left(1 - \frac{1}{3^n}\right) \lambda(B_\epsilon(0))$$

Thus the ratio $\frac{\mu(B_\epsilon(0)}{\lambda(B_\epsilon(0))}$ can be as small as $\frac{1}{3^n}$ and as large as $1 - \frac{1}{3^n}$ for arbitrarily small $\epsilon > 0$. Since $1 - \frac{1}{3^n} > \frac{1}{3^n}$ holds (for all $n \geq 1$), these bounds are incompatible with the existence of the limit above. $\square$

[1] E is in fact *locally-closed*. That is, it is the intersection of an open and a closed subset of $\mathbb{R}^n$.

[2] We write $\overline{B}_r(x)$ to denote the *closed* ball of radius $r > 0$ centred at $x \in \mathbb{R}^n$.

Question 2

Let $\mu : \mathcal{B}(\mathbb{S}^1) \to \mathbb{C}$ be a complex measure on the unit circle, which by abuse of notation we will identify with the topological quotient of the interval $[0, 2\pi]$ obtained by identifying the two endpoints via the map $t \mapsto (\cos t, \sin t)$. We define $\hat{\mu} : \mathbb{Z} \to \mathbb{C}$ via

$$\mu(n) := \int_{\mathbb{S}^1} e^{-int} \, d\mu(t)$$

(a) Show that if $\hat{\mu}(n) \longrightarrow 0$ as $n \longrightarrow +\infty$ then also $\hat{\mu}(n) \longrightarrow 0$ as $n \longrightarrow -\infty$.

(b) Give a criterion which guarantees that $\hat{\mu}$ is periodic.

Solution to (a). We define a *good* subset of $L^1(\mathbb{S}^1 \to \mathbb{C}, \mu) \equiv L^1(\mathbb{S}^1 \to \mathbb{C}, |\mu|)$ by

$$\mathcal{G} := \left\{ f \in L^1(\mathbb{S}^1 \to \mathbb{C}, \mu) \;\middle|\; \lim_{n \to +\infty} \int_{\mathbb{S}^1} e^{-int} f(t) \, d\mu(t) = 0 \right\}$$

Note that if $f, g \in \mathcal{G}$ and $z, w \in \mathbb{C}$ are arbitrary, then

$$\lim_{n \to +\infty} \int_{\mathbb{S}^1} e^{-int} (zf + wg)(t) \, d\mu(t) = z \lim_{n \to +\infty} \int_{\mathbb{S}^1} e^{-int} f(t) \, d\mu(t) + w \lim_{n \to +\infty} \int_{\mathbb{S}^1} e^{-int} g(t) \, d\mu(t) = z \cdot 0 + w \cdot 0 = 0$$

It follows that $\mathcal{G}$ is a $\mathbb{C}$-vector subspace of $L^1(\mathbb{S}^1 \to \mathbb{C}, \mu)$. In fact, $\mathcal{G}$ is a *closed* subspace of $L^1(\mathbb{S}^1 \to \mathbb{C}, \mu)$. To see this, suppose $f_1, f_2, \dots$ is a sequence of $L^1(|\mu|)$-functions converging in $\mathcal{G}$ to a function $f \in L^1(|\mu|)$. For each $k \in \mathbb{N}$,

$$\begin{aligned}
\left| \int_{\mathbb{S}^1} e^{-int} f(t) \, d\mu(t) \right| &\le \left| \int_{\mathbb{S}^1} e^{-int} f_k(t) \, d\mu(t) \right| + \left| \int_{\mathbb{S}^1} e^{-int} (f(t) - f_k(t)) \, d\mu(t) \right| \\
&\le \left| \int_{\mathbb{S}^1} e^{-int} f_k(t) \, d\mu(t) \right| + \int_{\mathbb{S}^1} |f(t) - f_k(t)| \; d\,|\mu(t)| \\
&= \left| \int_{\mathbb{S}^1} e^{-int} f_k(t) \, d\mu(t) \right| + \|f - f_k\|_{L^1(|\mu|)}
\end{aligned}$$

Taking limit suprema as $n \longrightarrow +\infty$, the last integral vanishes because $f_k \in \mathcal{G}$ and we get

$$\limsup_{n \to +\infty} \left| \int_{\mathbb{S}^1} e^{-int} f(t) \, d\mu(t) \right| \le \|f - f_k\|_{L^1(|\mu|)}$$

Since $\|f - f_k\|_{L^1(|\mu|)}$ gets arbitrarily small as we let $k \longrightarrow +\infty$, we conclude that $f \in \mathcal{G}$.

We now exhibit sufficiently many elements of $\mathcal{G}$ to be able to conclude that $\mathcal{G} = L^1(\mathbb{S}^1 \to \mathbb{C}, \mu)$. For $m \in \mathbb{N}$, define $T_m : \mathbb{S}^1 \to \mathbb{C}$ by $T_m(t) := e^{imt}$. Then

$$\lim_{n \to +\infty} \int_{\mathbb{S}^1} e^{-int} T_m(t) \, d\mu(t) = \lim_{n \to +\infty} \int_{\mathbb{S}^1} e^{-i(n-m)t} \, d\mu(t) = \lim_{n \to +\infty} \hat{\mu}(n - m) = \lim_{n \to +\infty} \hat{\mu}(n) = 0$$

It follows that $T_m \in \mathcal{G}$ for all $m \in \mathbb{N}$. Note that T_m is *continuous* on $\mathbb{S}^1$ since it is continuous on $[0, 2\pi]$ and satisfies $T_m(0) = T_m(2\pi)$. Let $\mathcal{A}$ be the $\mathbb{C}$-vector subspace of $C(\mathbb{S}^1) \subset L^1(\mathbb{S}^1 \to \mathbb{C}, \mu)$ generated by $\{T_m \mid m \in \mathbb{N}\}$. Note that $\mathcal{A} \subseteq \mathcal{G}$ by our observations above. Since $T_{m_1} T_{m_2} = T_{m_1+m_2}$, we see that $\mathcal{A}$ is actually a *subalgebra* of $C(\mathbb{S}^1)$. Since T_1 is injective, $\mathcal{A}$ separates points. Since T_0 is the constant 1, $\mathcal{A}$ vanishes nowhere. Finally, since $\overline{T_m} = T_{-m}$ for each m, $\mathcal{A}$ is closed under complex conjugation. It then follows from the Complex Stone-Weierstrass Theorem (Theorem 4.51 in Folland's *Real Analysis*) and the compactness of $\mathbb{S}^1$ that $\mathcal{A}$ is dense in $C(\mathbb{S}^1)$ (equipped with the sup-norm).

Consider the closure of $\mathcal{A}$ in $L^1(\mathbb{S}^1 \to \mathbb{C}, \mu)$ (which we know is contained in $\mathcal{G}$). Since any function in $C(\mathbb{S}^1)$ is a *uniform* limit of functions in $\mathcal{A}$ (and since $|\mu|$ is a finite measure) it is *a fortiori* an $L^1(|\mu|)$ limit of functions in $\mathcal{A}$. So $C(\mathbb{S}^1) \subseteq \overline{\mathcal{A}} \subseteq \mathcal{G}$. But by HW5Q15 we have that $C(\mathbb{S}^1)$ is *dense* in $L^1(\mathbb{S}^1 \to \mathbb{C}, \mu)$. It follows that $\mathcal{G} = L^1(\mathbb{S}^1 \to \mathbb{C}, \mu)$.

Now let $f = \overline{\left(\frac{d\mu}{d|\mu|}\right)}^2$. Then f has constant absolute value 1, so certainly $f \in L^1(\mathbb{S}^1 \to \mathbb{C}, \mu)$. Hence $f \in \mathcal{G}$, so

$$\begin{aligned}
0 &= \lim_{n\to+\infty} \int_{\mathbb{S}^1} e^{-int} f(t)\, d\mu(t) \\
&= \lim_{n\to+\infty} \int_{\mathbb{S}^1} e^{-int} \overline{\left(\frac{d\mu}{d|\mu|}(t)\right)}^2 \cdot \left(\frac{d\mu}{d|\mu|}(t)\right) d|\mu|(t) \\
&= \lim_{n\to+\infty} \int_{\mathbb{S}^1} e^{-int} \overline{\left(\frac{d\mu}{d|\mu|}(t)\right)} d|\mu|(t) \\
&= \overline{\lim_{n\to+\infty} \int_{\mathbb{S}^1} e^{int} \frac{d\mu}{d|\mu|}(t)\, d|\mu|(t)} \\
&= \overline{\lim_{n\to+\infty} \int_{\mathbb{S}^1} e^{int} d\mu(t)} \\
&= \overline{\lim_{n\to-\infty} \int_{\mathbb{S}^1} e^{-int} d\mu(t)} \\
&= \overline{\lim_{n\to-\infty} \hat{\mu}(n)}
\end{aligned}$$

where in an intermediate step we used Theorem 5.62 in the lecture notes. We deduce $\lim_{n\to-\infty} \hat{\mu}(n) = 0$. □

Solution to (b). Consider the subset of $\mathbb{S}^1$ determined by

$$\Psi := [0, 2\pi] \cap \pi\,\mathbb{Q} = \{t \in [0, 2\pi] \mid t/\pi \text{ is rational}\}$$

We claim that if μ is concentrated on a *finite subset* of Ψ then $\hat{\mu}$ is k-periodic for some k. To see this, choose $t_1, \ldots, t_m$ in $[0, 2\pi]$ such that t_j/π is rational for each j and μ is supported on $\{t_1, \ldots, t_m\}$.[3] Let N be a positive integer such that $Nt_1/\pi, \ldots, Nt_m/\pi$ are all integers.[4] Then the functions $t \mapsto e^{-2iNt}$

[3]This implies that $\mu = \sum_{j=1}^{m} a_j \delta_{t_j}$ for some $a_1, \ldots, a_m \in \mathbb{C}$.

[4]For instance, one can take the least-common-multiple of the denominators of $t_1/\pi, \ldots, t_m/\pi$ expressed in lowest terms.

and $t \mapsto 1$ are equal $|\mu|$-almost everywhere—because they agree on $\{t_1, \ldots, t_m\}$. This is because for each j, $n_j := Nt_j/\pi$ is an integer, so we get

$$e^{-2iNt_j} = e^{-2\pi i \cdot (Nt_j/\pi)} = e^{-2\pi i n_j} = \left(e^{2\pi i}\right)^{-n_j} = 1$$

Since integrals relative to μ do not change when we change the integrand by a $|\mu|$-almost everywhere equivalent function, it follows that letting $k = 2N$, we have for all $n \in \mathbb{N}$

$$\hat{\mu}(n+k) = \int_{\mathbb{S}^1} e^{-i(n+k)t}\, d\mu(t) = \int_{\mathbb{S}^1} e^{-int} e^{-ikt}\, d\mu(t) = \int_{\mathbb{S}^1} e^{-int} e^{-2iNt}\, d\mu(t) = \int_{\mathbb{S}^1} e^{-int} \cdot 1\, d\mu(t) = \hat{\mu}(n)$$

□

Question 3

If $f \in L^1(\mathbb{R}^n \to \mathbb{C}, \lambda)$ and $x \in \mathbb{R}^n$ is a Lebesgue point of f then $|f(x)| \leq (\mathcal{M}_\lambda f)(x)$.

Solution. For any $\epsilon > 0$, we have, using the definition of $\mathcal{M}_\lambda f$ followed by the triangle inequality

$$\begin{aligned}
(\mathcal{M}_\lambda f)(x) &\geq \frac{1}{\lambda(B_\epsilon(x))} \int_{B_\epsilon(x)} |f(y)|\, d\lambda(y) \\
&\geq \frac{1}{\lambda(B_\epsilon(x))} \int_{B_\epsilon(x)} |f(x)|\, d\lambda(y) - \frac{1}{\lambda(B_\epsilon(x))} \int_{B_\epsilon(x)} |f(y) - f(x)|\, d\lambda(y) \\
&= |f(x)| - \frac{1}{\lambda(B_\epsilon(x))} \int_{B_\epsilon(x)} |f(y) - f(x)|\, d\lambda(y)
\end{aligned}$$

Taking the limit as $\epsilon \longrightarrow 0^+$, the last integral term has limit 0 by hypothesis, giving $(\mathcal{M}_\lambda f)(x) \geq |f(x)|$. □

Question 4

Construct a continuous monotonic function $f : \mathbb{R} \to \mathbb{R}$ which is not constant on any interval but whose derivative vanishes λ-almost-everywhere.

Solution. Let $c : [0, 1] \to [0, 1]$ be the Cantor function constructed in HW3Q12. Recall the following properties of c:

- $c(0) = 0$, $c(1) = 1$.
- $c(t) > 0$ if $t > 0$ and $c(t) < 1$ if $t < 1$.
- c is (weakly) increasing.
- c is continuous on its domain.

- c is constant on every subinterval of its domain which lies in the complement of the Cantor set C.

We extend c to a function $\tilde{c} : \mathbb{R} \to \mathbb{R}$ by letting

$$\tilde{c}(t) := \begin{cases} 0 & \text{if } t < 0 \\ c(t) & \text{if } 0 \le t \le 1 \\ 1 & \text{if } t > 1 \end{cases}$$

Then $\tilde{c}$ satisfies all the above properties of c and also still takes values in $[0, 1]$.

Let $\gamma : \mathscr{B}(\mathbb{R}) \to \mathbb{R}$ be the Lebesgue-Stieltjes measure of $\tilde{c}$ (c.f. HW3Q5). Then γ satisfies:

- $\gamma([0, \epsilon)) > 0$ for all $\epsilon > 0$.
- γ is concentrated on the Cantor set.[5]
- Consequently, $\gamma \perp \lambda$.

Let $q_1, q_2, \ldots$ be an enumeration of $\mathbb{Q}$. For each j, let γ_j be the pushforward measure under the homeomorphism $t \mapsto t + q_j$ of $\mathbb{R}$. In other words, γ_j is the Lebesgue-Stieltjes measure of the function $\tilde{c}_j : t \mapsto \tilde{c}(t - q_j)$.

Define

$$\gamma_\infty := \sum_{k=1}^{\infty} 2^{-j} \gamma_j$$

Then γ_∞ is the Lebesgue-Stieltjes measure of

$$f := \sum_{j=1}^{\infty} 2^{-j} \tilde{c}_j$$

Since each $\tilde{c}_j$ is pointwise bounded by 1 in absolute value, the series defining f converges uniformly. Since each summand is continuous, it follows that f is continuous.[6] Since f is a sum of (weakly) monotonic functions, it too is (weakly) monotonic. Also, for any $x < y$ in $\mathbb{R}$ we can find a rational number q_k such that $x < q_k < y$. Then

$$f(y) - f(x) = \gamma_\infty((x, y]) \ge 2^{-k} \gamma_k((x, y]) \ge 2^{-k} \gamma_k([q_k, y)) = 2^{-k} \gamma([0, y - q_k)) > 0$$

So f is not constant on any intervals (i.e. strictly monotonic).

Since γ is concentrated on the Cantor set C, each γ_j is concentrated on the translated copy $C + q_j$. It follows that γ_∞ is concentrated on the countable union $\bigcup_{j=1}^{\infty} (C + q_j)$. In particular, this is a λ-measure 0 set. So $\gamma_\infty \perp \lambda$. By

[5] This is seen by noting that any point in the complement of the Cantor set is contained in an open interval where c is constant, hence an open interval of γ-measure 0.

[6] This can also be seen on the measure side: a sum of atomless measures is again atomless.

the form of the Lebesgue Differentiation Theorem found as Theorem 7.15 in Rudin's *Real and Complex Analysis*, one has

$$D\gamma_\infty(x) \equiv \lim_{r\to 0^+} \frac{\gamma_\infty([x-r, x+r])}{2r} = 0$$

for λ-almost all $x \in \mathbb{R}$. This forces

$$\lim_{r\to 0^+} \frac{\gamma_\infty((x-r, x])}{2r} = 0, \qquad \lim_{r\to 0^+} \frac{\gamma_\infty((x, x+r])}{2r} = 0$$

Multiplying by 2 and writing things in terms of f, we get

$$\lim_{r\to 0^+} \frac{f(x)-f(x-r)}{r} = 0, \qquad \lim_{r\to 0^+} \frac{f(x+r)-f(x)}{r} = 0$$

Thus we conclude that $f'(x) = 0$ holds λ-almost everywhere. □

Question 5

Construct an everywhere-differentiable monotonic function $f : \mathbb{R} \to \mathbb{R}$ whose derivative is not continuous.

Solution. Define

$$f(x) := \begin{cases} 2x + x^2 \cdot (1 + \sin(x^{-1})) & \text{if } x > 0 \\ -f(-x) & \text{if } x < 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Then on $(0, \infty)$ the function f is visibly differentiable and

$$f'(x) = 2 + 2x(1 + \sin(x^{-1})) - \cos(x^{-1})$$

Since x, $1 + \sin(x^{-1})$ and $1 - \cos(x^{-1})$ are all non-negative on this interval, we see that $f'(x) > 1$ for $x > 0$. For $x < 0$, we have

$$f'(x) = (-f(-x))' = f'(-x) > 1$$

It follows that f is strictly monotonic on each of the intervals $(-\infty, 0)$ and $(0, \infty)$.

Since $f(x) = 2x + O(x^2)$ as $x \longrightarrow 0^+$ and also $f(0) = 0$, it follows that f admits a right-derivative at 0 with value $f'_+(0) = 2$. Since $f(-x) = -f(x)$ (by construction), we also get $f'_-(0) = 2$. Thus f is differentiable—in particular continuous—at 0. It follows that f is differentiable and (strictly) monotonic on $\mathbb{R}$.

Now to see that f' is *not* continuously differentiable at 0, note that the term $2 + 2x(1 + \sin(x^{-1}))$ has limit 2 as $x \longrightarrow 0$. Therefore, if $f'(x)$ had a limit as $x \longrightarrow 0^+$ then so would the remaining term $\cos(x^{-1})$. However, this last term clearly does *not* have a limit as $x \longrightarrow 0^+$ since it takes both values ± 1 arbitrarily close to 0. (Specifically, this happens at the points $x_n := \frac{1}{\pi n}$ for $n \in \mathbb{N}$.) □

Question 6

Let f be a non-zero element of $L^1(\mathbb{R}^n \to \mathbb{C}, \lambda)$. Show that $\mathcal{M}_\lambda f \notin L^1(\mathbb{R}^n \to \mathbb{C})$.

Solution. By assumption $\int_{\mathbb{R}^n} |f| \; d\lambda \neq 0$. It follows (e.g. by monotone convergence) that $C_r := \int_{B_r(0)} |f| \; d\lambda > 0$ for some large enough radius $r > 0$. For any $x \in \mathbb{R}^n$, we have $B_r(0) \subseteq B_{r+\|x\|}(x)$ (by the triangle inequality). Hence

$$(\mathcal{M}_\lambda f)(x) = \sup_{\epsilon>0} \frac{\int_{B_\epsilon(x)} |f| \; d\lambda}{\lambda(B_\epsilon(x))} \geq \frac{\int_{B_{r+\|x\|}(x)} |f| \; d\lambda}{\lambda(B_{r+\|x\|}(x))} \geq \frac{\int_{B_r(0)} |f| \; d\lambda}{\lambda(B_{r+\|x\|}(x))} = \frac{C_r}{\lambda(B_1(0))}(r+\|x\|)^{-n}$$

Now note that the integral $\int_{\|x\|\geq 1} \frac{d\lambda(x)}{\|x\|^n}$ on $\mathbb{R}^n$ does *not* have finite value. We can see this by applying the change-of-variables formula for the scaling function $x \mapsto 2x$. We get

$$\int_{\|x\|\geq 1} \frac{d\lambda(x)}{\|x\|^n} = 2^n \int_{\|2x\|\geq 1} \frac{d\lambda(x)}{\|2x\|^n} = \int_{\|x\|\geq \frac{1}{2}} \frac{d\lambda(x)}{\|x\|^n} = \int_{\|x\|\geq 1} \frac{d\lambda(x)}{\|x\|^n} + \int_{\frac{1}{2}\leq\|x\|\leq 1} \frac{d\lambda(x)}{\|x\|^n}$$

Since the last term is clearly positive, this forces the first integral to be infinite.

Now observe that the expression for $\mathcal{M}_\lambda f$ is, up to a positive scalar, asymptotic to the function $x \mapsto \|x\|^{-n}$. It then follows from the above discussion that $\mathcal{M}_\lambda f$ is *not* in L^1. □

Question 7

Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) := \begin{cases} \frac{1}{x(\log x)^2} & \text{if } x \in \left(0, \frac{1}{2}\right) \\ 0 & \text{otherwise} \end{cases}$$

Then $f \in L^1(\mathbb{R} \to \mathbb{R})$. Show that moreover

$$(\mathcal{M}_\lambda f)(x) \geq \frac{1}{|2x \log 2x|} \quad \text{for } x \in \left(0, \frac{1}{4}\right)$$

and deduce

$$\int_0^1 \mathcal{M}_\lambda f \, d\lambda = \infty$$

Solution. To see that $f \in L^1(\mathbb{R} \to \mathbb{R})$, it of course suffices to show $f \in L^1((0, 1/2) \to \mathbb{R})$. We start by noting that for $\epsilon > 0$,

$$\int_\epsilon^{1/2} f \, d\lambda = \int_\epsilon^{1/2} \frac{dx}{x(\log x)^2} = \left[-\frac{1}{\log x}\right]_{x=\epsilon}^{x=1/2} = \frac{1}{\log 2} - \frac{1}{\log \epsilon}$$

Taking the limit as $\epsilon \longrightarrow 0^+$ and using the monotone convergence theorem, we get

$$\int_0^{1/2} f\, d\lambda = \frac{1}{\log 2}$$

So $f \in L^1$. For the second claim, we use the definition of $\mathcal{M}_\lambda f$ to get (for any $x \in (0, 1/4)$)

$$(\mathcal{M}_\lambda f)(x) \geq \frac{1}{\lambda(B_x(x))} \int_{B_x(x)} f\, d\lambda = \frac{1}{2x} \int_0^{2x} f\, d\lambda = \frac{1}{2x} \int_0^{2x} \frac{dt}{t(\log t)^2} = \frac{1}{2x} \left[-\frac{1}{\log t} \right]_{t=0}^{t=2x} = \frac{1}{2x\,|\log 2x|}$$

For the final conclusion, we write (for $\epsilon > 0$)

$$\int_0^1 (\mathcal{M}_\lambda f)(x)\, d\lambda(x) \geq \int_\epsilon^{1/4} \frac{dx}{2x\,|\log 2x|} = [-\log|\log 2x|]_\epsilon^{1/4} = \log|\log 2\epsilon| - \log\log 2$$

Since $\log|\log 2\epsilon|$ is unbounded as $\epsilon \longrightarrow 0^+$, it follows that $\mathcal{M}_\lambda f \notin L^1$. □

Question 8

Calculate the symmetric derivative and Hardy-Littlewood maximal functions for the following measures:

(a) δ_{x_0} for some $x_0 \in \mathbb{R}^n$.

(b) The counting measure c on $\mathbb{R}^n$.

(c) $\phi_{\lambda,f}$ defined by $d\phi_{\lambda,f} = f\, d\lambda$.

Solution to (a). Let β_n be the volume of the unit ball in $\mathbb{R}^n$. We claim that

$$(\mathcal{M}_\lambda \delta_{x_0})(x) = \begin{cases} \frac{\beta_n^{-1}}{\|x - x_0\|^n} & \text{if } x \neq x_0 \\ \infty & \text{if } x = x_0 \end{cases}$$

To see this, note that for $x \in \mathbb{R}^n$ and $\epsilon > 0$

$$\frac{1}{\lambda(B_\epsilon(x))} \int_{B_\epsilon(x)} \delta_{x_0} = \begin{cases} 0 & \text{if } \epsilon \leq \|x - x_0\| \\ \beta_n^{-1}\epsilon^{-n} & \text{if } \epsilon > \|x - x_0\| \end{cases}$$

Clearly, this value is maximized by letting ϵ be ever-so-slightly greater than $\|x - x_0\|$. This gives the values above. As for the symmetric derivative, we get

$$(\mathcal{D}_\lambda \delta_{x_0})(x) = \begin{cases} 0 & \text{if } x \neq x_0 \\ \infty & \text{if } x = x_0 \end{cases}$$

Since δ_{x_0} is a positive measure, this is seen by simply letting $\epsilon \longrightarrow 0+$ in the formula above. □

Solution to (b). For all $\epsilon > 0$ and $x \in \mathbb{R}^n$, we have

$$\frac{1}{\lambda(B_\epsilon(x))} \int_{B_\epsilon(x)} c = \infty$$

since every open ball contains infinitely many points. It follows that both $\mathcal{M}c$ and $\mathcal{D}_\lambda c$ are identically infinite. □

Solution to (c). We have

$$\mathcal{M}_\lambda \phi_{\lambda,f} = \mathcal{M}_\lambda f = \left(x \mapsto \sup_{r>0} \frac{1}{\beta_n r^n} \int_{B_r(x)} |f(t)| \, d\lambda(t) \right)$$

and

$$\mathcal{D}_\lambda \phi_{\lambda,f} = \mathcal{D}_\lambda f = \left(x \mapsto \lim_{r \to 0^+} \frac{1}{\beta_n r^n} \int_{B_r(x)} f(t) \, d\lambda(t) \right)$$

If f is continuous at $x \in \mathbb{R}^n$ (e.g. if f is a continuous function) then for any fixed $\epsilon > 0$ we can find $r > 0$ such that $|f(t) - f(x)| < \epsilon$ for all $t \in B_r(x)$. Then

$$\left| \frac{1}{\beta_n r^n} \int_{B_r(x)} f(t) \, d\lambda(t) - f(x) \right| \leq \frac{1}{\beta_n r^n} \int_{B_r(x)} |f(t) - f(x)| \, d\lambda(t) \leq \frac{1}{\beta_n r^n} \int_{B_r(x)} \epsilon \, d\lambda(t) = \epsilon$$

Since ϵ can be made arbirarily small, we conclude

$$(\mathcal{D}_\lambda \phi_{\lambda,f})(x) = \lim_{r \to 0^+} \frac{1}{\beta_n r^n} \int_{B_r(x)} f(t) \, d\lambda(t) = f(x)$$

□

Question 9

Define

$$\begin{aligned} \phi : [0, \infty) \times [0, 2\pi) &\to \mathbb{R}^2 \\ (r, \theta) &\mapsto (r \cos\theta, r \sin\theta) \end{aligned}$$

(a) Show that ϕ is continuously differentiable and injective when restricted to $(0, \infty) \times [0, 2\pi)$.

Solution. The continuous-differentiability of ϕ follows from the fact that its component functions are smooth.

For injectivity, suppose $\phi(r_1, \theta_1) = \phi(r_2, \theta_2)$ where $r_1, r_2 > 0$ and $\theta_1, \theta_2 \in [0, 2\pi)$. This means

$$r_1 \cos\theta_1 = r_2 \cos\theta_2 \quad \text{and} \quad r_1 \sin\theta_1 = r_2 \sin\theta_2$$

Then

$$r_1^2 = (r_1 \cos\theta_1)^2 + (r_1 \sin\theta_1)^2 = (r_2 \cos\theta_2)^2 = (r_2 \sin\theta_2)^2 = r_2^2$$

So $r_1 = r_2$ since they are both positive. It then follows from the equations above that $\cos\theta_1 = \cos\theta_2$ and $\sin\theta_1 = \sin\theta_2$. Since the cosine function is strictly monotonic (hence injective) on each of the intervals $[0, \pi]$ and $[\pi, 2\pi]$ and since $\cos(2\pi - \theta) = \cos(\theta)$, the equality $\cos\theta_1 = \cos\theta_2$ forces

$$\theta_1 = \theta_2 \quad \text{or} \quad \theta_1 = 2\pi - \theta_2$$

If $\theta_1 = 2\pi - \theta_2$ then

$$\sin(\theta_2) = \sin(\theta_1) = \sin(2\pi - \theta_2) = \sin(-\theta_2) = -\sin(\theta_2)$$

forces $\sin(\theta_2) = 0$. But given $\theta_2 \in [0, 2\pi)$ this means $\theta_2 \in \{0, \pi\}$. The first is impossible since then $\theta_1 = 2\pi$. The latter gives $\theta_1 = \theta_2 = \pi$. In all cases $\theta_1 = \theta_2$. □

(b) The total derivative of ϕ is given by

$$D\phi = \begin{pmatrix} \frac{\partial}{\partial r} r\cos\theta & \frac{\partial}{\partial \theta} r\cos\theta \\ \frac{\partial}{\partial r} r\sin\theta & \frac{\partial}{\partial \theta} r\sin\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

Hence

$$|\det(D\phi)| = (\cos\theta)(r\cos\theta) - (\sin\theta)(-r\sin\theta) = r(\cos^2\theta + \sin^2\theta) = r$$

(c) Let $f : [0, \infty) \times [0, 2\pi) \to \mathbb{C}$ be given. Calculate the right-hand-side of the change-of-variables formula

$$\int_{\mathbb{R}^2} f \, d\lambda = \int_{[0,\infty)\times[0,2\pi)} (f \circ \phi)(r, \theta) \, |\det(D\phi)| \, (r, \theta) \, d\lambda(r, \theta)$$

Solution. Substituting gives

$$\int_{\mathbb{R}^2} f \, d\lambda = \int_{[0,\infty)\times[0,2\pi)} r \cdot f(r\cos\theta, r\sin\theta) \, d\lambda(r, \theta)$$

□

Question 10

A map $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is called a *contraction* if it satisfies

$$L_\phi := \sup_{x \neq y} \frac{\|\phi(x) - \phi(y)\|}{\|x - y\|} < 1$$

(a) Let $\mathbb{1} : \mathbb{R}^n \to \mathbb{R}^n$ be the identity map. Show that if $\mathbb{1} - \phi$ is a contraction then ϕ is injective.

(b) Show that if ϕ is differentiable and $\|\mathbb{1} - D\phi\| < 1$ holds pointwise on $\mathbb{R}^n$ then ϕ is injective.

Solution to (a). Let $x \neq y$ be distinct points in $\mathbb{R}^n$. Then the assumption on $\mathbb{1} - \phi$ implies

$$\|(\mathbb{1} - \phi)(x) - (\mathbb{1} - \phi)(y)\| < \|x - y\|$$

That is,

$$\|(x - y) - (\phi(x) - \phi(y))\| < \|x - y\|$$

Using the triangle inequality, we deduce

$$\|\phi(x) - \phi(y)\| \geq \|x - y\| - \|(x - y) - (\phi(x) - \phi(y))\| > 0$$

So $\phi(x) \neq \phi(y)$. Since x, y we arbitrary, the injectivity of ϕ results. □

Solution to (b). Fix $x \neq y$ in $\mathbb{R}^n$. We will show $f(x) \neq f(y)$. Consider the function $f : \mathbb{R} \to \mathbb{R}^n$ given by

$$f(t) := \phi(tx + (1 - t)y)$$

The (multivariate) chain rule gives

$$f'(t) = D\phi(tx + (1 - t)y) \cdot (x - y) = (x - y) - (\mathbb{1} - D\phi(tx + (1 - t)y)) \cdot (x - y)$$

Define $g : \mathbb{R} \to \mathbb{R}$ by $g(t) := \langle x - y, f(t) \rangle$. Then

$$g'(t) = \langle x - y, f'(t) \rangle = \|x - y\|^2 - \langle x - y, (\mathbb{1} - D\phi(tx + (1 - t)y)) \cdot (x - y) \rangle$$

If we can show that $g'(t) > 0$ for $t \in (0, 1)$, then (by the mean value theorem) this will imply that g is strictly increasing on $[0, 1]$ which then implies that

$$\langle x - y, \phi(y) \rangle = \langle x - y, f(0) \rangle < \langle x - y, f(1) \rangle = \langle x - y, \phi(x) \rangle$$

which in turn implies $\phi(x) \neq \phi(y)$.

All that now remains is to prove the inequality

$$\|x - y\|^2 > \langle x - y, (\mathbb{1} - D\phi(tx + (1 - t)y)) \cdot (x - y) \rangle$$

pointwise for $t \in (0, 1)$. Thinking of $(x - y)$ as a column vector, and writing $A := \mathbb{1} - D\phi(tx + (1 - t)y)$

$$\begin{aligned} \langle x - y, (\mathbb{1} - D\phi(tx + (1 - t)y)) \cdot (x - y) \rangle &= (x - y)^\top A(x - y) \\ &\leq \|x - y\| \|A\| \|x - y\| \\ &< \|x - y\|^2 \end{aligned}$$

Here $^\top$ is the transpose operator, and the last inequality is obtained from the hypothesis $\|A\| < 1$.[7] □

[7] The second step requires some justification. The justification depends on the precise norm

Question 11

Let $(X, \mathfrak{M})$ be a measurable space. If $\mu : \mathfrak{M} \to \mathbb{C}$ is a complex measure satisfying $\mu(X) = |\mu|(X)$ then $\mu = |\mu|$.

Solution. It suffices to show that μ takes values in $[0, \infty)$ (c.f. Theorem 5.54(3) in the lecture notes).

For any complex number z, we write z^+ for the positive part of the real part of z. That is, $z^+ = \Re(z)$ if $\Re(z) > 0$ and $z^+ = 0$ otherwise. Note that $|z| \geq z^+$ with equality if and only if z is a non-negative real. Also for $z, w \in \mathbb{C}$ we have $z^+ + w^+ \geq (z + w)^+$.

Let $S \in \mathfrak{M}$ be arbitrary. We will show that $\mu(S)$ is a non-negative real number. Let $T = X - S$. Then $S, T, \varnothing, \varnothing, \ldots$ is an admissible partition of X. It follows by definition of $|\mu|$ that

$$|\mu|(X) \geq |\mu(S)| + |\mu(T)| \geq \mu(S)^+ + \mu(T)^+ \geq (\mu(S) + \mu(T))^+ = \mu(X)^+ = |\mu(X)|$$

Since the first and last terms of this sequence of inequalities are equal, each inequality must in fact be an equality. In particular, we must have $|\mu(S)| + |\mu(T)| = \mu(S)^+ + \mu(T)^+$. Since $|\mu(S)| \geq \mu(S)^+$ and $|\mu(T)| \geq \mu(T)^+$, we are forced to conclude $|\mu(S)| = \mu(S)^+$ and $|\mu(T)| = \mu(T)^+$. It follows that $\mu(S)$ is a non-negative real number. $\square$

Question 12

Let $\mu : \mathfrak{M} \to \mathbb{C}$ be a complex measure on a measurable space $(X, \mathfrak{M})$. Then for all $A \in \mathfrak{M}$,

$$\begin{aligned} |\mu|(A) &= \sup \left\{ \sum_{i=1}^{n} |\mu(A_i)| \;\middle|\; A_1, A_2, \ldots, A_n \text{ form a partition of } A, \text{ and } A_i \in \mathfrak{M} \text{ for each } i \right\} \\ &= \sup \left\{ \left| \int_A f \, d\mu \right| \;\middle|\; |f| \leq 1 \text{ pointwise} \right\} \end{aligned}$$

Solution. We prove the first equality first. Suppose $A_1, \ldots, A_n$ form a finite partition of A. Then $A_1, \ldots, A_n, \varnothing, \varnothing, \ldots$ form a countable partition of A and

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that is used for A. One possibility is that this is the usual Euclidean norm on $\mathrm{Mat}_{n\times n}$. In this case, we argue via

$$(x-y)^\top A(x-y) = \mathrm{tr}((x-y)^\top A(x-y)) = \mathrm{tr}(A(x-y)(x-y)^\top) \leq \sqrt{\mathrm{tr}(AA^\top)}\sqrt{\mathrm{tr}((x-y)(x-y)^\top(x-y)(x-y)^\top)} = \|A\|\|x-y\|^2$$

where in the second-to-last step we used the Cauchy-Schwartz inequality in $\mathrm{Mat}_{n\times n}$. Note that the usual Euclidean norm on $\mathrm{Mat}_{n\times n}$ is given by $M \mapsto \mathrm{tr}(MM^\top)$ where tr is the trace operator. Also, in the above we used the linear-algebraic fact $\mathrm{tr}(XY) = \mathrm{tr}(YX)$.

One could instead consider the (more natural) operator norm on $\mathrm{Mat}_{n\times n}$. In this case, we get $\|A(x-y)\| \leq \|A\|\|x-y\|$ *by definition* and then using Cauchy-Schwartz in $\mathbb{R}^n$ we get

$$\langle x-y, A(x-y)\rangle \leq \|x-y\|\|A(x-y)\| \leq \|x-y\|\|A\|\|x-y\| = \|A\|\|x-y\|^2$$

this gives

$$|\mu(A)| \geq \sum_{i=1}^{n} |\mu(A_i)|$$

Since this holds for all finite partitions of A, we get that $|\mu(A)|$ is at least as large as the first supremum.

For the opposite direction, fix $\epsilon > 0$. The definition of $|\mu|$ ensures we can find a countable partition $A_1, A_2, \ldots$ of A such that

$$|\mu|(A) \leq \sum_{i=1}^{\infty} |\mu(A_i)| + \epsilon$$

Since this sum converges, we can choose $n \in \mathbb{N}$ such that

$$\sum_{i=1}^{\infty} |\mu(A_i)| \leq \sum_{i=1}^{n} |\mu(A_i)| + \epsilon$$

Combining the two, we get

$$|\mu|(A) \leq \sum_{i=1}^{n} |\mu(A_i)| + 2\epsilon$$

Since ϵ can be made arbitrarily small, we get that $|\mu|(A)$ is no less than the first supremum.
Now for the second supremum, note that

$$\left|\int_A f\, d\mu\right| = \left|\int_A f \frac{d\mu}{d|\mu|}\, d|\mu|\right| \leq \int_A \left|f \frac{d\mu}{d|\mu|}\right| d|\mu| = \int_A |f|\; d|\mu| \leq \int_A 1\, d|\mu| = |\mu|(A)$$

This shows that $|\mu|(A)$ is at least as big as the second supremum. For the reverse inequality, let $f := \overline{\frac{d\mu}{d|\mu|}}$ then

$$\left|\int_A f\, d\mu\right| = \left|\int_A f \frac{d\mu}{d|\mu|}\, d|\mu|\right| = \left|\int_A \left|\frac{d\mu}{d|\mu|}\right|^2 d|\mu|\right| = \left|\int_A 1\, d|\mu|\right| = |\mu|(A)$$

So the second supremum is actually a maximum, and the maximum value is $|\mu|(A)$. □