Princeton University
Spring 2025 MAT425: Measure Theory
HW5
Mar 9th 2025

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1. Let $a,b\in\mathbb{R}$ with $a<b$. Let $f:[a,b]\to\mathbb{C}$ be bounded.
   (a) Prove that if f is Riemann integrable then it is measurable w.r.t. the σ-algebras $\mathscr{L}([a,b])$ and $\mathscr{B}(\mathbb{C})$ on its domain and codomain respectively.
   (b) Prove that if f is Riemann integrable then its Riemann integral $\int_{x=a}^{b} f(x)\,\mathrm{d}x$ equals its Lebesgue integral $\int_{[a,b]} f\mathrm{d}\lambda$.
   (c) Find an example of such a bounded f which has finite Lebesgue measure yet is *not* Riemann integrable.
2. Let $f:I\to\mathbb{C}$ where $I\subseteq\mathbb{R}$, a (possibly unbounded, not necessarily proper) interval. We say that f is improperly Riemann integrable on I iff there exists a increasing sequence $\{\, I_n \,\}_{n\in\mathbb{N}}$ of bounded intervals such that:
   - $I=\bigcup_{n\in\mathbb{N}} I_n$.
   - $f\chi_{I_n}:I_n\to\mathbb{C}$ is bounded and Riemann integrable.
   - $\lim_n \int_{I_n} f\chi_{I_n}$ exists and is finite.

   (a) Show that if $\operatorname{im}(f)\subseteq(0,\infty)$ and f is improperly Riemann integrable then it is Lebesgue measurable and has a finite Lebesgue integral.
   (b) Find an example of a function $f:I\to\mathbb{C}$ that is improperly Riemann integrable *yet* $f\notin L^1(I\to\mathbb{C},\lambda)$.
3. Let $x_0\in\mathbb{R}$ and δ_{x_0} be the Dirac measure on $\mathscr{B}(\mathbb{R})$. Let $\varphi:\mathbb{R}\to\mathbb{R}$ be measurable. Calculate the push-forward measure
$$(\delta_{x_0})_\varphi$$
   on $\mathscr{B}(\mathbb{R})$.
4. Let c be the counting measure on $\mathbb{N}$ w.r.t. the σ-algebra $\mathscr{P}(\mathbb{N})$. Let $\varphi:\mathbb{N}\to\mathbb{N}$ be a measurable bijection. Calculate the push-forward measure
$$c_\varphi\,.$$
5. Let $\varphi:(a,b)\to\mathbb{R}$ be convex. Prove that it is Lebesgue measurable.
6. (*Jensen's inequality*) Let $\varphi:(a,b)\to\mathbb{R}$ be convex and $\mu:\operatorname{Msrbl}(X)\to[0,\infty]$ a measure on a measure space X. Assume that μ is a finite measure.
   (a) Prove
$$\varphi\left(\frac{1}{\mu(X)}\int_X f\mathrm{d}\mu\right)\leq\frac{1}{\mu(X)}\int\varphi\circ f\mathrm{d}\mu\qquad\left(f\in L^1(X\to(a,b),\mu)\right)\,. \tag{0.1}$$
   (b) Assume $\mu(X)\neq 1$. Find an example for $f\in L^1(X\to(a,b),\mu)$ such that
$$\varphi\left(\int_X f\mathrm{d}\mu\right)>\int\varphi\circ f\mathrm{d}\mu\,.$$
   (c) Find an example of (0.1) where the RHS equals $+\infty$.

7. (*Minkowski*) Let $p \in (1,\infty)$ and $(X, \mathrm{Msrbl}(X), \mu)$ a measure space. Let $f, g : X \to \mathbb{C}$ be measurable. Recall

$$\|h\|_{L^p} \equiv \left(\int_X |h|^p \,\mathrm{d}\mu\right)^{\frac{1}{p}} \qquad (h : X \to \mathbb{C} \text{ msrbl.})\ .$$

Prove

$$\|f+g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}\ .$$

8. (Hölder) Let $(X, \mathrm{Msrbl}(X), \mu)$ be a measure space with $\mu : \mathrm{Msrbl}(X) \to [0,\infty]$.

   (a) Let $p \in (1,\infty)$ and $q := \frac{p}{p-1}$ its conjugate. Prove

   $$\|fg\|_{L^1} \leq \|f\|_{L^p} \|g\|_{L^q} \qquad (f, g : X \to \mathbb{C} \text{ msrbl.})\ .$$

   (b) Let $r \in (0,\infty]$ and $p_1, \cdots, p_n \in (0,\infty]$ such that $\sum_{j=1}^n \frac{1}{p_j} = \frac{1}{r}$. Prove

   $$\left\|\prod_{j=1}^n f_j\right\|_{L^r} \leq \prod_{j=1}^n \|f_j\|_{L^{p_j}} \qquad (f_j : X \to \mathbb{C} \text{ msrbl. for all } j = 1, \cdots, n)\ .$$

   (c) Let $p_1, \cdots, p_n \in (0,\infty]$ and $\theta_1, \cdots, \theta_n \in (0,1)$ such that $\sum_{j=1}^n \theta_j = 1$. Let $p := \left(\sum_{j=1}^n \frac{\theta_j}{p_j}\right)^{-1}$. Prove

   $$\left\|\prod_{j=1}^n |f_j|^{\theta_j}\right\|_{L^p} \leq \prod_{j=1}^n \left(\|f_j\|_{L^{p_j}}\right)^{\theta_j} \qquad (f : X \to \mathbb{C} \text{ msrbl.})\ .$$

   (d) Let $p \in (1,\infty)$ and assume $\mu(X) \neq 0$. Then

   $$\|fg\|_{L^1} \geq \|f\|_{L^{\frac{1}{p}}} \|g\|_{L^{\frac{-1}{p-1}}} \qquad (f, g : X \to \mathbb{C} \text{ msrbl. and } g \neq 0\ \mu-\text{a.e.})\ .$$

9. (*Young*) Let $p, q, r \in [1,\infty]$ are such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$. Let $X := \mathbb{R}^d$ with the σ-algebra $\mathscr{L}(\mathbb{R}^d)$ and $\mu := \lambda$. Then

$$\|f * g\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q} \qquad (f, g : X \to \mathbb{C} \text{ msrbl.})$$

where $f * g$ is the convolution,

$$(f * g)(x) \equiv \int_{y \in \mathbb{R}^d} f(y)\, g(x-y)\, \mathrm{d}\lambda(y) \qquad \left(x \in \mathbb{R}^d\right)\ .$$

10. Let $(X, \mathrm{Msrbl}(X), \mu)$ be a measure space. Prove that if $p \in (1,\infty)$ and L^p is understood as equivalence classes of functions that are equal μ-a.e., then $\|\cdot\|_{L^p}$ is a *complete* norm which makes L^p into a Banach space.

11. Show that iff $p = 2$ then the above norm satisfies the parallelogram identity (i.e. for $p \neq 2$ you need to find a counter-example).

12. Show that iff a norm satisfies the parallelogram identity then there exists a *unique* inner product $\langle \cdot, \cdot \rangle$ such that

$$\|v\| = \sqrt{\langle v, v \rangle}\ .$$

13. (*Riesz*) A bounded linear functional on a Banach space V is a $\mathbb{C}$-linear map $A : V \to \mathbb{C}$ such that

$$\left(\sup_{v \in V : \|v\|_V \leq 1} |Av|\right) < \infty\ .$$

Prove that if V is a Hilbert space (A Banach space whose norm obeys the parallelogram identity) then for any bounded linear functional $A : V \to \mathbb{C}$ there exists a unique vector $u_A \in V$ such that

$$A = \langle u, \cdot \rangle_V\ .$$

14. Let $(X, \mathrm{Msrbl}(X), \mu)$ be a measure space. Let $f : X \to \mathbb{C}$ be measurable. A number $M \geq 0$ is an *essential* upper bound on f iff

$$\mu\left(|f|^{-1}((M,\infty))\right) = 0\,.$$

The *essential supremum* of a measurable function is its *least* essential upper bound. We define $L^\infty(X \to \mathbb{C}, \mu)$ as (equivalence classes of) functions which are measurable and which are essentially bounded. The L^∞ norm is the essential supremum.

(a) Show that it is indeed a complete norm making L^∞ also into a Banach space.

(b) Extend Hölder's and Minkowski's inequalities to the case $p = \infty$.

15. Let X be a locally compact Hausdorff space and $\mu : \mathcal{B}(X) \to [0,\infty]$ be σ-finite. Let $p \in [1,\infty)$ and $C_c(X \to \mathbb{C})$ be the set of continuous functions whose (closure of their) support

$$\overline{f^{-1}(\{0\}^c)}$$

is compact. Show that

$$\overline{C_c(X \to \mathbb{C})}^{L^p(X\to\mathbb{C},\mu)} = L^p(X \to \mathbb{C}, \mu)\,.$$