Princeton University
Spring 2025 MAT425: Measure Theory
HW6
Mar 22nd 2025

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1. Prove that if $\mu, \nu : \mathfrak{M} \to \mathbb{C}$ are two measures and $\alpha \in \mathbb{C}$ then both $\alpha\mu$ and $\mu + \nu$ are measures as well.

2. Prove the chain rule for the Radon-Nikodym derivative. Let $(X, \mathfrak{M})$ be a measurable space. Let $\mu : \mathfrak{M} \to [0, \infty]$ be σ-finite, $\nu : \mathfrak{M} \to [0, \infty)$ and $\eta : \mathfrak{M} \to \mathbb{C}$ be measures such that
$$\eta \blacktriangleleft \nu \blacktriangleleft \mu \,.$$
Prove that $\eta \blacktriangleleft \mu$ and that the Radon-Nikodym derivatives satisfy
$$\frac{\mathrm{d}\eta}{\mathrm{d}\mu} = \frac{\mathrm{d}\eta}{\mathrm{d}\nu}\frac{\mathrm{d}\nu}{\mathrm{d}\mu} \,.$$

3. Prove that if $\mu \blacktriangleleft \nu$ and $\nu \blacktriangleleft \mu$ then
$$\frac{\mathrm{d}\mu}{\mathrm{d}\nu} = \frac{1}{\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right)} \,.$$

4. Let $(X, \mathfrak{M})$ be a measurable space. Let $\mu : \mathfrak{M} \to [0, \infty]$ be σ-finite and let $\nu_1, \cdots, \nu_n : \mathfrak{M} \to \mathbb{C}$ be measures such that
$$\nu_i \blacktriangleleft \mu \qquad (i \in \{\, 1, \cdots, n \,\}) \,.$$
Show that the sum measure $\sum_{i=1}^n \nu_i$ obeys
$$\sum_{i=1}^n \nu_i \blacktriangleleft \mu$$
too and calculate
$$\frac{\mathrm{d}\sum_{i=1}^n \nu_i}{\mathrm{d}\mu} \,.$$

5. Let $(X_i, \mathfrak{M}_i)$ be measurable spaces for $i = 1, \cdots, n$ and let $\mu_i : \mathfrak{M}_i \to [0, \infty]$ be σ-finite, $\nu_i : \mathfrak{M}_i \to \mathbb{C}$ be given. Assume that $\nu_i \blacktriangleleft \mu_i$ for all $i = 1, \cdots, n$. Show that
$$\prod_{i=1}^n \nu_i \blacktriangleleft \prod_{i=1}^n \mu_i$$
and that
$$\frac{\mathrm{d}\prod_{i=1}^n \nu_i}{\mathrm{d}\prod_{i=1}^n \mu_i}\left(x_1, \cdots, x_n\right) = \prod_{i=1}^n \frac{\mathrm{d}\nu_i}{\mathrm{d}\mu_i}\left(x_i\right) \qquad \left(x \in \prod_{i=1}^n X_i\right) \,.$$

6. Show that if δ_0 is a Dirac measure then $\delta_0 \perp \lambda$.

7. Consider both the counting measure c and Lebesgue measure λ on the σ-algebra $\mathcal{B}\left([0,1]\right)$. Show that $\lambda \blacktriangleleft c$ but that $\frac{\mathrm{d}\lambda}{\mathrm{d}c}$ does not exist (which hypothesis of the Radon-Nikodym derivative theorem does not hold?).

8. Let a Borel measure $\mu : \mathcal{B}(\mathbb{R}) \to [0,\infty]$ be given such that $\mu(\mathbb{R}) < \infty$ and $H_\mu : \mathbb{C}_+ \to \mathbb{C}_+$ is Stieltjes transform (see the lecture notes for the definition). Let

$$\mu = \mu_{\mathrm{ac}} + \mu_{\mathrm{sc}} + \mu_{\mathrm{pp}}$$

be the Lebesgue decomposition of μ w.r.t. the Lebesgue measure $\lambda : \mathcal{B}(\mathbb{R}) \to [0,\infty]$ (you may refer to the lecture notes for the notion of singular continuous and pure point). Then there exists some $M \in (0,\infty)$ such that

$$|H_\mu(x + \mathrm{i}\varepsilon)| \leq \frac{M}{\varepsilon}\,.$$

Moreover, if the support of any measure ν is defined as

$$\operatorname{supp}(\nu) := \{\, x \in X \mid \forall U \in \operatorname{Open}(X) : x \in U, \mu(U) > 0 \,\}$$

then

$$\operatorname{supp}(\mu_{\mathrm{ac}}) = \left\{ x \in \mathbb{R} \,\middle|\, \lim_{\varepsilon \to 0^+} \mathbb{Im}\{H_\mu(x + \mathrm{i}\varepsilon)\} \in (0,\infty) \right\}$$

and

$$\frac{\mathrm{d}\mu_{\mathrm{ac}}}{\mathrm{d}\lambda}(x) = \lim_{\varepsilon \to 0^+} \frac{1}{\pi} \mathbb{Im}\{H_\mu(x + \mathrm{i}\varepsilon)\} \qquad (x \in \mathbb{R})\,.$$

Moreover,

$$\operatorname{supp}(\mu_{\mathrm{s}}) = \left\{ x \in \mathbb{R} \,\middle|\, \lim_{\varepsilon \to 0^+} \mathbb{Im}\{H_\mu(x + \mathrm{i}\varepsilon)\} = \infty \right\}, \qquad \operatorname{supp}(\mu_{\mathrm{pp}}) = \left\{ x \in \mathbb{R} \,\middle|\, \lim_{\varepsilon \to 0^+} \varepsilon\, \mathbb{Im}\{H_\mu(x + \mathrm{i}\varepsilon)\} \in (0,\infty) \right\}.$$