Princeton University
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HW5 Sample Solutions
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Question 1

Let $a, b \in \mathbb{R}$ with $a < b$. Let $f : [a, b] \to \mathbb{C}$ be bounded.

(a) Prove that if f is Riemann integrable then it is measurable with respect to the σ-algebras $\mathcal{L}([a, b])$ and $\mathcal{B}(\mathbb{C})$ on its domain and codomain respectively.

Solution. See (the proof of) Claim 4.12 in the lecture notes. □

(b) Prove that if f is Riemann integrable then its Riemann integral $\int_a^b f(x)\,dx$ and its Lebesgue integral $\int_{[a,b]} f\,d\lambda$ are equal.

Solution. See (the proof of) Theorem 4.13 in the lecture notes. □

(c) Find a Lebesgue integrable function which is bounded but not Riemann integrable.

Solution. See Theorem 1.3 in the lecture notes and examples (3) and (4) that immediately follow. □

Question 2

Let $f : I \to \mathbb{C}$ where $I \subseteq \mathbb{R}$ is a (possibly unbounded, not necessarily proper) interval. We say that f is *improperly* Riemann integrable on I iff there exists an *increasing* sequence $\{I_n\}_{n\in\mathbb{N}}$ of bounded intervals such that

- $I = \bigcup_{n\in\mathbb{N}} I_n$
- The restriction of f to I_n is bounded and Riemann integrable for each $n \in \mathbb{N}$

- $\lim_{n\to\infty} \int_{I_n} f$ exists and is finite.

(a) Show that if the image of f lies in $[0,\infty)$ and f is improperly Riemann integrable then it is Lebesgue measurable with finite Lebesgue integral.[1]

Solution. We apply to a general setting the ideas of Example 4.14 in the lecture notes.

Define $f_n := f \cdot \chi_{I_n}$. By part (a) of the previous problem, the second assumption implies that each f_n is Lebesgue measurable. These functions converge to f pointwise by the first assumption. Hence f is measurable by Corollary 2.23(1).

By part (b) of the previous problem, the Lebesgue integrals $\int f_n$ coincide with the corresponding Riemann integrals. The non-negativity of f and the fact that the intervals are increasing imply that $f_1, f_2, \ldots$ is an increasing sequence of non-negative functions. Hence, by the monotone convergence theorem, the *Lebesgue* integrals $\int_{I_n} f = \int f_n$ converge to the Lebesgue integral $\int_I f$. By the third assumption, this limit is finite. □

(b) Construct a function $f : I \to \mathbb{C}$ (for *some* I) such that $f : I \to \mathbb{C}$ is improperly Riemann integrable but $f \notin L^1(I \to \mathbb{C}, \lambda)$.

Solution. See Example 4.15 in the lecture notes. The second example, there defined on $\mathbb{N}$, can be "continuized" by replacing it with the function $x \mapsto \frac{(-1)^{\lfloor x \rfloor}}{\lfloor x \rfloor}$ with domain $[1,\infty)$. □

Question 3

Let $x_0 \in \mathbb{R}$ and let δ_{x_0} be the Dirac measure on $\mathscr{B}(\mathbb{R})$. Let $\phi : \mathbb{R} \to \mathbb{R}$ be measurable. Then the push-forward measure $(\delta_{x_0})_\phi$ is equal to $\delta_{\phi(x_0)}$.

Solution. By definition, we have, for every $S \in (\mathbb{R})$

$$\begin{aligned}
(\delta_{x_0})_\phi(S) &= \delta_{x_0}(\phi^{-1}(S)) \\
&= \begin{cases} 1 & \text{if } x_0 \in \phi^{-1}(S) \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} 1 & \text{if } \phi(x_0) \in S \\ 0 & \text{otherwise} \end{cases} \\
&= \delta_{\phi(x_0)}(S)
\end{aligned}$$

□

[1] It is then also true that its Lebesgue and Riemann integrals are equal.

Question 4

Let c be the counting measure on $\mathbb{N}$ with respect to the σ-algebra $\mathscr{P}(\mathbb{N})$. Let $\phi : \mathbb{N} \to \mathbb{N}$ be a bijection.[2] Then the push-forward measure c_ϕ is just c again.

Solution. Since ϕ^{-1} is an injective function, it satisfies $|\phi^{-1}(S)| = |S|$ for all $S \subseteq \mathbb{N}$. Hence for any $S \in \mathscr{P}(\mathbb{N})$,

$$c_\phi(S) = c(\phi^{-1}(S)) = |\phi^{-1}(S)| = |S| = c(S)$$

□

[2] Note that every function with domain $\mathbb{N}$ is measurable if we allow all subsets of $\mathbb{N}$ to be measurable.

Question 5

Let $\phi : (a, b) \to \mathbb{R}$ be convex. Then ϕ is Lebesgue measurable.

Solution. In fact, any convex function on an open interval of $\mathbb{R}$ is continuous, as we show below. Any continuous function is measurable by HW1Q9. C.f. Theorem 3.2 in Rudin's *Real and Complex Analysis*. □

Claim. *Let $f : (a, b) \to \mathbb{R}$ be convex. Then f is left- and right-differentiable at each point $x \in (a, b)$. It follows that f is continuous on (a, b).*

Proof. By symmetry, it suffices to prove right-differentiability.[3] Fix $x \in (a, b)$ and consider the difference quotient

$$\Delta(y) := \frac{f(y) - f(x)}{y - x} \qquad (y \in (x, b))$$

Convexity implies that $\Delta(y) < \Delta(y')$ for $x < y < y' < b$. This means that Δ is an increasing function on (x, b). The limit $f'_+(x) := \lim_{y \to x^+} \Delta(y)$ therefore exists in $\mathbb{R} \cup \{-\infty\}$.[4] To see that it is finite, choose some (arbitrary) $z \in (a, x)$. Then convexity gives $\Delta(y) \geq \frac{f(x)-f(z)}{x-z}$ for all $y \in (x, b)$. □

Question 6: Jensen's Inequality

Let $(X, \mathfrak{M}, \mu)$ be a *finite* measure space.[5] Let $\phi : (a, b) \to \mathbb{R}$ be convex ($a, b \in \overline{\mathbb{R}}$). Suppose $f \in L^1(X \to (a, b), \mu)$.

(a) Show that[6]

$$\phi\left(\frac{1}{\mu(X)} \int_X f \, d\mu\right) \leq \frac{1}{\mu(X)} \int_X \phi \circ f \, d\mu$$

Solution. The inequality is invariant under multiplying μ by a constant. Since μ is finite, we may therefore suppose $\mu(X) = 1$.[7] With this assumption we are in the situation of Theorem 3.3 in Rudin's *Real and Complex Analysis*. See the proof there. □

(b) Show that if $\mu(X) \neq 1$ then it is possible to get

$$\phi\left(\int_X f \, d\mu\right) > \int_X \phi \circ f \, d\mu$$

[3]For left-differentiability, one can consider $t \mapsto f(-t)$ to reduce to this case.
[4]It is equal to $\inf_{y \in (x,b)} \Delta(y)$.
[5]That is, $\mu(X) < \infty$.
[6]In particular, that the integral $\int_X \phi \circ f \, d\mu$ has a well-defined value in $\overline{\mathbb{R}}$.
[7]Simply replace μ with $\frac{1}{\mu(X)}\mu$.

Solution. Let $X := [-1,1]$ with $\mu = \lambda$ the Lebesgue measure, $(a,b) := \mathbb{R}$. Let $f(x) := 1$ and let $\phi(x) := x^2$. Since ϕ is smooth and has positive second derivative, it is convex. However,

$$\phi\left(\int_X f\,d\mu\right) = \left(\int_{-1}^{1} 1\,d\mu\right)^2 = 4 > 2 = \int_{-1}^{1} 1^2\,d\mu = \int_X \phi \circ f\,d\mu$$

□

(c) Show that it is possible to have $\frac{1}{\mu(X)} \int_X \phi \circ f\,d\mu = +\infty$.

Solution. Let $X := [0,1]$ with $\mu = \lambda$ the Lebesgue measure, $(a,b) := \mathbb{R}$. Let $f(x) = x^{-1/2}$ and let $\phi(x) = x^2$ as above.[8] We are in the situation of Q2(a), so we can assume all integrals are Riemann integrals. As is well-known, $\int_0^1 x^{-1/2}\,dx$ is finite[9] so indeed $f \in L^1([0,1] \to \mathbb{R}, \lambda)$. However, $\int_0^1 \phi(f(x))\,dx = \int_0^1 x^{-1}\,dx = +\infty$. □

[8] The reader who insists that functions ought to be defined everywhere can set $f(0)$ to be their favourite real number.

[9] More precisely, it is equal to 2.

Question 7 : Minkowski

Let $p \in (1, \infty)$ and $(X, \mathfrak{M}, \mu)$ be a measure space. Let $f, g : X \to \mathbb{C}$ be measurable. Prove

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p$$

where

$$\|h\|_p := \left(\int_X |h|^p \, d\mu \right)^{1/p} \qquad (h : X \to \mathbb{C} \text{ measurable})$$

Solution. The case where f, g are non-negative is handled in Theorem 3.5 of Rudin's *Real and Complex analysis.*[10] See the proof there. In general, we have $|f+g| \leq |f| + |g|$ pointwise and thus

$$\|f+g\|_p = \left(\int_X |f+g|^p \, d\mu \right)^{1/p} \leq \left(\int_X (|f|+|g|)^p \, d\mu \right)^{1/p} = \||f|+|g|\|_p \leq \||f|\|_p + \||g|\|_p = \|f\|_p + \|g\|_p$$

using that we know the inequality in the non-negative case. □

Question 8 : Hölder

Let $(X, \mathfrak{M}, \mu)$ be a measure space.

(a) Let $p \in (1, \infty)$ and let $q := \frac{p}{p-1}$ be its conjugate. Prove that

$$\|fg\|_1 \leq \|f\|_p \|g\|_q \qquad (f, g : X \to \mathbb{C} \text{ measurable})$$

Proof. We may assume without loss of generality that f and g are non-negative functions. This case is handled in Theorem 3.5 of Rudin's *Real and Complex analysis.* See the proof there.[11] □

(b) Let $n \in \mathbb{N}$. Let $r \in (0, \infty]$ and $p_1, \ldots, p_n \in (0, \infty]$ such that $\sum_{j=1}^n p_j^{-1} = r^{-1}$

$$\left\| \prod_{j=1}^n f_j \right\|_r \leq \prod_{j=1}^n \|f_j\|_{p_j} \qquad (f_1, \ldots, f_n : X \to \mathbb{C} \text{ measurable})$$

Proof. If $r = \infty$ then we must have $p_j = \infty$ for all j. This special case follows from the statement: if for each $1 \leq j \leq n$ we have $|f_j| < M_j$ a.e. for some $M_j > 0$, then also $|f_1 f_2 \cdots f_n| \leq M_1 M_2 \cdots M_n$ a.e. We may therefore assume that r is finite. We also assume in the remainder that the Hölder result in 14(b) is known.

[10] Note that Rudin uses the result of Q8(a) to prove this.

[11] See also Proposition 6.2 in Folland's *Real Analysis.*

We proceed by induction on n. If $n < 2$, there is nothing to prove. For the case $n = 2$, we make use of the easy but useful observation that for $r \in (0, \infty)$, $p \in (0, \infty]$ and f a measurable function we have

$$\||f|^r\|_{p/r} = \|f\|_p^r$$

Returning to the proof, suppose $p_1^{-1} + p_2^{-1} = r^{-1}$ and $f_1 \in L^{p_1}$, $f_2 \in L^{p_2}$. Then $(p_1/r)^{-1} + (p_2/r)^{-1} = 1$. This in particular implies $p_1/r, p_2/r > 1$. Hence by part (a) and the above discussion

$$\|f_1 f_2\|_r^r = \||f_1 f_2|^r\|_1 = \||f_1|^r |f_2|^r\|_1 \leq \||f_1|^r\|_{p_1/r} \||f_2|^r\|_{p_2/r} = \|f_1\|_{p_1}^r \|f_2\|_{p_2}^r$$

Taking r-th roots gives the desired result.

Assume the validity of the statement for n, suppose we are given $p_1, \dots, p_{n+1} \in (0, \infty]$ and $r \in (0, \infty]$ such that $r^{-1} := \sum_{j=1}^{n+1} p_j^{-1}$. Define $\tilde{p}_j := p_j$ for $j < n$ and define $\tilde{p}_n$ by $\tilde{p}_n^{-1} = p_n^{-1} + p_{n+1}^{-1}$. Then by construction $\sum_{j=1}^{n} (\tilde{p}_j)^{-1} = r^{-1}$. Similarly define $\tilde{f}_j := f_j$ for $j < n$ and $\tilde{f}_n := f_n f_{n+1}$. Then the induction hypothesis gives

$$\left\| \prod_{j=1}^{n} \tilde{f}_j \right\|_r \leq \prod_{j=1}^{n} \|\tilde{f}_j\|_{\tilde{p}_j}$$

Translating back gives

$$\left\| \prod_{j=1}^{n+1} f_j \right\|_r \leq \left(\prod_{j=1}^{n-1} \|f_j\|_{p_j} \right) \cdot \|f_n f_{n+1}\|_{\tilde{p}_n}$$

But $\tilde{p}_n^{-1} = p_n^{-1} + p_{n+1}^{-1}$ implies by the $n = 2$ case that

$$\|f_n f_{n+1}\|_{\tilde{p}_n} \leq \|f_n\|_{p_n} \|f_{n+1}\|_{p_{n+1}}$$

Combining the two inequalities completes the induction step. □

(c) Let $p_1, \dots, p_n \in (0, \infty]$ and $\theta_1, \dots, \theta_n \in (0, 1)$ such that $\sum_{j=1}^{n} \theta_j = 1$. Define r by $r^{-1} := \left(\sum_{j=1}^{n} \theta_j p_j^{-1} \right)$. Prove

$$\left\| \prod_{j=1}^{n} |f_j|^{\theta_j} \right\|_r \leq \prod_{j=1}^{n} \|f_j\|_{p_j}^{\theta_j}$$

Proof. We set $\tilde{f}_j := |f_j|^{\theta_j}$, $\tilde{p}_j := p_j/\theta_j$. Then $\sum_{j=1}^{n} \tilde{p}_j^{-1} = r^{-1}$ so by part (b), we get

$$\left\| \prod_{j=1}^{n} \tilde{f}_j \right\|_r \leq \prod_{j=1}^{n} \|\tilde{f}_j\|_{\tilde{p}_j}$$

Translating back and using the observation made in part (b) we get

$$\left\| \prod_{j=1}^{n} |f_j|^{\theta_j} \right\|_r \le \prod_{j=1}^{n} \| |f_j|^{\theta_j} \|_{p_j/\theta_j} = \prod_{j=1}^{n} \|f_j\|_{p_j}^{\theta_j}$$

□

(d) Let $p \in (1, \infty)$ and assume $\mu(X) \neq 0$. Then

$$\|fg\|_1 \ge \|f\|_{1/p} \|g\|_{-\frac{1}{p-1}} \qquad (f, g : X \to \mathbb{C} \text{ measurable and } |g| > 0 \ \mu\text{-a.e.})$$

Proof. Set $\tilde{f} := |fg|$ and $\tilde{g} := |g|^{-1}$. (One might worry about the measurability of $\tilde{g}$. But this can be handled by writing it as the pointwise limit of the measurable functions $t \mapsto i_\epsilon(|g|(t) + \epsilon)$ where i_ϵ is a continuous function which agrees with $t \mapsto 1/t$ outside of $(-\epsilon, \epsilon)$.[12]) Then the inequality at hand can be written

$$\int_X \tilde{f} \, d\mu \ge \left(\int_X \left(\tilde{f}\tilde{g} \right)^{1/p} d\mu \right)^p \left(\int_X \tilde{g}^{1/(p-1)} \, d\mu \right)^{1-p}$$

If the right-most integral is infinite then the right side of the inequality is 0 (because $1 - p < 0$) and the whole thing trivializes. Otherwise, the right-most integral must be a finite positive number[13] Thus the above is *equivalent* to

$$\int_X \tilde{f} \, d\mu \left(\int_X \tilde{g}^{1/(p-1)} \, d\mu \right)^{p-1} \ge \left(\int_X \left(\tilde{f}\tilde{g} \right)^{1/p} d\mu \right)^p$$

But this is just the special case of (b) where $n = 2$, $r = 1/p$, $p_1 = 1$, $p_2 = 1/(p-1)$, $f_1 = \tilde{f}$, $f_2 = \tilde{g}$. □

[12] One should also change g on a measure 0 set to make it Borel-measurable.
[13] The assumption on g ensures that the integral is well-defined and non-zero.

Question 9 : Young

Suppose $p, q, r \in [1, \infty]$ are such that $p^{-1} + q^{-1} = r^{-1} + 1$. Let $X := \mathbb{R}^d$ with the σ-algebra $\mathscr{L}(\mathbb{R}^d)$ and $\mu := \lambda$. Then

$$\|f * g\|_r \leq \|f\|_p \|g\|_q \qquad (f, g : X \to \mathbb{C} \text{ measurable})$$

where

$$(f * g)(x) := \int_{y \in \mathbb{R}^d} f(y) g(x - y) \, d\lambda(y) \qquad (x \in \mathbb{R}^d)$$

Solution. See Proposition 8.9 in Folland's *Real Analysis* for a proof employing the very useful technique of "interpolation". Here we provide a proof that assumes no such background material.

It is not even immediately clear that $f * g$ is measurable! To see that it is, let $F_1, F_2 : X \times X \to \mathbb{C}$ be defined by

$$F_1(x, y) = f(x)g(y), \quad F_2(x, y) = f(y)g(x - y)$$

See the sample solution for Q20 on the midterm for a proof that $F_1 \in L^1(X \times X, \lambda \times \lambda)$. Now note that the function

$$\begin{aligned} \psi : \mathbb{R}^d \times \mathbb{R}^d &\to \mathbb{R}^d \times \mathbb{R}^d \\ (x, y) &\mapsto (y, x - y) \end{aligned}$$

is measure-preserving (it is a smooth function whose Jacobian determinant is constantly -1). So from $F_1 \in L^1$ it follows that $F_2 = f \circ \psi \in L^1$. The measurability of $f * g$ then follows from Fubini's theorem applied to F_2 (c.f. Theorem 2.37(b) in Folland).[14]

To get the norm bound, we start by using the version of Hölder's inequality in Q8(c) with $f_1(y) := |f(y)|$, $f_2(y) := |g(x - y)|$, $f_3(y) := |f(y)|^p |g(x - y)|^q$ (here x is fixed), $p_1 = p$, $p_2 = q$, $p_3 = 1$, $\theta_1 = 1 - p/r$, $\theta_2 = 1 - q/r$ and $\theta_3 = 1/r$.[15] We get (noting that $y \mapsto g(y)$ and $y \mapsto g(x - y)$ have the same norm)

$$\begin{aligned} \int_{y \in \mathbb{R}^d} |f(y)g(x - y)| \, d\lambda(y) &= \int_{y \in \mathbb{R}^d} \left(|f(y)|\right)^{1-p/r} \left(|g(x - y)|\right)^{1-q/r} \left(|f(y)|^p |g(x - y)|^q\right)^{1/r} \, d\lambda(y) \\ &\leq \|f\|_p^{1-p/r} \|g\|_q^{1-q/r} \left(\int_{y \in \mathbb{R}^d} |f(y)^p g(x - y)^q| \, d\lambda(y) \right)^{1/r} \end{aligned}$$

[14] We are only using the measurability portion of that theorem.

[15] Noting that $\frac{\theta_1}{p_1} + \frac{\theta_2}{p_2} + \frac{\theta_3}{p_3} = p^{-1} - r^{-1} + q^{-1} - r^{-1} + r^{-1} = 1$.

Applying this, and using Fubini's theorem, we get

$$\begin{aligned}
\|f * g\|_r &= \left(\int_{x\in\mathbb{R}^d} \left|\int_{y\in\mathbb{R}^d} f(y)^p g(x-y)^q \, d\lambda(y)\right|^r \, d\lambda(x)\right)^{1/r} \\
&\leq \left(\int_{x\in\mathbb{R}^d} \|f\|_p^{r-p} \|g\|_q^{r-q} \left(\int_{y\in\mathbb{R}^d} |f(y)^p g(x-y)^q| \, d\lambda(y)\right) \, d\lambda(x)\right)^{1/r} \\
&= \|f\|_p^{1-p/r} \|g\|_q^{1-q/r} \left(\int_{y\in\mathbb{R}^d} \left(\int_{x\in\mathbb{R}^d} |f(y)^p g(x-y)^q| \, d\lambda(x)\right) \, d\lambda(y)\right)^{1/r} \\
&= \|f\|_p^{1-p/r} \|g\|_q^{1-q/r} \left(\int_{y\in\mathbb{R}^d} |f(y)|^p \left(\int_{x\in\mathbb{R}^d} |g(x-y)|^q \, d\lambda(x)\right) \, d\lambda(y)\right)^{1/r} \\
&= \|f\|_p^{1-p/r} \|g\|_q^{1-q/r} \left(\int_{y\in\mathbb{R}^d} |f(y)|^p \cdot \|g\|_q^q \, d\lambda(y)\right)^{1/r} \\
&= \|f\|_p^{1-p/r} \|g\|_q^{1-q/r} \|g\|_q^{q/r} \|f\|_p^{p/r} = \|f\|_p \|g\|_q
\end{aligned}$$

□

Question 10

Let $(X, \mathfrak{M}, \mu)$ be a measure space. Prove that if $p \in (1, \infty)$ and L^p is understood as consisting of *equivalence classes* of functions with respect to the equivalence relation of being equal μ-a.e., then $\|\cdot\|_p$ is a complete norm, giving rise to an L^p Banach space.

Solution. We first need to check that $\|\cdot\|_p$ is a norm. C.f. Claim 5.29 in the lecture notes. We check each of the criteria in Definition C.1.

0. Changing a function on a measure 0 set does not change its integral. From this it follows that $\|\cdot\|_p$ is well-defined. It should be clear that $\|\cdot\|_p$ takes values in $[0, \infty)$.

1. For $f \in L^p$ and $\alpha \in \mathbb{C}$, we have

$$\|\alpha f\|_p = \left(\int_X |\alpha f|^p \, d\mu\right)^{1/p} = |\alpha| \left(\int_X |f|^p \, d\mu\right)^{1/p} = |\alpha| \cdot \|f\|_p$$

2. The triangle inequality for $\|\cdot\|_p$ is the content of Question 7.

3. Clearly $\|x \mapsto 0\|_p = 0$. On the other hand, if $f \in L^p(X, \mu)$ is *non-zero* then $V_\epsilon := \mu(\{x \in X \mid |f(x)| > \epsilon\}$ must be positive for some $\epsilon > 0$.[16] It follows that

$$\|f\|_p^p = \int_X |f|^p \, d\mu \geq \int_{V_\epsilon} \epsilon^p \, d\mu > 0$$

For completeness of the norm, the proof of Theorem 5.31 in the lecture notes goes through by simply changing suitable exponents and subscripts from 1 to p. See also Theorem 6.6 in Folland's *Real Analysis* and Theorem 3.11 in Rudin's *Real and Complex Analysis*. □

[16] Otherwise, $\{x \in X \mid f(x) \neq 0\} = \bigcup_{n \in \mathbb{N}} V_{1/n}$ would have measure 0.

Question 11

Show that the norm $\|\cdot\|_p$ satisfies the parallelogram identity (for general measure spaces X) if and only if $p = 2$.

Remark. *If a norm $\|\cdot\|$ on a vector space V is induced by an inner product $\langle\cdot,\cdot\rangle$, then for any $v, w \in V$ we have*

$$\begin{aligned}\|v+w\|^2 + \|v-w\|^2 &= \langle v+w, v+w\rangle + \langle v-w, v-w\rangle \\ &= \langle v, v+w\rangle + \langle w, v+w\rangle + \langle v, v-w\rangle - \langle w, v-w\rangle \\ &= \langle v, v+w+(v-w)\rangle + \langle w, v+w-(v-w)\rangle \\ &= 2\langle v,v\rangle + 2\langle w,w\rangle = 2\|v\|^2 + 2\|w\|^2\end{aligned}$$

I.e. $\|\cdot\|$ satisfies the parallelogram identity.

Remark. *If a norm $\|\cdot\|$ on a vector space V obeys the parallelogram* inequality

$$\|v+w\|^2 + \|v-w\|^2 \le 2\|v\|^2 + 2\|w\|^2 \qquad (v, w \in V)$$

Then by substituting $v' := v + w$, $w' := v - w$ into the inequality we get

$$\begin{aligned}\|v'+w'\|^2 + \|v'-w'\|^2 &\le \|v'\|^2 + \|w'\|^2 \\ 4\|v\|^2 + 4\|w\|^2 &\le 2\|v+w\|^2 + 2\|v-w\|^2 \\ 2\|v\|^2 + 2\|w\|^2 &\le \|v+w\|^2 + \|v-w\|^2\end{aligned}$$

So the norm satisfies the a priori stronger parallelogram identity

$$\|v+w\|^2 + \|v-w\|^2 = 2\|v\|^2 + 2\|w\|^2 \qquad (v, w \in V)$$

Solution. The fact that for $p = 2$ the parallelogram identity is satisfied follows from the first remark above applied to the inner product presented in Claim 5.34 in the lecture notes. We now focus on the case $p \neq 2$.

To show that the parallelogram law fails for *some* measure space, consider in $L^p(\mathbb{R})$ the example (with $p \neq 2$)

$$\|\chi_{(0,1)} + \chi_{(1,2)}\|_p^2 + \|\chi_{(0,1)} - \chi_{(1,2)}\|_p^2 = 2 \cdot 2^{2/p} \neq 4 = 2\|\chi_{(0,1)}\|_p^2 + 2\|\chi_{(1,2)}\|_p^2$$

In fact, the parallelogram law fails for *any* measure space $(X, \mathfrak{M}, \mu)$ with $\dim L^p(X) > 1$, as we now show. Given such a space, let $Y_1, Y_2 \subseteq X$ be *disjoint* subsets with $0 < \mu(Y_i) < \infty$ for $i = 1, 2$.[17] Then

$$\|\chi_{Y_1} + \chi_{Y_2}\|_p^2 = \|\chi_{Y_1} - \chi_{Y_2}\|_p^2 = (\mu(Y_1) + \mu(Y_2))^{2/p}$$

For $p > 2$, the function $t \mapsto t^{2/p}$ is strictly concave[18] and thus strictly subadditive on $(0, \infty)$ (see lemma below):

$$(\mu(Y_1) + \mu(Y_2))^{2/p} < \mu(Y_1)^{2/p} + \mu(Y_2)^{2/p}$$

[17] If such subsets cannot be found, then the space of integrable functions is it most 1-dimensional and all L^p spaces coincide.

[18] as is readily verified by checking that its second derivative is negative on $(0, \infty)$

This gives

$$\|\chi_{Y_1}+\chi_{Y_2}\|_p^2+\|\chi_{Y_1}-\chi_{Y_2}\|_p^2 < 2\|\chi_{Y_1}\|_p^{2/p}+2\|\chi_{Y_2}\|_p^{2/p}$$

Thus the parallelogram inequality fails. If instead $p < 2$ then $t^{2/p}$ is strictly convex and thus strictly superadditive on $(0,\infty)$. It follows that the opposite inequality holds strictly. This means that the parallelogram identity fails (which by the second remark above implies that the parallelogram inequality still fails). □

Lemma. *Suppose $f : [0,\infty) \to \mathbb{R}$ is a function with $f(0) = 0$. If f is concave (resp. strictly concave, resp. convex, resp. strictly convex) then it is also subadditive (resp. strictly subadditive, resp. superadditive, resp. strictly superadditive).*[19]

Proof. By changing f to $-f$ we exchange concavity with convexity. It therefore suffices to prove the concave half of the claim. Let $x, y \in [0,\infty)$. Concavity gives

$$f(x) = f\left(\frac{y}{x+y}\cdot 0+\frac{x}{x+y}\cdot(x+y)\right) \geq \frac{y}{x+y}f(0)+\frac{x}{x+y}f(x+y) = \frac{x}{x+y}f(x+y)$$

Similarly, $f(y) \geq \frac{y}{x+y}f(x+y)$. Adding the two, we get

$$f(x)+f(y) \geq \frac{x}{x+y}f(x+y)+\frac{y}{x+y}f(x+y) = f(x+y)$$

If f is *strictly* concave then the first inequality is strict unless either $0 = x+y$ or $y/(x+y) = 0$ or $x/(x+y) = 0$. Thus equality holds only if $x = 0$ or $y = 0$. □

Question 12

Suppose $\|\cdot\|$ is a norm on a complex vector space V. Show that $\|\cdot\|$ satisfies the parallelogram identity if and only if there exists an inner product $\langle\cdot,\cdot\rangle$ on V such that $\|v\|^2 = \langle v, v\rangle$ for all $v \in V$. In this case, the inner product in question is uniquely determined by $\|\cdot\|$.

Solution. C.f. Prop. 2.1.8 in Kadison and Ringrose's *Fundamentals of the Theory of Operator Algebras, Volume I.* See also Theorem 4.3.6 in Istrăţescu's *Inner Product Structures.*

Suppose that $\langle\cdot,\cdot\rangle$ were such an inner product, then for any $v, w \in V$ we would have

$$\|v+w\|^2-\|v-w\|^2 = \langle v+w, v+w\rangle-\langle v-w, v-w\rangle = 2\langle v, w\rangle+2\langle w, v\rangle = 2\langle v, w\rangle+2\overline{\langle v, w\rangle} = 4\Re(\langle v, w\rangle)$$

(where $\Re$ is the real-part function). Note that $(u, v) \mapsto \Re(\langle u, v\rangle)$ is a *real* inner-product on V.

[19]A function on $[0,\infty)$ or $\mathbb{R}$ is subadditive $f(x+y) \leq f(x)+f(y)$ whenever x, y lie in its domain. It is superadditive if the reverse inequality always holds. The adverb *strictly* here means that the inequality holds strictly whenever x, y are both non-zero.

Proposition. *Let V be a complex normed vector space whose norm satisfies the parallelogram law. Define*

$$[v, w] := \frac{1}{4}\left(\|v + w\|^2 - \|v - w\|^2\right) \quad (v, w \in V)$$

The bracket $[\cdot, \cdot]$ has the following properties:

(o) $[u, v] \in \mathbb{R}$ *for all* $u, v \in V$.

(i) $[v, v] = \|v\|^2$ *for all* $v \in V$.

(ii) $[v, w] = [w, v]$ *for all* $v, w \in V$.

(iii) $[iv, iw] = [v, w]$ *for all* $v, w \in V$.

(iv) $[\cdot, \cdot]$ *is continuous as a function* $V \times V \to \mathbb{R}$.

(v) $[\cdot, \cdot]$ *is additive in each argument, i.e.*

$$[u + v, w] = [u, w] + [v, w], \quad [u, v + w] = [u, v] + [u, w] \quad (u, v, w \in V)$$

(vi) $[\cdot, \cdot]$ *is* $\mathbb{R}$*-homogeneous in each argument, i.e.*

$$[\lambda u, v] = [u, \lambda v] = \lambda[u, v] \quad (u, v \in V,\ \lambda \in \mathbb{R})$$

Furthermore, $[\cdot, \cdot]$ *is the unique real inner-product on* V *satisfying (i).*

Proof. Claims (o) through (iii) are immediate from the definition. Claims (o) through (vi) imply that $[\cdot, \cdot]$ is an inner product.[20] The uniqueness condition follows by an argument similar to the preamble above.

(iv) The continuity of $[\cdot, \cdot]$ follows from the fact that it is a composition of continuous functions.

(v) By symmetry, it clearly suffices to prove additivity on the left.

The parallelogram law applied to the vectors u and $v + w$ gives

$$2\|u\|^2 + 2\|v + w\|^2 = \|u + v + w\|^2 + \|u - v - w\|^2$$

Applying it to the vectors $u - w$ and v gives

$$2\|u - w\|^2 + 2\|v\|^2 = \|u + v - w\|^2 + \|u - w - v\|^2$$

Subtracting the one from the other gives

$$2\|u\|^2 - 2\|v\|^2 + 2\|v + w\|^2 - 2\|u - w\|^2 = \|u + v + w\|^2 - \|u + v - w\|^2$$

Symmetrically (exchanging the roles of u and v), we get

$$2\|v\|^2 - 2\|u\|^2 + 2\|u + w\|^2 - 2\|v - w\|^2 = \|v + u + w\|^2 - \|v + u - w\|^2$$

[20] Note that positive-definiteness follows from (i).

Adding the two equations above gives

$$2\|u+w\|^2-2\|u-w\|^2+2\|v+w\|^2-2\|v-w\|^2 = 2\|u+v+w\|^2-2\|u+v-w\|^2$$

The left-hand-side here is $8[u,w]+8[v,w]$ and the right-hand-side is $8[u+v,w]$.

(vi) Again by symmetry, it suffices to show that $[\lambda u, v] = \lambda[u,v]$. Fix $u, v \in V$. We consider the function

$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R} \\ \lambda &\mapsto [\lambda u, v] \end{aligned}$$

Note that f is continuous as the composition of continuous functions. From part (v) it is clear that for $\kappa, \lambda \in \mathbb{R}$ we have $f(\kappa+\lambda) = f(\kappa)+f(\lambda)$. It then follows from the proposition below that $f(\lambda) = \lambda f(1) = \lambda[u,v]$. □

Proposition. *Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is a continuous function satisfying $f(x+y) = f(x)+f(y)$ for all $x, y \in \mathbb{R}^n$. Then f is $\mathbb{R}$-linear. That is, $f(\lambda x) = \lambda f(x)$ for all $x \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$.*

Proof. Consider the *good* set

$$\mathcal{G} := \{t \in \mathbb{R} \mid f(tx) = tf(x) \quad \text{for all } x \in \mathbb{R}^n\}$$

We will show that $\mathcal{G} = \mathbb{R}$, which suffices. Clearly $1 \in \mathcal{G}$. Now note that

$$f(0) = f(0+0) = f(0)+f(0)$$

implies $f(0) = 0$. So for any $x \in \mathbb{R}^n$, we have $f(0 \cdot x) = f(0) = 0 \cdot f(x)$. This shows $0 \in \mathcal{G}$. Next suppose $s, t \in \mathcal{G}$ then for any $x \in \mathbb{R}^n$,

$$f((s+t)x) = f(sx+tx) = f(sx)+f(tx) = sf(x)+tf(x) = (s+t)f(x)$$

So $s+t \in \mathcal{G}$. Also if $s \in \mathcal{G}$ then for all $x \in \mathbb{R}^n$

$$0 = f(0 \cdot x) = f(sx - sx) = f(sx) + f(-sx)$$

Hence $f(-sx) = -f(sx) = -sf(x)$. This means $-s \in \mathcal{G}$.

These properties imply that $\mathcal{G}$ is an (additive) *subgroup* of $\mathbb{R}$. Since $1 \in \mathcal{G}$, the subgroup generated by 1, namely $\mathbb{Z}$ must also be contained in $\mathcal{G}$.

Suppose again that $s, t \in \mathcal{G}$. Then for all $x \in \mathbb{R}^n$

$$f(stx) = sf(tx) = stf(x)$$

This shows that $st \in \mathcal{G}$. Also if $s \in \mathcal{G}$ is nonzero then for all $x \in \mathbb{R}^n$

$$f(x) = f(ss^{-1}x) = sf(s^{-1}x)$$

shows that $f(s^{-1}x) = s^{-1}f(x)$. This implies $s^{-1} \in \mathcal{G}$.

The above properties now imply that $\mathcal{G}$ is a *subfield* of $\mathbb{R}$. In particular, this implies $\mathbb{Q} \subseteq \mathcal{G}$.

For the final nail in the coffin, we define for each $x \in \mathbb{R}^n$, the function $\Delta_x : \mathbb{R} \to \mathbb{R}^m$ by $t \mapsto f(tx) - tf(x)$. Note that (for each x) the function Δ_x is continuous (as a composition of continuous functions) and

$$\mathcal{G} = \bigcap_{x \in \mathbb{R}^n} \Delta_x^{-1}(\{0\})$$

This shows that $\mathcal{G}$ is an intersection of closed subsets of $\mathbb{R}$, and must therefore be closed. But the only closed subset of $\mathbb{R}$ containing $\mathbb{Q}$ is $\mathbb{R}$ itself. So $\mathcal{G} = \mathbb{R}$. $\square$

To define the inner product in terms of $[\cdot,\cdot]$, we use a standard trick. Note that if $[\cdot,\cdot] = \Re \circ \langle \cdot,\cdot \rangle$ for some inner product $\langle \cdot,\cdot \rangle$ then (letting $\Im$ denote the imaginary-part function)

$$[u,v]+i[iu,v] = \Re(\langle u,v \rangle)+i\Re(\langle iu,v \rangle) = \Re(\langle u,v \rangle)+i\Re(-i\langle u,v \rangle) = \Re(\langle u,v \rangle)+i\Im(\langle u,v \rangle) = \langle u,v \rangle$$

Proposition. *Suppose V is a complex vector space and $[\cdot,\cdot] : V \times V \to \mathbb{R}$ is a* real *inner product on V. Suppose $\langle iu, iv \rangle = \langle u, v \rangle$ for all $u, v \in V$. Then the function $\langle \cdot,\cdot \rangle : V \times V \to \mathbb{C}$ defined by*

$$\langle u,v \rangle := [u,v] + i[iu,v]$$

is the unique *inner product on V which satisfies $\langle u,u \rangle = [u,u]$ for all $u \in V$.*

Proof. Once we show $\langle \cdot,\cdot \rangle$ is in fact an inner product, the uniqueness condition will follow by the discussion above. For conjugate-symmetry, note that

$$\langle v,u \rangle = [v,u]+i[iv,u] = [u,v]+i[u,iv] = [u,v]+i[i{\cdot}u,i{\cdot}iv] = [u,v]-i[iu,v] = \overline{\langle u,v \rangle}$$

Right-additivity of $\langle \cdot,\cdot \rangle$ follows immediately from the right-additivity of $[\cdot,\cdot]$. The same is true of $\mathbb{R}$-homogeneity. For multiplication by i, we have

$$\langle u,iv \rangle = [u,iv] + i[iu,iv] = [i \cdot u, i \cdot iv] + i[u,v] = i([u,v] + i[iu,v]) = i\langle u,v \rangle$$

Hence for any $z = a + bi \in \mathbb{C}$ (with $a, b \in \mathbb{R}$) we have

$$\langle u,zv \rangle = \langle u,(a+ib)v \rangle = \langle u,av+ibv \rangle = \langle u,av \rangle+\langle u,ibv \rangle = \langle u,av \rangle+i\langle u,bv \rangle = a\langle u,v \rangle+ib\langle u,v \rangle = z\langle u,v \rangle$$

Thus $\langle \cdot,\cdot \rangle$ is $\mathbb{C}$-linear in its second argument. Now observe that (for $u \in V$)

$$[u,iu] = [iu, i \cdot iu] = [iu,-u] = -[u,iu]$$

This implies $[u,iu] = 0$. This means that

$$\langle u,u \rangle = [u,u] + i[iu,v] = [u,u]$$

This proves both the last claim in the proposition and the positive-definiteness of $\langle \cdot,\cdot \rangle$. $\square$

Returning to our original problem, we combine the two propositions above to get an inner-product $\langle \cdot,\cdot \rangle$ such that $\langle u,u \rangle = \|u\|^2$ for all $u \in V$. Its uniqueness is a consequence of the uniqueness conditions in these propositions. $\square$

Question 13 : Riesz

A bounded linear functional on a Banach space V is (by definition) a $\mathbb{C}$-linear map $A : V \to \mathbb{C}$ such that

$$\sup_{\substack{v \in V \\ \|v\|_V \leq 1}} |Av| < \infty$$

Show that if $\|\cdot\|_V$ is complete and satisfies the parallelogram identity then every bounded linear functional on V has the form $v \mapsto \langle u, - \rangle$ for a unique vector $u \in V$.

Solution. This is essentially a restatement of Theorem D.10 in the lecture notes.[21] See the proof there. See also Theorem 5.3 in Stein and Shakarchi's *Real Analysis.* □

[21] The definition of "Hilbert space" there uses the notion of inner product instead of the parallelogram law. But it is precisely the content of the previous question that the two notions are equivalent.

Question 14

Let $(X, \mathfrak{M}, \mu)$ be a measure space. Let $f : X \to \mathbb{C}$ be measurable. A number $M \geq 0$ is an *essential* upper bound on f iff the set

$$\{x \in X \mid |f(x)| > M\}$$

has measure 0 in X. For any such function f, the set of essential upper bounds on f admits a *minimum* in $[0, \infty]$, which is by definition the *essential supremum* of f (c.f. Definition 3.7 in Rudin's *Real and Complex Analysis*). We define $L^\infty(X \to \mathbb{C}, \mu)$ to be the collection of equivalence classes of essentially bounded[22] measurable functions $X \to \mathbb{C}$ under the equivalence relation of being equal μ-a.e. The map $\|\cdot\|_\infty$ taking a function to its essential supremum defines a norm on this space.

(a) $L^\infty(X \to \mathbb{C}, \mu)$ is a Banach space. That is, the essential supremum gives a complete norm.

Solution. See (the end of the proof of) Theorem 3.11 in Rudin's *Real and Complex Analysis.* □

(b) Extend the theorems of Hölder and Minkowski to the case $p = \infty$.

Solution. For the simplest version of Hölder's inequality, see Theorem 3.8 in Rudin's *Real and Complex Analysis.* The other cases follow formally as in Question 8 parts (b) and (c).

For Minkowski's inequality (which logically precedes part (a)), suppose f, g are measurable functions. Then $|f| \leq \|f\|_\infty$ holds pointwise on the complement of a measure 0 set E. Similarly, $|g| \leq \|g\|_\infty$ holds pointwise on the complement of a measure 0 set E'. Hence

$$|f + g| \leq |f| + |g| \leq \|f\|_\infty + \|g\|_\infty$$

holds pointwise on the complement of the measure 0 set $E \cup E'$. It follows that $\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$ □

[22] That is, those whose essential supremum is finite.

Question 15

Let X be a locally-compact, σ-compact, T6 Hausdorff space and $\mu : \mathcal{B}(X) \to [0, \infty]$ a locally-finite, σ-finite Borel measure. Let $p \in [1, \infty)$ and $C_c(X \to \mathbb{C})$ the set of continuous functions whose closed support $\overline{f^{-1}(\mathbb{C} - \{0\})}$ is compact. Show that $C_c(X \to \mathbb{C})$ is dense in $L^p(X \to \mathbb{C}, \mu)$.

Proof. Consider

Theorem. *Suppose X is a locally-compact, σ-compact, T6 Hausdorff space and $\mu : \mathcal{B}(X) \to [0, \infty]$ a locally-finite, σ-finite Borel measure. Let f be a complex measurable function on X supported on a set of finite measure. Then for each $\epsilon > 0$ there exists $g \in C_c(X \to \mathbb{C})$ such that*

$$\mu(\{x \in X \mid f(x) \neq g(x)\}) < \epsilon$$

We can further choose g such that $\sup_{x \in X} |g(x)| \leq \sup_{x \in X} |f(x)|$.

This is theorem 2.24 in Rudin's *Real and Complex Analysis*.[23] See the proof there.

With this tool in hand, we can make quick work of our problem.

Fix $\epsilon > 0$ and $f \in L^p(X \to \mathbb{C}, \mu)$. It suffices to show that we can find $g \in C_c(X \to \mathbb{C})$ such that $\|f - g\|_p < 2\epsilon$. For each $M \in \mathbb{N}$, define

$$\begin{aligned} X_M &:= \{x \in X \mid M^{-1} \leq |f(x)| \leq M\} \\ f_M &:= f|_{X_M} = f \cdot \chi_{X_M} \end{aligned}$$

Note that X_M has finite measure by the Tschebyschev inequality. Also $f_M \longrightarrow f$ pointwise as $M \longrightarrow \infty$.[24] The inequality $|f - f_M| \leq |f|$ holds pointwise. It follows by the dominated convergence theorem that $\|f - f_M\|_p \longrightarrow 0$ as $M \longrightarrow 0$. In particular, we can choose M_0 large enough that $\|f - f_{M_0}\|_p < \epsilon$. We can then choose $g \in C_c(X \to \mathbb{C})$ such that $|g| \leq M_0$ pointwise and

$$\mu(\{x \in X \mid f_{M_0}(x) \neq g(x)\}) < (\epsilon/2M_0)^p$$

Let $D := \{x \in X \mid f_{M_0}(x) \neq g(x)\}$. Then

$$\|f_{M_0} - g\|_p^p = \int_X |f_{M_0} - g|^p \, d\mu \leq \int_D (|f_{M_0}| + |g|)^p \, d\mu \leq \mu(D) \cdot (2M_0)^p < \epsilon^p$$

Thus $\|f_{M_0} - g\|_p < \epsilon$ and so (using Minkowksi's inequality)

$$\|f - g\|_p \leq \|f - f_{M_0}\|_p + \|f_{M_0} - g\|_p < \epsilon + \epsilon < 2\epsilon$$

□

[23] The hypotheses in that theorem are a bit different. Rudin asks that (i) μ be finite on compact sets, (ii) μ be regular and (iii) μ be complete. The third hypothesis is never used and since we can always pass to the completion (and then come back) it is harmless. The assumption (i) follows from the hypotheses listed above by applying Claim 3.21. The assumption (ii) follows from the same hypotheses by applying Theorem 3.22.

[24] At every point $x \in X$, the sequence $f_M(x)$ is eventually constant with limiting value $f(x)$. Note that the case $f(x) = 0$ needs to be argued for separately.