Princeton University
Spring 2025 MAT425: Measure Theory
HW3
Feb 16th 2025

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1. Prove Claim 2.67 in the lecture notes.

2. Prove Claim 2.71 in the lecture notes.

3. Prove that if $\mathscr{F} \subseteq \mathscr{P}(X)$ is countable then $\sigma(\mathscr{F})$, the σ-algebra generated by $\mathscr{F}$, is either finite or has cardinality $2^{\aleph_0}$.

4. Let C be the standard $\frac{1}{3}$-Cantor set.
   (a) Prove that the Lebesgue measure λ of the Cantor set C is zero:
   $$\lambda(C) = 0$$
   using the definition of the Lebesgue measure λ given in Definition 4.6 in the lecture notes.
   (b) Prove that $|C| = 2^{\aleph_0}$.

5. (*The Lebesgue-Stieltjes measure*) Let $F : \mathbb{R} \to \mathbb{R}$ be increasing and right continuous. Let $\mathcal{A}_0$ and $\mathcal{A}$ be the elementary set and the algebras defined in Claim 4.3 and Claim 4.4 in the lecture notes. Define now
   $$\rho_F(S) := \begin{cases} \sum_{n=1} F(b_j) - F(a_j) & S = \bigcup_{j=1}^{n} (a_j, b_j] \\ 0 & S = \varnothing \end{cases} \qquad (S \in \mathcal{A}) .$$
   Show that ρ_F is a premeasure on $\mathcal{A}$. With the choice $F = x \mapsto x$ we get, using Caratheodory's construction to get $\mu_{\varphi_{\rho_F}}$ the Lebesgue measure and more generally this is called *the Lebesgue-Stieltjes measure*. Show that each of the following equations hold:
   $$\begin{aligned} \mu_{\varphi_{\rho_F}}(\{ a \}) &= F(a) - \lim_{\varepsilon \to 0^+} F(a - \varepsilon) \\ \mu_{\varphi_{\rho_F}}([a, b)) &= \lim_{\varepsilon \to 0^+} [F(b - \varepsilon) - F(a - \varepsilon)] \\ \mu_{\varphi_{\rho_F}}([a, b]) &= F(b) - \lim_{\varepsilon \to 0^+} F(a - \varepsilon) \\ \mu_{\varphi_{\rho_F}}((a, b)) &= \lim_{\varepsilon \to 0^+} F(b - \varepsilon) - F(a) . \end{aligned}$$

6. Show there exists a Borel set $A \subseteq [0, 1]$ such that
   $$0 < \lambda(A \cap I) < \lambda(I) \qquad (I \subseteq [0, 1] \text{ is an interval}) .$$
   Hint: Every subinterval of $[0, 1]$ contains Cantor-type sets of positive measure.

7. Give an example of an open set $U \in \text{Open}(\mathbb{R})$ such that $\lambda\left(\partial \overline{U}\right) > 0$.
   Hint: Consider the set obtained by taking the union of open intervals which are deleted at the odd steps in the construction of a Cantor-like set.

8. Let

$$A := \{ x \in [0,1] \mid x \text{ does not have the digit 4 in its decimal expansion} \} .$$

Show that A is Lebesgue measurable and calculate $\lambda(A)$.

9. (*The Borel-Cantelli lemma*) Let $\{ E_k \}_{k=1}^{\infty}$ be a sequence of Lebesgue measurable subsets of $\mathbb{R}$ such that

$$\sum_{k=1}^{\infty} \lambda(E_k) < \infty .$$

Define

$$E := \{ x \in \mathbb{R} \mid x \in E_k \text{ for infinitely many } k \} .$$

Show that

(a) E is Lebesgue measurable.

(b) $\lambda(E) = 0$.
Hint: $E = \bigcap_{n=1}^{\infty} \bigcup_{k \geq n} E_k$.

(c) Let $f_n : [0,1] \to \mathbb{C}$ be a sequence of measurable functions with

$$\lambda(\{ x \in [0,1] \mid |f_n(x)| = \infty \}) = 0 \qquad (n \in \mathbb{N}) .$$

Show there is a sequence $\{ c_n \}_{n \in \mathbb{N}} \subseteq (0, \infty)$ such that

$$\lambda\left(\left\{ x \in [0,1] \,\middle|\, \lim_{n \to \infty} \frac{f_n(x)}{c_n} \text{ does not exist or does not equal } 0 \right\}\right) = 0 .$$

10. Show that for every measurable function $f : \mathbb{R} \to \mathbb{C}$ there exists a sequence of continuous functions $f_n : \mathbb{R} \to \mathbb{C}$ such that

$$\lambda\left(\left\{ x \in \mathbb{R} \,\middle|\, \lim_{n \to \infty} f_n(x) \text{ does not exist or does not equal } f(x) \right\}\right) = 0 .$$

11. Show there exist closed sets $A, B \in \text{Closed}(\mathbb{R})$ such that $\lambda(A) = \lambda(B) = 0$ yet $\lambda(A + B) > 0$.

12. Show there exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that there exists a set $A \subseteq \mathbb{R}$ which is Lebesgue measurable and yet $f(A)$ is *not* Lebesgue measurable.

13. Show that does not exist a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that

$$\lambda\left(\left\{ x \in \mathbb{R} \,\middle|\, f(x) \neq \chi_{[0,1]}(x) \right\}\right) = 0 .$$

14. Assume $A \subseteq E \subseteq B \subseteq \mathbb{R}$ with A, B Lebesgue measurable such that $\lambda(A) = \lambda(B)$. Show that E is Lebesgue measurable.

15. (*The Hausdorff measure*) Let X be a metric space. Define for any $S \subseteq X$, $\delta > 0$ and $d \in [0, \infty)$,

$$H_\delta^d(S) := \inf\left(\left\{ \sum_{i=1}^{\infty} (\text{diam}(U_i))^d \,\middle|\, \{ U_i \}_{i=1}^{\infty} \subseteq X \text{ s.t. } \bigcup_{i=1}^{\infty} U_i \supseteq S \wedge \text{diam}(U_i) < \delta \right\}\right) .$$

Here $\text{diam}(U_i)$ is the diameter of the set defined via the metric on X.

(a) Show that $\delta \mapsto H_\delta^d(S)$ is monotone decreasing.

(b) Define

$$H^d(S) := \lim_{\delta \to 0^+} H_\delta^d(S) .$$

Show that $S \mapsto H^d(S)$ is an outer measure. The induced measure is called *the d-dimensional Hausdorff measure on X*.

(c) Show that all Borel subsets $\mathcal{B}(X)$ (w.r.t. the metric topology on X) are H^d-measurable (in the sense of Caratheodory).

(d) Show that for any given set S,

$$[0,\infty) \ni d \mapsto H^{d}(S)$$

is monotone decreasing. If it is not always infinite, its image equals $\left\{ \infty, H^{d_\star(S)}(S), 0 \right\}$ for some $d_\star(S) \in [0,\infty)$ such that $H^{d_\star(S)}(S) \in (0,\infty)$. The number

$$d_\star(S) := \inf\left(\left\{ d \geq 0 \,\middle|\, H^{d}(S) = 0 \right\}\right)$$

is called *the Hausdorff dimension of S.*

(e) Let $d \in \mathbb{N}$. Show that if $X = \mathbb{R}^d$ and $\lambda : \mathcal{B}\left(\mathbb{R}^d\right) \to [0,\infty]$ is the Lebesgue measure on it then

$$\lambda(E) = \beta_d H^{d}(E) \qquad \left(E \in \mathcal{B}\left(\mathbb{R}^d\right)\right)$$

for some constant β_d which depends only on d. Calculate the constant.