Princeton University

Spring 2025 MAT425: Measure Theory

HW4 Sample Solutions

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Problem 1.

Solution. We introduce the notation $f_\delta(x) = f(\delta x)$. For any $\epsilon > 0$, we prove that for all $|\delta - 1|$ small enough we have $\|f - f_\delta\|_{L^1} \leq 4\epsilon$, which will imply that $f_\delta \to f$ in $L^1(\mathbb{R}^d)$. We recall that continuous functions of compact support are dense in $L^1(\mathbb{R}^d)$, so let g be a continuous function of compact support such that $\|f - g\|_{L^1} \leq \epsilon$. We can write:

$$f - f_\delta = (f - g) + (g - g_\delta) + (f_\delta - g_\delta)$$

By a change of variables we have:

$$\int_{\mathbb{R}^d} |f_\delta - g_\delta|(x)dx = \int_{\mathbb{R}^d} |f - g|(\delta x)dx = \delta^{-n}\|f - g\|_{L^1} < 2\epsilon$$

because for $|\delta - 1|$ small enough we have $\delta^{-n} < 2$. Finally, since g is continuous and has compact support we have that for $|\delta - 1|$ small enough:

$$\int_{\mathbb{R}^d} |g(x) - g(\delta x)|dx < \epsilon$$

We use the triangle inequality and the previous bounds to conclude.

□

Problem 2.

Solution. We first define the function $h : [0, b] \times [0, b] \to \mathbb{R}$ given by $h(t, x) = \mathbf{1}_{[x,b]}(t) \cdot \frac{f(t)}{t}$. We now show that h is measurable on $[0, b] \times [0, b]$. For this, we use the following result from Corollary 3.7 in Chapter 2 of Stein and Shakarchi Real analysis: Suppose that F is measurable on $\mathbb{R}^{d_1}$, then $\tilde{F}(x, y) = F(x)$ is measurable on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$. The proof of the result is immediate from the definition of measurable functions. Applying this result, we get that $(t,x) \mapsto \frac{f(t)}{t}$ is measurable on $[0,b]\times[0,b]$. Similarly, we also have that $(t,x)\mapsto t$ and $(t,x)\mapsto x$ are measurable. Thus, we get that $(t,x)\mapsto t-x$ is measurable, so the set $\{t-x\geq 0\}\cap\{t\leq b\}$ is measurable. Finally, this implies that $(t,x)\mapsto \mathbf{1}_{[x,b]}(t)$ is measurable, so h is measurable on $[0,b]\times[0,b]$. In conclusion, the function used below is measurable and we can apply Tonelli's theorem (and later Fubini's theorem).

By the triangle inequality and Tonelli's theorem we have in the extended sense:

$$\int_0^b |g(x)|dx \leq \int_0^b\int_x^b \frac{|f(t)|}{t}dtdx = \int_0^b\int_0^t \frac{|f(t)|}{t}dxdt = \int_0^b |f(t)|dt$$

Since f is integrable, we obtain that $g\in L^1([0,b])$ as well. We then use Fubini's theorem to conclude:

$$\int_0^b g(x)dx \leq \int_0^b\int_x^b \frac{f(t)}{t}dtdx = \int_0^b\int_0^t \frac{f(t)}{t}dxdt = \int_0^b f(t)dt.$$

□

Problem 3.

Solution. We denote $E_\alpha = f^{-1}(\alpha,\infty) = \{x:\ f(x)>\alpha\}$. For every $x\in\mathbb{R}^d$ we have $f(x)\geq \alpha\mathbf{1}_{E_\alpha}$. We conclude by monotonicity of the integral:

$$\alpha\lambda(E_\alpha) = \int \alpha\mathbf{1}_{E_\alpha}d\lambda \leq \int f d\lambda$$

□

Problem 4.

Solution. We write $f=f^+-f^-$ where f^+,f^- are non-negative functions. For any $\alpha>0$ we have that $\{f>\alpha\}=\{f^+>\alpha\}$. Using Chebyshev's inequality we get:

$$\lambda\big(\{f>\alpha\}\big) = \lambda\big(\{f^+>\alpha\}\big) \leq \frac{1}{\alpha}\int f^+d\lambda = \frac{1}{\alpha}\int_{\{f\geq 0\}} fd\lambda = 0$$

Since $\{f>0\}=\bigcup_n\{f>1/n\}$, we get that $\lambda(\{f>0\})=0$. We use the same argument for $-f$ to get $\lambda(\{f<0\})=0$. □

Problem 5.

Solution. We first construct the example in the case of $[0,1]\subset\mathbb{R}$. We set $f=0$ and for every $0\leq k\leq n-1$ we set:

$$g_{n,k} = \mathbf{1}_{I_{n,k}},\ I_{n,k} = \left[\frac{k}{n},\frac{k+1}{n}\right]$$

We have that $\|g_{n,k}\|_{L^1} = 1/n \to 0$ as $n\to\infty$. However, for any $x\in[0,1]$ we have that $\limsup_{n,k} g_{n,k}(x)=1$ and $\liminf_{n,k} g_{n,k}(x)=0$. We now consider the sequence $\{a_i\}$ to be an enumeration of $\{(k,n):\ 0\leq k\leq n-1\}$

and we set $f_i = g_{a_i}$. We proved that the sequence $\{f_i\}$ converges to f in $L^1([0,1])$ but does not converge to f pointwise anywhere. Translating the functions $\{f_i\}$ and using an additional relabeling argument we can construct an example on $\mathbb{R}$ as well. □

Problem 6.

Solution. For any $x \in \mathbb{R}^d$ we can write:

$$|f(x)| = \int_0^{|f(x)|} 1 d\alpha = \int_0^{\infty} \mathbf{1}_{\{|f(x)|>\alpha\}} d\alpha$$

Since $f \in L^1(\mathbb{R}^d)$ we can use Fubini to get:

$$\int_{\mathbb{R}^d} |f| d\lambda = \int_{\mathbb{R}^d} \int_0^{\infty} \mathbf{1}_{\{|f(x)|>\alpha\}} d\alpha d\lambda = \int_0^{\infty} \int_{\mathbb{R}^d} \mathbf{1}_{\{|f(x)|>\alpha\}} d\lambda d\alpha = \int_0^{\infty} \lambda\{|f(x)| > \alpha\} d\alpha$$

□

Problem 7.

Solution. Let $f_n \to f$ in L^1. For any $\epsilon > 0$ we get by Chebyshev's inequality:

$$\lambda\Big(\big\{x \in \mathbb{R}^d : \ |f_n(x) - f(x)| > \epsilon\big\}\Big) \leq \frac{1}{\epsilon} \int_{\mathbb{R}^d} |f_n(x) - f(x)| dx \to 0 \text{ as } n \to \infty$$

Thus, we showed that convergence in L^1 norm implies convergence in measure.

We consider now the case $d = 1$. We set $f = 0$ and $f_n(x) = n\mathbf{1}_{[-1/n,1/n]}$. For any $1 > \epsilon > 0$ we have:

$$\lambda\Big(\big\{x \in \mathbb{R} : \ |f_n(x) - f(x)| > \epsilon\big\}\Big) = \frac{2}{n} \to 0 \text{ as } n \to \infty$$

As a result, f_n converges to f in measure. However, we see that $\{f_n\}$ does not converge in L^1. □

Problem 8.

Solution. The proof is on pages 24-25 of Chapter 1 in Stein and Shakarchi: Real analysis. □