Princeton University

Spring 2025 MAT425: Measure Theory

HW6 Sample Solutions

Mar 27 2025

Serban Cicortas

March 27, 2025

Problem 1.

Solution. Let $\mu, \nu : \mathfrak{M} \to \mathbb{C}$ be two measures, and let $\alpha \in \mathbb{C}$. We first have that $\alpha\mu(\emptyset) = 0$ and $(\mu+\nu)(\emptyset) = 0$. Also for any sequence $\{E_j\}$ of disjoint sets in $\mathfrak{M}$ we have:

$$\alpha\mu\Big(\bigcup_j E_j\Big) = \alpha \sum_j \mu(E_j) = \sum_j \alpha\mu(E_j)$$

$$(\mu+\nu)\Big(\bigcup_j E_j\Big) = \mu\Big(\bigcup_j E_j\Big) + \nu\Big(\bigcup_j E_j\Big) = \sum_j \mu(E_j) + \sum_j \nu(E_j) = \sum_j (\mu+\nu)(E_j)$$

We conclude that $\alpha\mu$ and $\mu+\nu$ are also measures. □

Problem 2.

Solution. (See Proposition 3.9 in Folland) Let $\eta \ll \nu \ll \mu$. Let any $E \in \mathfrak{M}$ such that $\mu(E) = 0$. Since $\nu \ll \mu$ we get that $\nu(E) = 0$. Then using $\eta \ll \nu$, we also get that $\eta(E) = 0$. Thus, $\eta \ll \mu$.

We claim that for any $g \in L^1(\nu)$ we have $g\frac{d\nu}{d\mu} \in L^1(\mu)$ and:

$$\int g d\nu = \int g \frac{d\nu}{d\mu} d\mu$$

To prove this we first see that for any $E \in \mathfrak{M}$ and $g = \mathbf{1}_E$, by the definition of the Radon-Nikodym derivative we have:

$$\int \mathbf{1}_E d\nu = \nu(E) = \int_E \frac{d\nu}{d\mu} d\mu = \int \mathbf{1}_E \frac{d\nu}{d\mu} d\mu$$

By linearity, the above is true for simple functions and we can then apply the monotone convergence theorem to prove it for nonnegative integrable functions. Finally, we then get that the claim holds for all $g \in L^1(\nu)$.

We apply our claim for $g = \mathbf{1}_E \frac{d\nu}{d\mu}$ to get:

$$\int_E \frac{d\nu}{d\mu} d\mu = \int_E \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda} d\lambda$$

We then get that:

$$\int_E \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda} d\lambda = \int_E \frac{d\nu}{d\mu} d\mu = \nu(E) = \int_E \frac{d\nu}{d\lambda} d\lambda$$

Since this holds for any $E \in \mathfrak{M}$, we conclude that $\frac{d\nu}{d\mu}\frac{d\mu}{d\lambda} = \frac{d\nu}{d\lambda}$ a.e. w.r.t. λ. □

Problem 3.

Solution. We assume that $\mu \ll \nu \ll \mu$. We first notice that $\frac{d\mu}{d\mu} = 1$ by the definition of the Radon-Nikodym derivative. Applying the chain rule proved in the previous exercise we conclude that:

$$\frac{d\mu}{d\nu}\frac{d\nu}{d\mu} = \frac{d\mu}{d\mu} = 1$$

□

Problem 4.

Solution. Let $\nu_i \ll \mu$ for all $i \in \{1, \ldots, n\}$. Let any $E \in \mathfrak{M}$ such that $\mu(E) = 0$. We have that $\nu_i(E) = 0$ for all $i \in \{1, \ldots, n\}$. Thus, $\sum_{i=1}^n \nu_i(E) = 0$, so $\sum_{i=1}^n \nu_i \ll \mu$.

For any $E \in \mathfrak{M}$ we have that:

$$\sum_{i=1}^n \nu_i(E) = \sum_{i=1}^n \int_E \frac{d\nu_i}{d\mu} d\mu = \int_E \sum_{i=1}^n \frac{d\nu_i}{d\mu} d\mu$$

By uniqueness of the Radon-Nikodym derivative we obtain that:

$$\frac{d\sum_{i=1}^n \nu_i}{d\mu} = \sum_{i=1}^n \frac{d\nu_i}{d\mu}$$

□

Problem 5.

Solution. We note that it suffices to prove the problem for $n = 2$, as in general one can iterate this result for finite n. We assume that $\nu_1 \ll \mu_1$ and $\nu_2 \ll \mu_2$. Consider any E which has measure zero with respect to $\mu_1 \times \mu_2$. For any $x_2 \in X_2$ we denote $E_{x_2} = \{x_1 \in X_1 : (x_1, x_2) \in E\}$. By Fubini's theorem we have that:

$$\mu_1 \times \mu_2(E) = \int_{X_2}\int_{X_1} \mathbf{1}_E(x_1, x_2) d\mu_1 d\mu_2 = \int_{X_2} \mu_1(E_{x_2}) d\mu_2.$$

Since $\mu_1 \times \mu_2(E)$ we get that $\mu_1(E_{x_2}) = 0$ for a.e. x_2 w.r.t. μ_2. Using $\nu_2 \ll \mu_2$, this implies that $\mu_1(E_{x_2}) = 0$ for a.e. x_2 w.r.t. ν_2. By absolute continuity we also get that $\nu_1(E_{x_2}) = 0$ for a.e. x_2 w.r.t. ν_2. Applying Fubini as above for $\nu_1 \times \nu_2$, we then obtain that $\nu_1 \times \nu_2(E) = 0$, so $\nu_1 \times \nu_2 \ll \mu_1 \times \mu_2$.

For any E measurable we have by Fubini and our claim in Problem 2:

$$\nu_1 \times \nu_2(E) = \int_{X_2} \nu_1(E_{x_2}) d\nu_2(x_2) = \int_{X_2} \nu_1(E_{x_2}) \frac{d\nu_2}{d\mu_2}(x_2) d\mu_2(x_2)$$

We also have by the definition of the Radon-Nikodym derivative for any $x_2 \in X_2$:

$$\nu_1(E_{x_2}) = \int_{E_{x_2}} \frac{d\nu_1}{d\mu_1}(x_1) d\mu_1(x_1)$$

As a result we get:

$$\nu_1 \times \nu_2(E) = \int_{X_2} \int_{E_{x_2}} \frac{d\nu_1}{d\mu_1}(x_1) \frac{d\nu_2}{d\mu_2}(x_2) d\mu_1(x_1) d\mu_2(x_2) = \int_E \frac{d\nu_1}{d\mu_1}(x_1) \frac{d\nu_2}{d\mu_2}(x_2) d\mu_1(x_1) d\mu_2(x_2)$$

Thus, we conclude that:

$$\frac{d(\nu_1 \times \nu_2)}{d(\mu_1 \times \mu_2)}(x_1, x_2) = \frac{d\nu_1}{d\mu_1}(x_1) \frac{d\nu_2}{d\mu_2}(x_2)$$

□

Problem 6.

Solution. Since δ_0 is concentrated at $\{0\}$ and $\lambda(\{0\}) = 0$ (so λ is concentrated on $\{0\}^c$), we get $\delta_0 \perp \lambda$. □

Problem 7.

Solution. The counting measure of a set is given by the number of elements in the set. Thus, if $c(E) = 0$ then $E = \emptyset$ so $\lambda(E) = 0$. This implies $\lambda \ll c$.

We notice that c is not σ-finite, since $[0, 1]$ is uncountable and counting measure of a set is finite iff the set has finitely many elements. Thus, we cannot apply the Radon-Nikodym derivative theorem.

Finally, we show that $\frac{d\lambda}{dc}$ does not exist. We assume the contrary, so let $f \in L^1(c)$ such that for any E measurable:

$$\lambda(E) = \int_E f(x) dc(x) = \sum_{x \in E} f(x),$$

where we also used the definition of the counting measure. If $E = \{x_0\}$ for any $x_0 \in [0, 1]$ we have that $\lambda(\{x_0\}) = 0$, so the above implies that $f(x_0) = 0$. We get that $f \equiv 0$ on $[0, 1]$, which of course contradicts the fact that $\lambda([0, 1]) = 1$. □

Problem 8.

Solution. (See *Teschl. Mathematical methods in quantum mechanics.*) We have that μ is a finite Borel measure, so we set $M = \mu(\mathbb{R})$. We first prove the growth estimate for the Stieltjes transform:

$$|H_\mu(x + i\epsilon)| \leq \int_{\mathbb{R}} \frac{1}{|x - y + i\epsilon|} d\mu(y) \leq \frac{1}{Im(x - y + i\epsilon)} \int_{\mathbb{R}} d\mu(y) \leq \frac{\mu(\mathbb{R})}{\epsilon} = \frac{M}{\epsilon}$$

We recall the lecture notes definition of the symmetric derivative:

$$\mathcal{D}\mu(x) = \lim_{\epsilon\to 0^+} \frac{\mu(B_\epsilon(x))}{\lambda(B_\epsilon(x))}$$

We similarly define $\underline{\mathcal{D}}\mu(x)$ and $\overline{\mathcal{D}}\mu(x)$ by considering lim inf respectively lim sup in the above definition. Our first claim is (See Theorem A.37 of Teschl) that $\mathcal{D}\mu$ exists a.e. w.r.t. Lebesgue measure and:

$$\mu_{ac}(E) = \int_E \mathcal{D}\mu(x)dx$$

To prove this we use the Lebesgue decomposition $\mu = \mu_{ac} + \mu_s$, and note that $N = supp(\mu_s)$ has $\lambda(N) = 0$. In the lecture notes we prove that $\mathcal{D}\mu_{ac} = \frac{d\mu_{ac}}{d\lambda}$. To prove the claim it suffices to show that $\overline{\mathcal{D}}\mu_s(x) = 0$ a.e. on N^c. We note this is the content of Lemma A.33 in Teschl and it's an application of Theorem 6.4 in the lecture notes.

Next, we claim that (See Theorem A.38 of Teschl):

$$supp(\mu_s) = \big\{x:\ (\mathcal{D}\mu)(x) = \infty\big\},\ supp(\mu_{ac}) = \big\{x:\ (\mathcal{D}\mu)(x) \in (0,\infty)\big\}$$

It suffices to show that for every $k \in \mathbb{N}$ we have that the set $O_k = \{x \in supp(\mu_s):\ \underline{\mathcal{D}}\mu(x) < k\}$ satisfies $\mu(O_k) = 0$. Let $K \subset O_k$ be compact and V_j an open set s.t. $\lambda(K\backslash V_j) \le 1/j$. For every $x \in K$ there exists $\epsilon > 0$ such that $B_\epsilon(x) \subset V_j$ and $\mu(B_\epsilon(x)) \le k\lambda(B_\epsilon(x))$. Since K is compact we can cover it with a finitely many $B_{\epsilon_i}(x_i)$ and we also have $\mu(K) < k\sum_i \lambda(B_{\epsilon_i}(x_i))$. Using Vitali's covering, we can select disjoint balls $B_{\epsilon_l}(x_l)$ such that $\mu(K) < k3^n \sum_l \lambda(B_{\epsilon_l}(x_l)) \le k3^n\lambda(V_j)$. Letting $j \to \infty$ we proved that $\mu(K) < k3^n\lambda(K)$. This implies that μ is absolutely continuous on $O_k \subset supp(\mu_s)$, so $\mu(O_k) = 0$ as desired. We conclude that $supp(\mu_s) = \big\{x:\ (\mathcal{D}\mu)(x) = \infty\big\}$. Additionally, we also get that $supp(\mu_{ac}) = \big\{x:\ (\mathcal{D}\mu)(x) \in [0,\infty)\big\}$. Using the above formula we also have $\mu_{ac}\big(\{(\mathcal{D}\mu)(x) = 0\}\big) = 0$, so finally $supp(\mu_{ac}) = \big\{x:\ (\mathcal{D}\mu)(x) \in (0,\infty)\big\}$.

Our next claim is that (See Theorem 3.22 of Teschl):

$$\underline{\mathcal{D}}\mu(x) \le \liminf_{\epsilon\to 0^+} \frac{1}{\pi} Im(H_\mu(x+i\epsilon)) \le \limsup_{\epsilon\to 0^+} \frac{1}{\pi} Im(H_\mu(x+i\epsilon)) \le \overline{\mathcal{D}}\mu(x)$$

To prove this we first note that:

$$Im(H_\mu(x+i\epsilon)) = \int_{\mathbb{R}} \frac{\epsilon}{|y-x-i\epsilon|^2} d\lambda(y) = \int_{\mathbb{R}} K_\epsilon(y)d\lambda(y),\ K_\epsilon(t) := \frac{\epsilon}{t^2+\epsilon^2}$$

We denote $I_\delta = (x-\delta, x+\delta)$ and split the integral as:

$$Im(H_\mu(x+i\epsilon)) = \int_{I_\delta} K_\epsilon(y-x)d\lambda(y) + \int_{\mathbb{R}\backslash I_\delta} K_\epsilon(y-x)d\lambda(y)$$

The second integral is bounded by $K_\epsilon(\delta)\mu(\mathbb{R}) = M \cdot K_\epsilon(\delta) \to 0$ as $\epsilon \to 0^+$. For the first integral we notice that:

$$\int_0^\delta \mu(I_s)K_\epsilon'(s)ds = \int_{I_\delta} K_\epsilon(\delta) - K_\epsilon(y-x)d\mu(y) = \mu(I_\delta)K_\epsilon(\delta) - \int_{I_\delta} K_\epsilon(y-x)d\lambda(y)$$

Suppose there are constants $c_\delta, C_\delta > 0$ such that $c_\delta \leq \mu(I_s)/2s \leq C_\delta$ for all $s \in [0, \delta]$. We note that:

$$2\delta K_\epsilon(\delta) - \int_0^\delta 2sK'_\epsilon(s)ds = 2\arctan(\delta/\epsilon) \to \pi \text{ as } \epsilon \to 0^+$$

As a result, we conclude the claim since we get that:

$$\liminf_{\epsilon \to 0^+} \frac{1}{\pi} Im(H_\mu(x+i\epsilon)) \geq \liminf_{\delta \to 0^+} \inf_{0<s<\delta} \frac{\mu(I_s)}{2s} \geq \underline{\mathcal{D}}\mu(x)$$

$$\limsup_{\epsilon \to 0^+} \frac{1}{\pi} Im(H_\mu(x+i\epsilon)) \leq \limsup_{\delta \to 0^+} \sup_{0<s<\delta} \frac{\mu(I_s)}{2s} \leq \overline{\mathcal{D}}\mu(x)$$

Combining our previous claims we get that:

$$supp(\mu_{ac}) = \left\{x : \lim_{\epsilon \to 0^+} \frac{1}{\pi} Im(H_\mu(x+i\epsilon)) \in (0,\infty)\right\},\ supp(\mu_s) = \left\{x : \lim_{\epsilon \to 0^+} \frac{1}{\pi} Im(H_\mu(x+i\epsilon)) = \infty\right\}$$

Finally, we also use the above claim that $\mathcal{D}\mu = \frac{d\mu_{ac}}{d\lambda}$ to conclude that:

$$\frac{d\mu_{ac}}{d\lambda}(x) = \lim_{\epsilon \to 0^+} \frac{1}{\pi} Im(H_\mu(x+i\epsilon))$$

□