Princeton University
Spring 2025 MAT425: Measure Theory
HW4
Feb 24th 2025

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1. In this question we consider the space $L^1\left(\mathbb{R}^d\right)$ w.r.t. to the Lebesgue measure on $\mathbb{R}^d$, and endow it with a norm
$$\|f\|_{L^1} \equiv \int_{\mathbb{R}^d} |f| \, \mathrm{d}\lambda \, .$$
Prove that if $f \in L^1\left(\mathbb{R}^d\right)$ and $\delta > 0$ then $\{\, x \mapsto f(\delta x) \,\}_{\delta>0}$ converges to f in the L^1-norm as $\delta \to 1$.
2. Let $f \in L^1([0,b])$ and set
$$g(x) := \int_x^b \frac{f(t)}{t} \mathrm{d}t \qquad (x \in (0,b]) \, .$$
Show that $g \in L^1([0,b])$ and
$$\int_0^b g = \int_0^b f \, .$$
3. (*Chebyshev inequality*) Let $f \in L^1\left(\mathbb{R}^d \to [0,\infty)\right)$. Show that
$$\lambda\left(f^{-1}((\alpha,\infty))\right) \leq \frac{1}{\alpha} \int f \mathrm{d}\lambda \qquad (\alpha > 0) \, .$$
4. Show that for $f \in L^1\left(\mathbb{R}^d \to \mathbb{R}\right)$, if
$$\int_A f \mathrm{d}\lambda = 0 \qquad (A \text{ Lebesgue measurable})$$
then
$$\lambda\left(f^{-1}(\{\, 0 \,\}^c)\right) = 0 \, .$$
5. Find a function $f \in L^1\left(\mathbb{R}^d\right)$ and a sequence $\{\, f_n \,\}_n \subseteq L^1\left(\mathbb{R}^d\right)$ such that
$$\|f - f_n\|_{L^1} \to 0$$
yet $f_n(x) \to f(x)$ does not hold for *any* $x \in \mathbb{R}^d$.
6. (*The Layer-Cake Representation*) Let $f \in L^1\left(\mathbb{R}^d\right)$. Show that
$$\int_{\mathbb{R}^d} |f| \, \mathrm{d}\lambda = \int_{\alpha=0}^{\infty} \lambda\left(\left\{\, x \in \mathbb{R}^d \,\middle|\, |f(x)| > \alpha \,\right\}\right) \mathrm{d}\alpha \, .$$
7. A sequence $\left\{\, f_n : \mathbb{R}^d \to \mathbb{C} \,\right\}_n$ of measurable functions is called *Cauchy in measure* iff for any $\varepsilon > 0$,
$$\lambda\left(\left\{\, x \in \mathbb{R}^d \,\middle|\, |f_n(x) - f_k(x)| > \varepsilon \,\right\}\right) \to 0$$
as $n, k \to \infty$. Moreover, the sequence *converges in measure* to a measurable function $f : \mathbb{R}^d \to \mathbb{C}$ iff for any $\varepsilon > 0$,
$$\lim_{n\to\infty} \lambda\left(\left\{\, x \in \mathbb{R}^d \,\middle|\, |f_n(x) - f(x)| > \varepsilon \,\right\}\right) = 0 \, .$$
Prove that convergence in L^1 norm implies convergence in measure, and provide a counter example to the converse.

8. In this exercise, you will construct a Vitali set in $[0, 1]$ and prove that it cannot be Lebesgue measurable.

   (a) **The Equivalence Relation.**

      i. Let $\sim$ be the relation on $[0, 1]$ defined by

      $$x \sim y \quad \text{if and only if} \quad x - y \in \mathbb{Q}.$$

      Prove that $\sim$ is an equivalence relation on $[0, 1]$.

      ii. Show that each equivalence class is countable. *(Hint: For any fixed $x \in [0, 1]$, the equivalence class of x is given by $\{x + q : q \in \mathbb{Q}\} \cap [0, 1]$, and $\mathbb{Q}$ is countable.)*

   (b) **Existence of a Vitali Set.** Using the Axiom of Choice, show that there exists a subset $V \subset [0, 1]$ such that V contains exactly one element from each equivalence class defined by $\sim$. This set V is called a *Vitali set.*

   (c) **Translates of the Vitali Set.** For each rational number $r \in \mathbb{Q} \cap [-1, 1]$, define the translated set

   $$V_r = \{v + r : v \in V\}.$$

   Prove that the sets $\{V_r : r \in \mathbb{Q} \cap [-1, 1]\}$ are pairwise disjoint. *(Hint: Suppose that for $r_1 \neq r_2$, there exist $v_1, v_2 \in V$ such that $v_1 + r_1 = v_2 + r_2$. Use the definition of the equivalence classes to reach a contradiction.)*

   (d) **Covering a Finite Interval.** Show that

   $$\bigcup_{r \in \mathbb{Q} \cap [-1,1]} V_r \subset [-1, 2].$$

   (Hint: If $v \in V \subset [0, 1]$ and $r \in [-1, 1]$, then $v + r \in [-1, 2]$.)

   (e) **Non-measurability of the Vitali Set.** Assume, for the sake of contradiction, that V is Lebesgue measurable with measure $m(V)$. Using the translation invariance and countable additivity of Lebesgue measure, show that this assumption leads to a contradiction.

      i. Express the measure of the union

      $$U = \bigcup_{r \in \mathbb{Q} \cap [-1,1]} V_r,$$

      in terms of $m(V)$.

      ii. Explain why this leads to a contradiction given that U is contained in the finite interval $[-1, 2]$. *(Hint: Consider the cases $m(V) = 0$ and $m(V) > 0$, and show that each case contradicts the finiteness of the measure of $[-1, 2]$.)*

9. [*NOT FOR THE MIDTERM*] The Banach-Tarski paradox is one of the most striking results in measure theory and geometric group theory. In this exercise, you will explore several aspects of the paradox, including the notions of equidecomposability, the role of non-measurable sets, the use of the Axiom of Choice, and the connection to free groups and amenability.

   (a) **Equidecomposability and Measure Preservation.**
   Let $A, B \subset \mathbb{R}^3$. We say that A and B are *equidecomposable* if there exist finite partitions

   $$A = \bigcup_{i=1}^{n} A_i \quad \text{and} \quad B = \bigcup_{i=1}^{n} B_i,$$

   and isometries (rotations and translations) $T_1, \ldots, T_n$ of $\mathbb{R}^3$ such that

   $$T_i(A_i) = B_i \quad \text{for } i = 1, \ldots, n.$$

      i. Prove that if A and B are Lebesgue measurable and equidecomposable (via the above definition), then $\lambda(A) = \lambda(B)$.

      ii. Explain why the existence of a *paradoxical* decomposition (as in the Banach-Tarski paradox) implies that at least one of the pieces must be non-measurable.

   (b) **Statement and Intuition Behind the Paradox.**
   Let $B \subset \mathbb{R}^3$ be a solid ball. The Banach-Tarski paradox asserts that one can partition B into finitely many disjoint subsets $B_1, B_2, \ldots, B_n$ such that, using only rotations and translations, these pieces can be reassembled to form two solid balls, each congruent to B.

i. Write a precise statement of the Banach-Tarski paradox.

ii. Discuss which fundamental properties of Lebesgue measure (such as additivity and invariance under isometries) would be violated if all the pieces in the decomposition were measurable. You will need to appeal to Theorem 4.11 in the lecture notes.

(c) **The Role of the Axiom of Choice.**

i. Explain why the construction of the Banach-Tarski paradox relies on the Axiom of Choice.

ii. Describe briefly how the Axiom of Choice is used to select non-measurable sets that are critical in the paradoxical decomposition.

(d) **Free Groups and Paradoxical Decompositions.**
A crucial step in the proof of the Banach-Tarski paradox is to show that the rotation group $SO(3)$ contains a free subgroup on two generators.

i. Prove (or outline a proof) that $SO(3)$ contains a subgroup isomorphic to the free group on two generators, denoted F_2.

ii. Explain why the existence of such a free subgroup is essential for constructing a paradoxical decomposition of the sphere.

(e) **Amenability and the Dimensionality Issue.**

i. Define what it means for a group to be *amenable*.

ii. Explain why the Banach-Tarski paradox cannot occur in $\mathbb{R}^2$. (Hint: Relate this to the amenability of the rotation group in the plane.)

(f) **Consequences for Finitely Additive Measures.**
Assume for the sake of contradiction that there exists a finitely additive, rotation-invariant measure defined on *all* subsets of $\mathbb{R}^3$ that extends Lebesgue measure.

i. Show that, under this assumption, the Banach-Tarski paradox leads to a contradiction with the finite measure of the ball B.

ii. Conclude why no such finitely additive measure can exist on all subsets of $\mathbb{R}^3$.