Princeton University
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HW3 Sample Solutions
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Problem 1

(Claim 2.67) Suppose we are given a set X, a collection $\mathscr{E} \subseteq \mathscr{P}(X)$ and a function $\rho : \mathscr{E} \to [0, \infty]$ satisfying $\varnothing, X \in \mathscr{E}$ and $\rho(\varnothing) = 0$. Proposition 2.65[1] provides a construction of a corresponding outer measure φ_ρ on X. We are asked to show that φ_ρ can fail to coincide with ρ when restricted to $\mathscr{E}$ in general.

Solution 1. Let $X = \{x_1, x_2\}$ be a set of size 2. Set $\mathscr{E} = \mathscr{P}(X)$ and define the function $\rho : \mathscr{E} \to [0, \infty]$ by

$$\rho(S) := \begin{cases} 1 & \text{if } S = X \\ 0 & \text{otherwise} \end{cases}$$

Then setting $E_1 = \{x_1\}$, $E_2 = \{x_2\}$ and $E_i = \varnothing$ for $i > 2$ in the definition of φ_ρ found in Proposition 2.65, we get

$$\varphi_\rho(X) = \varphi_\rho(\{x_1, x_2\}) \le \rho(\{x_1\}) + \rho(\{x_2\}) = 0$$

In particular, $\varphi_\rho(X) \neq \rho(X)$. □

Solution 2. As suggested in the notes, we set $X = \mathbb{N}$,

$$\mathscr{E} := \{A \subseteq \mathbb{N} \mid |A| < \infty \lor |A^{\mathsf{c}}| < \infty\}$$

and

$$\rho(A) := \begin{cases} 1 & |A^{\mathsf{c}}| < \infty \\ 0 & |A| < \infty \end{cases}$$

[1] All unspecified references refer to the lecture notes.

It is clear that $\varnothing, X \in \mathcal{E}$ and that $\rho(\varnothing) = 0$, so φ_ρ is well-defined. Note that since every set forms a cover of itself, we have $\varphi_\rho(S) \leq \rho(S)$ for all $S \subseteq X$.[2] Since φ_ρ is an outer measure, it is countably subadditive. Hence

$$\varphi_\rho(\mathbb{N}) = \varphi_\rho\left(\bigcup_{i=1}^{\infty}\{i\}\right) \leq \sum_{i=1}^{\infty}\varphi_\rho(\{i\}) \leq \sum_{i=1}^{\infty}\rho(\{i\}) = \sum_{i=1}^{\infty}0 = 0$$

Thus $\varphi_\rho(X) = 0 \neq 1 = \rho(X)$.[3] □

[2] This holds in complete generality.

[3] Since φ_ρ is an outer measure, it will follow that $\varphi_\rho = 0$.

Problem 2

(Claim 2.71) With notation as in the previous exercise, we let $\mathscr{A}_{\varphi_\rho}$ be the φ_ρ-measurable subsets of X (c.f. Definition 2.68). Then it need not be the case that $A_{\varphi_\rho} = \sigma(\mathscr{E})$.

Solution 1. Consider any set X of cardinality ≥ 2 and set $\mathscr{E} = \{\varnothing, X\}$ and $\rho(\varnothing) = \rho(X) = 0$. Then $\varphi_\rho = 0$; so all subsets of X are φ_ρ-measurable.[4] Thus $\mathscr{A}_{\varphi_\rho} = \mathscr{P}(X) \neq \mathscr{E} = \sigma(\mathscr{E})$. □

Solution 2. The previous example shows that A_{φ_ρ} may strictly contain $\sigma(\mathscr{E})$. However, even the inclusion $\mathscr{E} \subseteq A_{\varphi_\rho}$ may fail. To see this, suppose $X = \{x_0, x_1\}$, $\mathscr{E} = \mathscr{P}(X)$ and $\rho : \mathscr{E} \to [0, \infty]$ is defined by

$$\rho(S) := \begin{cases} 0 & \text{if } S = \varnothing \\ 2 & \text{if } |S| = 1 \\ 3 & \text{if } |S| = 2 \end{cases}$$

Then ρ is subadditive, hence countably subadditive—since X is finite. It follows that $\varphi_\rho = \rho$. On the other hand, $\{x_0\}$ is *not* φ_ρ-measurable since

$$\varphi_\rho(\{x_0\} \cap X) + \varphi_\rho(\{x_0\}^{\mathsf{c}} \cap X) = \rho(\{x_0\}) + \rho(\{x_1\}) = 2 + 2 = 4 \neq 3 = \rho(X) = \varphi_\rho(X)$$

Thus $\mathscr{A}_{\varphi_\rho} = \{\varnothing, X\} \neq \mathscr{P}(X) = \sigma(\mathscr{E})$. □

[4] The condition of Carathéodory reads $0 = 0 + 0$ in all cases.

Problem 3.

Suppose X is a set and $\mathscr{F} \subseteq \mathscr{P}(X)$ a collection of subsets of X. If $|\mathscr{F}| \leq 2^{\aleph_0}$ then either $\sigma(\mathscr{F})$ is finite or $|\sigma(\mathscr{F})| = 2^{\aleph_0}$.

Solution. We split the proof into two lemmas and three propositions.

Lemma 1. *Suppose $f : Y \to X$ is a map of sets and X is equipped with a σ-algebra generated by a collection $\mathscr{F} \in \mathscr{P}(X)$. Then in the notation of Definition 2.11,*

$$\sigma(f) = \sigma\left(\{f^{-1}(S) \mid S \in \mathscr{F}\}\right)$$

Proof. Clearly the right-hand-side is contained in the left-hand-side. Conversely, let $\mathfrak{N} := \sigma\left(\{f^{-1}(S) \mid S \in \mathscr{F}\}\right)$ and consider the σ-algebra $\mathfrak{M}$ on X defined by

$$\mathfrak{M} := \{S \subseteq X \mid f^{-1}(S) \in \mathfrak{N}\}$$

Then clearly $\mathscr{F} \subseteq \mathfrak{M}$. Since $\mathfrak{M}$ is a σ-algebra, it follows that $\sigma(f) \subseteq \mathfrak{M}$. □

Lemma 2. *Let X be a set and $\mathfrak{M}$ a σ-algebra on X. For $A \subseteq X$ we define*

$$\mathfrak{M}|_A := \sigma(\imath) = \{S \cap A \mid S \in \mathfrak{M}\} \in \mathscr{P}(\mathscr{P}(A))$$

where $\imath : A \hookrightarrow X$ is the inclusion map and $\sigma(\imath)$ is in the sense of Definition 2.11. Then

$$\mathfrak{M} \subseteq \{U \cup V \mid U \in \mathfrak{M}|_A \ \wedge \ V \in \mathfrak{M}|_{A^c}\}$$

with equality if and only if $A \in \mathfrak{M}$.

Proof. Take $S \in \mathfrak{M}$. We have $S \cap A \in \mathfrak{M}|_A$ and $S \cap A^c \in \mathfrak{M}|_{A^c}$. Hence

$$S = (S \cap A) \cup (S \cap A^c) \in \{U \cup V \mid U \in \mathfrak{M}|_A \ \wedge \ V \in \mathfrak{M}|_{A^c}\}$$

The desired inclusion follows. We leave the equality case to the reader. □

Proposition 1. *If $\mathscr{F}$ is* finite *then so is $\sigma(\mathscr{F})$.*

Proof. We prove the result by induction on $|\mathscr{F}|$. If $|\mathscr{F}| = 0$, then $\sigma(\mathscr{F}) = \{0, X\}$. In general, write $\mathscr{F} = \mathscr{F}' \sqcup \{A\}$ so that $|\mathscr{F}'| < |\mathscr{F}|$. Let $\mathfrak{M} = \sigma(\mathscr{F})$. It follows from Lemma 1 that $\mathfrak{M}|_A$ and $\mathfrak{M}|_{A^c}$ are generated by $\mathscr{F}'|_A$ and $\mathscr{F}'|_{A^c}$.[5] By the induction hypothesis, $\mathfrak{M}|_A$ and $\mathfrak{M}|_{A^c}$ must be finite σ-algebras. Using Lemma 2,

$$\mathfrak{M} \subseteq \{U \cup V \mid U \in \mathfrak{M}|_A \ \wedge \ V \in \mathfrak{M}|_{A^c}\}$$

Since the right-hand-side is finite, so is $\mathfrak{M}$. □

Proposition 2. *If $\mathscr{F}$ is* infinite *then $|\sigma(\mathscr{F})| \geq 2^{\aleph_0}$.*

[5] This is obtained by applying Lemma 1 to the inclusion maps $\imath : A \hookrightarrow X$ and $\jmath : A^c \hookrightarrow X$ and noting that the generators $\imath^{-1}(A) = A$ and $\jmath^{-1}(A) = \varnothing$ are redundant.

Proof. Let $\mathfrak{M} := \sigma(\mathcal{F})$. Note that $\mathfrak{M}$ is infinite as $\mathcal{F} \subseteq \mathfrak{M}$. We will construct a sequence $S_0 \supsetneq S_1 \supsetneq S_2 \supsetneq \dots$ of sets in $\sigma(\mathcal{F})$ such that $\mathfrak{M}|_{S_i}$ is infinite for each i. We start by choosing a $S_0 = X$. Having chosen S_i, we pick $T \in \mathfrak{M}|_{S_i} - \{\varnothing, S_i\}$. Lemma 2 tells us that either $\mathfrak{M}|_T$ or $\mathfrak{M}|_{S_i - T}$ must be infinite.[6] Without loss of generality, we are in the former case and we define $S_{i+1} := T$.

We now define $\Delta_i := S_{i-1} - S_i$ for $i \geq 1$. Note that $\Delta_1, \Delta_2, \dots$ are countably many non-empty pairwise-disjoint subsets of X lying in $\sigma(\mathcal{F})$. Hence[7]

$$|\sigma(\mathcal{F})| \geq |\mathcal{P}(\{\Delta_i \mid i \in \mathbb{N}\})| = |\mathcal{P}(\mathbb{N})| = 2^{\aleph_0}$$

□

Proposition 3. *If* $|\mathcal{F}| \leq 2^{\aleph_0}$ *then* $|\sigma(\mathcal{F})| \leq 2^{\aleph_0}$.

Proof. We will employ transfinite induction to show that $\mathcal{F}$ is contained in some σ-algebra of cardinality at most $2^{\aleph_0}$.[8] We begin by defining an *augment* function $\mathcal{N} : \mathcal{P}(\mathcal{P}(X)) \to \mathcal{P}(\mathcal{P}(X))$ by

$$\mathcal{N}(\mathcal{A}) = \mathcal{A} \cup \{S^c \mid S \in \mathcal{A}\} \cup \left\{ \bigcup_{i=1}^{\infty} S_i \;\middle|\; S_1, S_2, \dots \in \mathcal{A} \right\}$$

Observe that $\mathcal{N}$ satisfies

1. $\mathcal{A} \subseteq \mathcal{N}(\mathcal{A})$ for all $\mathcal{A} \subseteq \mathcal{P}(X)$ with equality if and only if $\mathcal{A}$ is a σ-algebra.
2. If $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{P}(X)$ then $\mathcal{N}(\mathcal{A}) \subseteq \mathcal{N}(\mathcal{B})$.
3. If $|\mathcal{A}| \leq 2^{\aleph_0}$ then $|\mathcal{N}(\mathcal{A})| \leq 2^{\aleph_0}$.[9]

Let ω_1 be the first uncountable ordinal. Define $\mathcal{F}_\alpha \subseteq \mathcal{P}(X)$ for $\alpha \leq \omega_1$ inductively:

$$\mathcal{F}_\alpha := \begin{cases} \mathcal{F} \cup \{X\} & \text{if } \alpha = 0 \\ \mathcal{N}(\mathcal{F}_\beta) & \text{if } \alpha = \beta + 1 \\ \bigcup_{\beta < \alpha} \mathcal{F}_\beta & \text{if } \alpha \text{ is a limit ordinal} \end{cases}$$

We claim that $\mathcal{F}_{\omega_1}$ is a σ-algebra. To see this, note that $\mathcal{F}_{\omega_1} = \bigcup_{\alpha < \omega_1} \mathcal{F}_\alpha$. Hence

- $X \in \mathcal{F}_0 \subseteq \mathcal{F}_{\omega_1}$.
- If $S \in \mathcal{F}_{\omega_1}$ then $S \in \mathcal{F}_\alpha$ for some $\alpha < \omega_1$. So $S^c \in \mathcal{N}(\mathcal{F}_\alpha) = \mathcal{F}_{\alpha+1} \subseteq \mathcal{F}_{\omega_1}$.

[6] by imitating the argument in the induction step of Proposition 1

[7] c.f. Problem 8 on Homework 1

[8] For a quick introduction to transfinite induction, consult Chapter 0.4 in Folland or read `https://ericmoorhouse.org/handouts/transfinite.pdf`.

[9] The only tricky part of this estimate is bounding the size of $\{\bigcup_{i=1}^{\infty} S_i \mid S_1, S_2, \dots \in \mathcal{A}\}$. An upper bound is given by the size of the indexing set, which is the set of all infinite sequences with terms in $\mathcal{A}$. This is by definition $\mathcal{A}^{\mathbb{N}}$, so has cardinality $|\mathcal{A}|^{|\mathbb{N}|} \leq (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = 2^{\aleph_0}$.

- If $S_1, S_2, \ldots$ is a sequence in $\mathscr{F}_{\omega_1}$ then we can find $\alpha_1, \alpha_2, \cdots < \omega_1$ such that $S_i \in \mathscr{F}_{\alpha_i}$ for each i. Let $\alpha = \sup_i \alpha_i$. Since α is a countable limit of countable ordinals, it is countable. I.e. $\alpha < \omega_1$. Now $S_i \in \mathscr{F}_\alpha$ for all i, so

$$\bigcup_{i=1}^{\infty} S_i \in \mathcal{N}(\mathscr{F}_\alpha) = \mathscr{F}_{\alpha+1} \subseteq \mathscr{F}_{\omega_1}$$

Thus we see that $\mathscr{F}_{\omega_1}$ is a σ-algebra. It contains $\mathscr{F}_0$, so contains $\mathscr{F}$. To bound the cardinality of $\mathscr{F}_{\omega_1}$, we prove $|\mathscr{F}_\alpha| \leq 2^{\aleph_0}$ by induction on $\alpha \leq \omega_1$. For $\alpha = 0$, this is true by hypothesis. For α a successor ordinal, it follows from property (3) above. For α a limit ordinal, it follows from the induction hypothesis using

$$\left| \bigcup_{\beta<\alpha} \mathscr{F}_\beta \right| \leq \sum_{\beta<\alpha} 2^{\aleph_0} = |\alpha| \cdot 2^{\aleph_0} \leq 2^{\aleph_0} \cdot 2^{\aleph_0} = 2^{\aleph_0}$$

□

Now consider $\mathscr{F}$ as in the hypothesis of the original problem. If $\mathscr{F}$ is finite then we apply Proposition 1. Otherwise, we apply Proposition 2 to get $|\sigma(\mathscr{F})| \geq 2^{\aleph_0}$ and Proposition 3 to get $|\sigma(\mathscr{F})| \leq 2^{\aleph_0}$. Combining the two, we conclude.[10]

□

[10] Here we are implicitly using the Cantor-Schröder-Berstein Theorem (see `https://en.wikipedia.org/wiki/Schr%C3%B6der%E2%80%93Bernstein_theorem`).

Problem 4.

Let $C \subset \mathbb{R}$ be the standard (i.e. one-third) Cantor set.

Proposition 4 (Problem 4(a))**.** *Let λ be the Lebesgue measure. Then $\lambda(C) = 0$.*

Solution 1. Note that the Cantor set is contained in the set of real numbers that have a ternary[11] representation that does not contain the digit 1. The proof of Problem 8 shows that such sets have measure 0.[12] ☐

Solution 2. The Lebesgue measure of an interval is given by the difference of its endpoints.[13] The set C is the decreasing intersection of sets $C_0, C_1, \ldots$ where $C_0 = [0, 1]$ and C_{n+1} is is obtained from C_i by replacing each of its interval components by the disjoint union of two subintervals, each half the length of the original. It follows that $\lambda(C_n) = \frac{2^n}{3^n}$ and thus $\lambda(C) = 0$ by Theorem 2.29(4). ☐

Proposition 5 (Problem 4(b))**.** *The cardinality of C is $|C| = 2^{\aleph_0} = |\mathbb{R}|$.*

Solution 1. Ignoring the point 1, the set C consists of those real numbers in $[0, 1)$ that admit a ternary representation consisting only of the digits 0 and 2. These are in bijection with the set of all infinite sequences in the alphabet $\{0, 2\}$, which has cardinality

$$\left|\{0, 2\}^{\mathbb{N}}\right| = |\{0, 2\}|^{\aleph_0} = 2^{\aleph_0}$$

☐

Solution 2. We will prove in Problem 11 that $C + C = [0, 2]$. This means in particular that there is a surjection $C \times C \to [0, 2]$. Cardinality-wise we get

$$|C|^2 \geq 2^{\aleph_0}$$

But $|S|^2 = |S|$ whenever S is infinite.[14] So we get $|C| \geq 2^{\aleph_0}$. On the other hand $C \subset \mathbb{R}$, so

$$|C| \leq |\mathbb{R}| = 2^{\aleph_0}$$

☐

[11] i.e. base 3

[12] Though the proof provided for that problem is a specific case, the method of proof is completely general.

[13] This is by definition for intervals of the form $(a, b]$ and follows in general because singletons have Lebesgue measure 0.

[14] This fact requires the axiom of choice.

Problem 5.

(Part 1) Let $F : \mathbb{R} \to \mathbb{R}$ be increasing[15] and right-continuous. Let $\mathscr{A}$ be the algebra defined in Claim 4.4. For $S \in \mathscr{A}$ define[16]

$$\rho_F(S) := \sum_{j=1}^{n} (F(b_j) - F(a_j)) \quad \text{where } S = \bigsqcup_{j=1}^{n} (a_j, b_j]$$

In particular $\rho_F(\varnothing) = 0$.[17] Then ρ_F is a premeasure on $\mathscr{A}$.

Solution. The first step is showing that ρ_F is well-defined (c.f. the proof of Theorem 4.5). Note that by assumption $F(b) \geq F(a)$ whenever $b \geq a$, so $\rho_F(S) \geq 0$ for all $S \in \mathscr{A}$. Now we show that $\rho_F(S)$ is independent of the choice of representation of S as a finite union of right-closed intervals. Suppose

$$S = \bigsqcup_{j=1}^{n} (a_j, b_j] = \bigsqcup_{j=1}^{m} (a'_j, b'_j]$$

Without loss of generality, we have $a_1 < a_2 < \cdots < a_n$ and $a'_1 < a'_2 < \cdots < a'_m$. Since the intervals are disjoint, we must have $a_i < b_i \leq a_{i+1}$ and $a'_i < b'_i < a'_{i+1}$ for each i. Let $c_1 < c_2 < \cdots < c_k$ be such that

$$\{c_1, \ldots, c_k\} = \{a_1, \ldots, a_n\} \cup \{b_1, \ldots, b_n\} \cup \{a'_1, \ldots, a'_m\} \cup \{b'_1, \ldots, b'_m\}$$

For each j, we can find $s < t$ such that $a_j = c_s$ and $b_j = c_t$. Then

$$\begin{aligned} \rho_F((a_j, b_j]) &= -F(a_j) + F(b_j) \\ &= (-F(c_s) + F(c_{s+1})) + (-F(c_{s+1}) + F(c_{s+2})) + \cdots + (-F(c_{t-1}) + F(c_t)) \\ &= \rho_F((c_s, c_{s+1}]) + \rho_F((c_{s+1}, c_{s+2}]) + \cdots + \rho_F((c_{t-1}, c_t]) \end{aligned}$$

We can write this as

$$\rho_F((a_j, b_j]) = \sum_{(c_i, c_{i+1}] \subseteq (a_j, b_j]} \rho_F((c_i, c_{i+1}])$$

By construction, $(c_i, c_{i+1}] \subseteq S$ if and only if $(c_i, c_{i+1}] \subseteq (a_j, b_j]$ for some j, and this j is uniquely determined if it exists. Hence

$$\sum_{j=1}^{n} \rho_F((a_j, b_j]) = \sum_{j=1}^{n} \sum_{(c_i, c_{i+1}] \subseteq (a_j, b_j]} \rho_F((c_i, c_{i+1}]) = \sum_{(c_i, c_{i+1}] \subseteq S} \rho_F((c_i, c_{i+1}])$$

By symmetry, we also have $\sum_{j=1}^{m} \rho_F((a'_j, b'_j]) = \sum_{(c_i, c_{i+1}] \subseteq S} \rho_F((c_i, c_{i+1}])$ Hence the two sums defining $\rho_F(S)$ are equal and S is well-defined.

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[15] Here we do *not* mean strictly increasing.

[16] The rectangular union symbol means we are requiring the union to be *disjoint*. Implicitly, we are also requiring $a_j < b_j$.

[17] An empty sum has value 0.

Observe that the well-definedness of S gives us *finite* additivity of ρ_F on $\mathscr{A}$. It remains to show that ρ_F is countably additive in the restricted sense of Definition 2.72. That is, given $S_1, S_2, \dots \in \mathscr{A}$ pairwise disjoint such that $S := \bigsqcup_{i=1}^{\infty} S_i$ is in $\mathscr{A}$, we need to show that $\rho_F(S) = \sum_{i=1}^{\infty} \rho_F(S_i)$. Without loss of generality we may suppose that each S_i is an interval, i.e. $S_i = (a_i, b_i]$.[18] We may further suppose that S is also an interval $(a, b]$, for if not then it is a *finite* disjoint union of such intervals on which we may argue separately and then use finite additivity.

Let $\mathcal{B} := \{b_i \mid i \in \mathbb{N}\} \cup \{a\}$. Note that $a_i \in \mathcal{B}$ for each i since if $a_i > a$ then it must be contained in some interval S_j.[19] Note also that $\mathcal{B}$ is a closed set since

$$\mathcal{B} = [a, b] - \bigsqcup_{i=1}^{\infty} (a_i, b_i)$$

Define

$$\mathcal{G} := \left\{ x \in \mathcal{B} \;\middle|\; \rho_F((x, b]) = \sum_{a_i \geq x} \rho_F(S_i) \right\}$$

When $x = b$,[20] the condition $a_i \geq x$ is false for all i. So we get $b \in \mathcal{G}$ from

$$\sum_{a_i \geq b} \rho_F(S_i) = 0 = \rho_F(\varnothing) = \rho_F((b, b])$$

Let $y := \inf \mathcal{G}$. Clearly $a \leq x_0 \leq b$. We first show that $y \in \mathcal{G}$. If not then we can find a sequence $y_1 > y_2 > \dots$ of elements in $\mathcal{G}$ such that $\lim_n y_n = y$. It follows that $y \in \mathcal{B}$ since this set is closed. By right-continuity of F,

$$\rho_F((y, b]) = \lim_{n \to \infty} \rho_F((y_n, b]) = \lim_{n \to \infty} \sum_{a_i \geq y_n} \rho_F(S_i) = \sum_{a_i > x} \rho_F(S_i)$$

We cannot have $x = a_i$ for $i \in \mathbb{N}$ since then $y_n \geq b_i > a_i$ for all n. Hence the sum on the very right is the same as $\sum_{a_i \geq x} \rho_F(S_i)$, and it follows that $y \in \mathcal{G}$.

If $y = a$, then we are done, since the statement $a \in \mathcal{G}$ is equivalent to the additivity condition we are seeking. Otherwise $y = b_j$ for some $j \in \mathbb{N}$. Now for any i, the condition $a_i \geq a_j$ holds if and only if either $i = j$ or $a_i \geq b_j$. So we get

$$\sum_{a_i \geq a_j} \rho_F(S_i) = \rho_F(S_j) + \sum_{a_i \geq y} \rho_F(S_j) = \rho_F((a_j, y]) + \rho_F((y, b]) = \rho_F(a_j, b])$$

Since $a_j \in \mathcal{B}$ it follows that $a_j \in \mathcal{G}$. This contradicts the definition of y as the infimum of $\mathcal{G}$. This contradiction allows us to conclude that ρ_F is well-defined. □

[18] We simply expand each S_i into its finitely many constituent components to obtain a wider (but still countable) union.

[19] In contrast, it is very far from true that each b_i must coincide with some a_j.

[20] Note that b must be contained in some interval $(a_i, b_i]$, and since b is the maximum of S we are forced to then have $b = b_i$. So indeed $b \in \mathcal{B}$.

Problem 5

(Part 2) With notation as in the first part, let $\mu_F := \mu_{\varphi_{\rho_F}}$ be the induced measure. Then the following hold:

1. $\mu_F(\{a\}) = F(a) - F(a-)$
2. $\mu_F([a, b)) = F(b-) - F(a-)$
3. $\mu_F([a, b]) = F(b) - F(a-)$
4. $\mu_F((a, b)) = F(b-) - F(a)$

Here $F(x-) = \lim_{t \to x^-} F(t) = \lim_{\epsilon \to 0^+} F(t - \epsilon)$. Note that this limit always exists for increasing functions $\mathbb{R} \to \mathbb{R}$.

Solution. It is easy to see that $\sigma(\mathscr{A}) = \mathscr{B}(\mathbb{R})$. So all these sets are μ_F-measurable.

1. By Theorem 2.29(4),

$$\begin{aligned} \mu_F(\{a\}) &= \mu_F\left(\bigcap_{n=1}^{\infty} (a - 1/n, a]\right) \\ &= \lim_{n\to\infty} \mu_F((a - 1/n, a]) \\ &= \lim_{n\to\infty} (F(a) - F(a - 1/n)) \\ &= F(a) - F(a-) \end{aligned}$$

2.

$$\begin{aligned} \mu_F([a, b)) &= \mu_F(\{a\} \sqcup (a, b] - \{b\}) \\ &= \mu_F(\{a\}) + \mu_F((a, b]) - \mu_F(\{b\}) \\ &= F(a) - F(a-) + F(b) - F(a) - (F(b) - F(b-)) \\ &= F(b-) - F(a-) \end{aligned}$$

3.

$$\begin{aligned} \mu_F([a, b]) &= \mu_F(\{a\} \sqcup (a, b]) \\ &= \mu_F(\{a\}) + \mu_F((a, b]) \\ &= F(a) - F(a-) + F(b) - F(a) \\ &= F(b) - F(a-) \end{aligned}$$

4.

$$\begin{aligned} \mu_F((a, b)) &= \mu_F((a, b] - \{b\}) \\ &= \mu_F((a, b]) - \mu_F(\{b\}) \\ &= F(b) - F(a) - (F(b) - F(b-)) \\ &= F(b-) - F(a) \end{aligned}$$

□

Problem 6.

There exists a Borel set $A \subseteq [0,1]$ such that

$$0 < \lambda(A \cap I) < \lambda(I) \tag{$*$}$$

for all subintervals $I \subseteq [0,1]$. (Here λ is the Lebesgue measure.)

Solution. First note that $(*)$ holds for I (and fixed A) if and only if it holds for $\overline{I}$. Next observe that if $(*)$ *fails* for some interval $I \subseteq [0,1]$ then it fails for all subintervals $I' \subseteq I$. It therefore suffices (given A) to prove $(*)$ for closed intervals I with rational endpoints.

There are only countably many closed subintervals of $[0,1]$ whose endpoints are rational. Let $I_1, I_2, I_3, \ldots$ be an enumeration of all of them. Write $I_j = [a_j, b_j]$ with $a_j, b_j \in \mathbb{Q}$ and $\ell_j := b_j - a_j$. Let $\epsilon_1, \epsilon_2, \ldots$ be a sequence of positive reals *rapidly* converging to 0, by which we intend that $\epsilon_j < \ell_j/2$ and $\epsilon_j > \sum_{i>j} \epsilon_i$ for each j.[21]

We define sets $A_j \subseteq [0,1]$ for $j \in \mathbb{N}$ inductively as follows

$$A_0 = [0,1], \quad A_{j+1} = (A_j \cup [a_j, a_j + \epsilon_j]) - [b_j - \epsilon_j, b_j]$$

And we let $A := \bigcap_{n \in \mathbb{N}} \bigcup_{m \geq n} A_m$.[22]

Now suppose $I \subseteq [0,1]$ is a closed interval with rational endpoints. Then $I = I_j$ for some $j \in \mathbb{N}$. We have $\lambda(I \cap A_j) \geq \lambda([a_j, a_j + \epsilon_j]) = \epsilon_j$ and for each $k \geq j$

$$\lambda(I \cap A_k) \geq \lambda(I \cap A_j) - \sum_{i=j+1}^{k} \lambda([b_i - \epsilon_i, b_i]) \geq \epsilon_j - \sum_{i=j+1}^{k} \epsilon_i > \epsilon_j - \sum_{i>j} \epsilon_i$$

Also $\lambda(I \cap A_j) \leq \lambda([a_j, b_j - \epsilon_j]) = \lambda(I) - \epsilon_j$ and for $k \geq j$

$$\lambda(I) - \lambda(I \cap A_k) \geq \lambda(I) - \lambda(I \cap A_j) - \sum_{i=j+1}^{k} \lambda([a_i, a_i + \epsilon_i]) \geq \epsilon_j - \sum_{i=j+1}^{k} \epsilon_i > \epsilon_j - \sum_{i>j} \epsilon_i$$

Now using Theorem 2.29,

$$\lambda(A \cap I) = \lambda\left(\bigcap_{n \in \mathbb{N}} \bigcup_{m \geq n} A_m \cap I\right) = \inf_{n \in \mathbb{N}} \sup_{m \geq n} \lambda(A_m \cap I) = \limsup_{n \to \infty} \lambda(A_m \cap I)$$

So from $\lambda(I \cap A_k) > \epsilon_j - \sum_{i>j} \epsilon_i$ and $\lambda(I) - \lambda(I \cap A_k) \geq \epsilon_j - \sum_{i>j} \epsilon_i$ we get $\lambda(I \cap A) \geq \epsilon_j - \sum_{i>j} \epsilon_i > 0$ and $\lambda(I) - \lambda(I \cap A) \geq \epsilon_j - \sum_{i>j} \epsilon_i > 0$ by taking limits. $\square$

[21] One such sequence is given by $\epsilon_j := 3^{-j} \ell_1 \cdots \ell_j$.

[22] For intuition about what this means, see Problem 9.

Problem 7

Give an example of an open set $U \subseteq \mathbb{R}$ such that $\lambda(\partial\overline{U}) > 0$.

Solution. Let C be a closed subset of $\mathbb{R}$ of positive Lebesgue measure having *empty interior*.[23] Define for each positive integer m

$$V_m := \left\{ x \in \mathbb{R} \;\middle|\; \frac{1}{m+1} < d(x, C) < \frac{1}{m} \right\}$$

Note that each V_m is open because the distance function $x \mapsto d(x, C)$ is a continuous function.[24] Define

$$U := \bigcup_{m \text{ even}} V_m$$

We will show that $C \subseteq \partial\overline{U}$, from which the claim follows immediately. To see that $C \subset \overline{U}$, let $x \in C$ be arbitrary and let $J \ni x$ be an open interval containing x. Since C has empty interior, we can find $y \in C^{\mathsf{c}} \cap J$. We have $d(x, C) = 0$ and $d(y, C) > 0$. By the continuity of $z \mapsto d(z, C)$, this function attains every value in $[0, d(z, C)]$ on the interval J. In particular, it attains a value between $1/m$ and $1/(m+1)$ for sufficiently large even m. It follows that $J \cap U \neq \varnothing$. Since J was arbitrary we get $x \in \overline{U}$. Since $x \in C$ was arbitrary, we get $C \subset \overline{U}$.

To see that $C \subset \overline{U^{\mathsf{c}}}$, note that by symmetry, if we define $U' := \bigcup_{m \text{ odd}} V_m$ then also $C \subset \overline{U'}$. But clearly $U' \subseteq U^{\mathsf{c}}$. Thus we conclude. $\square$

[23] A fat Cantor set would be one such example. For another, let $\Omega \supset \mathbb{Q}$ be an open set of finite measure and consider the closed set Ω^{c}.

[24] In fact, it is uniformly continuous: $|d(x, C) - d(y, C)| \leq d(x, y)$.

Problem 8

Let A be the set of real numbers in the interval $[0, 1]$ whose decimal expansions do not contain the digit 4. Then $\lambda(A) = 0$. (In particular, A is Lebesgue-measurable.)

Solution. First note the following:

Claim (1)**.** *Let n be a positive integer. Then the number of* integers *in the interval* $[0, 10^n)$ *whose decimal expansion does not contain the digit 4 is exactly* 9^n.

The integers in the interval $[0, 10^n)$ are in bijection with the sequences of length n in the alphabet $\{0, 1, \dots, 9\}$. Those avoiding the digit 4 are in bijection with the sequences in the sub-alphabet $\{0, 1, 2, 3, 5, 6, 7, 8, 9\}$. Hence they number

$$|\{0, 1, 2, 3, 5, 6, 7, 8, 9\}^n| = |\{0, 1, 2, 3, 5, 6, 7, 8, 9\}|^n = 9^n$$

Claim (2)**.** *The set A_n of real numbers in the interval* $[0, 1)$ *whose first n decimal digits are not the digit 4 has Lebesgue measure precisely* $(9/10)^n$.

To see this, note that $x \in [0, 1)$ satisfies the property in the claim if and only if $\lfloor 10^n x \rfloor$ does not contain the digit 4. Hence

$$A_n = \bigsqcup_{\substack{m \in \mathbb{Z} \cap [0,10^n) \\ m \text{ has no digit 4}}} \left[\frac{m}{10^n}, \frac{m+1}{10^n} \right)$$

Using Claim (1), A_n is the union of 9^n pairwise disjoint intervals of length 10^{-n}. Claim (2) follows.

Now observe that $A = \{1\} \cup \bigcap_{n \in \mathbb{N}} A_n$.[25] Also, clearly $A_1 \supset A_2 \supset A_3 \supset \dots$. Hence by Theorem 2.29(4),

$$\lambda(A) = \lim_{n \to \infty} \lambda(A_n) = \lim_{n \to \infty} \left(\frac{9}{10} \right)^n = 0$$

□

[25] Note that this means A is Borel, not just Lebesgue-measurable.

Problem 9 (a) and (b)

Let $E_1, E_2, \ldots$ be a sequence of Lebesgue measurable subsets of $\mathbb{R}$ such that $\sum_{k=1}^{\infty} \lambda(E_k) < \infty$. We define

$$E := \{x \in \mathbb{R} \mid x \in E_k \text{ for infinitely many } k\}$$

Show (a) that E is measurable and (b) that $\lambda(E) = 0$.

Solution. For $x \in \mathbb{R}$, have the following sequence of equivalences:

$$\begin{aligned} x \in E_k \text{ for infinitely many } k &\iff \text{the set } \{k \in \mathbb{N} \mid x \in E_k\} \text{ is infinite} \\ &\iff \text{the set } \{k \in \mathbb{N} \mid x \in E_k\} \text{ is not bounded above} \\ &\iff \text{for any } n \in \mathbb{N}, \text{ there exists } k \geq n \text{ such that } x \in E_k \\ &\iff \text{for any } n \in \mathbb{N}, x \in \bigcup_{k \geq n} E_k \\ &\iff x \in \bigcap_{n \in \mathbb{N}} \bigcup_{k \geq n} E_k \end{aligned}$$

Hence $E = \bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} E_k$. Thus E is in the σ-algebra generated by the E_i. In particular, E is contained in the σ-algebra of λ-measurable sets. This proves (a). As for (b), note that for each n, $E \subseteq \bigcap_{k\geq n} E_k$. Hence by subadditivity

$$\lambda(E) \leq \lambda\left(\bigcap_{k\geq n} E_k\right) \leq \sum_{k\geq n} \lambda(E_k)$$

Since the series $\sum_{k=1}^{\infty} \lambda(E_k)$ converges, the "remainders" $\sum_{k\geq n}^{\infty} \lambda(E_k)$ tend to 0 as n grows. Taking the limit, we get $\lambda(E) \leq 0$. □

Problem 9(c)

Let $f_1, f_2, \cdots : [0,1] \to \mathbb{C}$ be a sequence of measurable functions. Then there exists a sequence of constants $c_1, c_2, \cdots \in (0, \infty)$ such that

$$\lim_{n\to\infty} \frac{f_n(x)}{c_n} = 0$$

for almost all $x \in [0,1]$, i.e. for all $x \in [0,1]$ excluding some set of measure 0.

Solution. After replacing f_n with $|f_n|$ we may suppose without loss of generality that each f_n has image lying in $[0, \infty)$. For each n, choose $M_n > 0$ such that $\lambda(\{x \in [0,1] \mid f_n(x) > M_n\}) < 2^{-n}$.[26] Let $E_n := \{x \in [0,1] \mid f_n(x) > M_n\}$ and let $c_n = nM_n$. Then $\lambda(E_n) < 2^{-n}$ implies that $\sum_{n=1}^{\infty} \lambda(E_n)$ converges. By the result above, there is a set E of measure 0 such that $x \notin E$ implies that

[26] This is made possible by applying Theorem 2.29(4) to $\bigcap_{M\in\mathbb{N}}\{x \in [0,1] \mid f_n(x) > M\} = \varnothing$.

x is in at most finitely many E_n's. For $x \in [0,1] - E$ and for large enough n (depending on x)

$$0 \leq \frac{f_n(x)}{c_n} \leq \frac{M_n}{c_n} \leq \frac{1}{n}$$

and the result follows. □

Problem 10

Suppose $f : \mathbb{R} \to \mathbb{C}$ is a measurable function.[27] Then there exist *continuous* functions $f_1, f_2, \dots : \mathbb{R} \to \mathbb{C}$ and a set E of measure 0 such that

$$\lim_{n\to\infty} f_n(t) = f(t)$$

for all $t \in E^{\mathsf{c}}$. (To describe this, we say that $f_n \to f$ *pointwise almost-everywhere.*)

Solution. We will say that a function $f : \mathbb{R} \to \mathbb{C}$ is *good* if it satisfies the property stated above. We will say that a *set* is good if its characteristic function is good. We proceed in six stages of increasing goodness.

Claim (1)**.** *Let $a \in \mathbb{R}$ be arbitrary. Then the interval (a, ∞) is good.*

Here one can be very explicit. We can simply define

$$f_n(t) := \begin{cases} 0 & \text{if } t \le a \\ n(t-a) & \text{if } a \le t \le a + 1/n \\ 1 & \text{if } t \ge a + 1/n \end{cases}$$

Then it is easy to see that each f_n is continuous and that $\lim_{n\to\infty} f_n(t) = \chi_{(a,\infty)}(t)$.

Lemma 3. *The collection of good functions is closed under finite sums, finite products and multiplication by complex scalars.*

Proof. If $f_1, f_2, \dots$ and $g_1, g_2, \dots$ are continuous and $f_n \longrightarrow f$ pointwise outside of a set E_1 of measure 0 and $g_n \longrightarrow g$ pointwise outside of a set E_2 of measure 0 then $h_n := f_n + g_n$ is continuous and $h_n \longrightarrow f + g$ outside of $E_1 \cup E_2$, which has measure 0. This proves the case of sums. The other cases are treated analogously. □

Claim (2)**.** *Let $\mathcal{A}$ be the algebra in defined in Claim 4.4 and used in Problem 5. Then every set in $\mathcal{A}$ is good.*

For any real numbers $a < b$, the identity $\chi_{(a,b]} = \chi_{(a,\infty)} - \chi_{(b,\infty)}$ shows that $\chi_{(a,b]}$ is good. Any set in $\mathcal{A}$ is a finite *disjoint* union of intervals of the form $(a, b]$. So the claim follows by taking sums and using Lemma 3.

Lemma 4. *Suppose that $f_1, f_2, \dots$ is a sequence of good functions converging pointwise to a function f. Then f is good.*

[27] One could also allow f to take the value ∞ on a set of measure 0. However, to recover this version, one can simply change the values on that set from ∞ to 0 (or anything else) without affecting measurability or the validity of the problem statement.

Proof. First note that if g is a good function then for any $\epsilon_1, \epsilon_2, M > 0$ one can find a continuous function h such that $\lambda(\{x \in [-M, M] \mid |g(x) - h(x)| > \epsilon_1\}) < \epsilon_2$. To see that this is so, consider a sequence of continuous functions $g_1, g_2, \ldots$ such that $g_n \longrightarrow g$ as $n \longrightarrow \infty$ pointwise outside a set E of measure 0. Then by assumption[28]

$$\bigcap_{n\in\mathbb{N}} \bigcup_{m\geq n} \{x \in [-M, M] \bigm| |g(x) - g_m(x)| > \epsilon_1\} \subseteq E$$

By Theorem 2.29(4), we can find $n \in \mathbb{N}$ such that

$$\lambda\left(\bigcup_{m\geq n} \{x \in [-M, M] \bigm| |g(x) - g_m(x)| > \epsilon_1\}\right) < \epsilon_2$$

In particular,

$$\lambda(\{x \in [-M, M] \bigm| |g(x) - g_n(x)| > \epsilon_1\}) < \epsilon_2$$

So we may let $h = g_n$.

Returning to the main claim, we now know that we can find continuous functions $h_1, h_2, \ldots$ such that for each $n \in \mathbb{N}$ the set

$$E_n := \left\{x \in [-n, n] \,\middle|\, |f_n(x) - h_n(x)| > \frac{1}{n}\right\}$$

satisfies $\lambda(E_n) < 2^{-n}$. Now applying the Borel-Cantelli Lemma (i.e. Problem 9 (b)), we get that almost all $x \in \mathbb{R}$ belong to only finitely many of the sets E_n. For such an x, we have $|f_n(x) - h_n(x)| < 1/n$ for all sufficiently large n. It follows that

$$\lim_{n\to\infty} h_n(x) = \lim_{n\to\infty} f_n(x) = f(x)$$

□

Claim (3)**.** *The collection $\mathcal{G}$ of good sets $S \subseteq \mathbb{R}$ forms a σ-algebra.*

To prove this, note that $\chi_\varnothing = 0$ and $\chi_\mathbb{R} = 1$ are themselves continuous, so $\varnothing, \mathbb{R} \in \mathcal{G}$. Next, if $S_1, S_2 \in \mathcal{G}$ then

$$\chi_{S_1\cap S_2} = \chi_{S_1} \cdot \chi_{S_2}, \quad \chi_{S_1^c} = 1 - \chi_{S_1}$$

It follows from this and Lemma 3 that $\mathcal{G}$ forms an algebra. It remains to show that $\mathcal{G}$ is closed under taking countable increasing unions. Suppose $S_1 \subseteq S_2 \subseteq S_3 \subseteq \ldots$ is an increasing sequence of elements of $\mathcal{G}$ and let $S = \bigcup_{n\in\mathbb{N}} S_n$. Then clearly

$$\chi_S(t) = \lim_{n\to\infty} \chi_{S_n}(t) \quad \text{for all } t \in \mathbb{R}$$

So $S \in \mathcal{G}$ by Lemma 4.

[28] To unpack the meaning of the intersection-union below, the reader may wish to compare this situation with the one in Problem 9.

Claim (4)**.** *Every Lebesgue measurable set is good.*

From the previous two claims it follows that every Borel set is good. Moreover, for any set S of Lebesgue measure 0, setting $f_n := 0$ we get $f_n \longrightarrow \chi_S$ almost everywhere. So measure-zero sets are in $\mathcal{G}$. The rest now follows from Claim (3).[29]

Claim (5)**.** *Let $f : \mathbb{R} \to [0, \infty)$ be a measurable function. Then f is good.*

By Theorem 2.27, f is the pointwise limit of simple functions $s_1 \leq s_2 \leq \ldots$. Each simple function is good by Lemma 3 applied to Claim (4). It follows that f is good by Lemma 4.

Claim (6)**.** *Let $f : \mathbb{R} \to \mathbb{C}$ be a measurable function. Then f is good.*

This is immediate from Lemma 3 applied to Claim (5) and the fact that any function $f : \mathbb{R} \to \mathbb{C}$ can be written in the form

$$f = f_0 + if_1 - f_2 - if_3$$

where f_0, f_1, f_2, f_3 are functions taking values in $[0, \infty)$.[30] □

[29] It is a good exercise to show that every Lebesgue-measurable set differs from a Borel set by a set of measure 0.

[30] One sets $f_m(t) := \max(0, \Re(i^{-m} f(t)))$ where $\Re : \mathbb{C} \to \mathbb{R}$ is the *real part* function.

Problem 11

Show there exist closed sets $A, B \subseteq \mathbb{R}$ such that $\lambda(A) = \lambda(B) = 0$ but $\lambda(A + B) > 0$.

Solution 1. We will prove the beautiful result that

$$C + C = [0, 2]$$

where C is the Cantor set. It will be more convenient to work with $C' := \frac{1}{2}C$. We need to show that $C' + C' = [0, 1]$. Note that C is the set of all real numbers in $[0, 1]$ with a ternary expansion of only 0's and 2's. It follows that $C' = \frac{1}{2}C$ is the set of all real numbers in $[0, 1)$ with a ternary expansion of only 0's and 1's.[31]

Any real number $x \in [0, 1]$ can be written the form

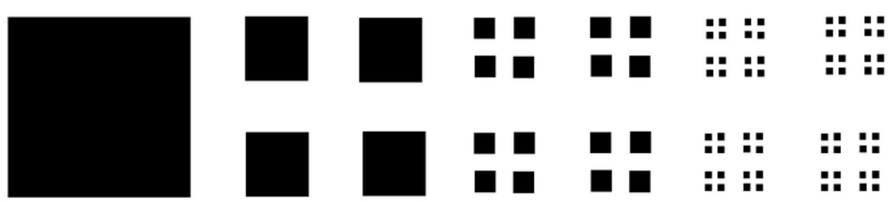
$$x = \sum_{k=1}^{\infty} \frac{a_k}{3^k}$$

where $a_k \in \{0, 1, 2\}$. Define $b_k := \min(a_k, 1)$ and $c_k := a_k - b_k$.[32] Then $b_k, c_k \in \{0, 1\}$ and $b_k + c_k = a_k$. Define

$$y := \sum_{k=1}^{\infty} \frac{b_k}{3^k}, \qquad z := \sum_{k=1}^{\infty} \frac{c_k}{3^k}$$

Then $y, z \in C'$ and $y + z = x$. □

Solution 2. For a visual proof, the reader should ponder the projections onto the antidiagonal of the figures below. □



[31] A priori, it should be the numbers in $[0, 1/2]$. However, every number in $(1/2, 1)$ contains a 2 in its ternary expansion.

[32] That is, $b_k = a_k, c_k = 0$ if $a_k \in \{0, 1\}$ and $b_k = c_k = 1$ if $a_k = 2$.

Problem 12

The image of a Lebesgue measurable set under a continuous function need not be Lebesgue measurable.

Lemma 5. *There exists a continuous function* $c : [0,1] \to [0,1]$ *which is surjective when restricted to the Cantor set, i.e. such that each* $x \in [0,1]$ *is of the form* $c(y)$ *for some* y *in the Cantor set.*

Proof. The Cantor set is the set of all real numbers which admit a ternary expansion containing only 0's and 2's. Even though a real number can have up to *two* ternary expansions, it is easy to see that at most one of these has no 1's. We define the Cantor *function* $c : C \to [0,1]$ by sending $x \in C$ to the unique real number admitting a binary expansion that coincides with a ternary expansion of $\frac{1}{2}x$. That is, if $x = 0.d_1d_2d_3\ldots$ is a ternary expansion of $\frac{1}{2}$ with $d_i \in \{0,2\}$ for each i. Then $c(x)$ is the unique real number represented in base 2 by $0.d_1'd_2'd_3'\ldots$ where $d_i' = d_i/2$ for each i. It is clear that c is *increasing* in the sense that $c(x) \leq c(y)$ whenever $x, y \in C$ and $x \leq y$. It should also be clear that c is *surjective* onto $[0,1]$.[33]

For any $x \in [0,1] - C$, we can write the ternary expansion of x as $x = 0.d_1d_2d_3\ldots$. Let j be the first index at which $d_j = 1$. Let x_- be the number represented in ternary by $0.d_1d_2\ldots d_{j-1}0222222\ldots$ and let x_+ be the number represented by $0.d_1d_2\ldots d_{j-1}20000000\ldots$. Then $x_-, x+ \in C$ and $C \cap (x_-, x_+) \cap C = \varnothing$.[34] It is not hard to see that $c(x_-) = c(x_+)$. We extend c to a function $[0,1] \to [0,1]$ by defining $c(x) = c(x_-) = c(x_+)$ in every such case. It is clear that by doing so we maintain the property that c is increasing and surjective. A surjective monotone function between intervals of $\mathbb{R}$ *must* be continuous, since increasing functions can only have *jump* discontinuities. Thus c is continuous. □

To prove the statement in the problem, we of course need to know that non-measurable subsets of $\mathbb{R}$ exist at all![35] Here we will take this for granted.

Let $X \subset [0,1]$ be a non-measurable set, let C be the Cantor set, let $c : [0,1] \to [0,1]$ be the Cantor function in the lemma above, and let $Y = c^{-1}(X) \cap C$. Since C has measure 0 and $Y \subset C$, the completeness of λ means that Y is measurable. On the other hand, since c is surjective when restricted to C, we have $c(Y) = X$.[36]

[33] A real number in $[0,1]$ represented in binary by the expansion $0.d_1d_2\ldots$ is the image of the ternary number $0.\tilde{d}_1\tilde{d}_2\ldots$ defined by $\tilde{d}_n := 2d_n$, which lies in the Cantor set.

[34] That is, x_+ and x_- the *endpoints* of the Cantor set nearest to x.

[35] For a proof of this fact, which requires the axiom of choice, see `https://e.math.cornell.edu/people/belk/measuretheory/NonMeasurableSets.pdf`.

[36] We remark that Y is also a Lebesgue-measurable set which is *not* Borel. The curious reader may have a go at proving this.

Problem 13

For any continuous function $f : \mathbb{R} \to \mathbb{R}$, letting

$$\Delta(f, \chi_{[0,1]}) := \{x \in \mathbb{R} \mid f(x) \neq \chi_{[0,1]}(x)\}$$

we get $\lambda(\Delta(f, \chi_{[0,1]})) > 0$.

Proof. It will suffice to show that $\Delta(f, \chi_{[0,1]})$ has *non-empty interior*.

Choose $x_0 \in (-1, 0)$, $x_1 \in [0, 1]$ such that $f(x_0) = \chi_{[0,1]}(x_0)$ and $f(x_1) = \chi_{[0,1]}(x_1)$. If either of these is impossible to find then either $(-1, 0)$ or $[0, 1]$ is contained in $\Delta(f, \chi_{[0,1]})$, giving the result immediately.

Since $f(x_0) = 0 < 1 = f(x_1)$ we can, by the intermediate value theorem, find $x_{1/2} \in [x_0, x_1]$ such that $f(x_{1/2}) = 1/2$. By continuity, there is some $\delta > 0$ such that $|f(y) - 1/2| < 1/2$ if $|y - x_{1/2}| < \delta$. It follows that

$$(x_{1/2} - \delta, x_{1/2} + \delta) \subseteq \Delta(f, \chi_{[0,1]})$$

□

Problem 14

Suppose $A \subseteq E \subseteq B \subseteq \mathbb{R}$ with A, B Lebesgue measurable and $\lambda(A) = \lambda(B)$. Then E is Lebesgue measurable.

Solution. Observe that by additivity of λ

$$\lambda(B - A) = \lambda(B) - \lambda(A) = 0$$

Since

$$E - A \subseteq B - A$$

and the Lebesgue measure is *complete*, it follows that $E - A$ is λ-measurable and thus so is $E = (E - A) \cup A$. $\square$

Problem 15

Let X be a metric space. For $S \subseteq X$, $\delta > 0$ and $d \in [0, \infty)$[37] define

$$H^d_\delta(S) := \inf\left(\left\{\sum_{i=1}^{\infty} (\operatorname{diam} U_i)^d \;\middle|\; U_1, U_2, \cdots \subseteq X \text{ s.t. } S \subseteq \bigcup_{i=1}^{\infty} U_i \,\wedge\, \operatorname{diam} U_i < \delta \text{ for all } i\right\}\right)$$

Note $H^d_\delta(S) \in [0, \infty]$. *Exceptionally*, we treat $(\operatorname{diam} U_i)^d$ as 0 if $\operatorname{diam} U_i = d = 0$.

Claim (a). *For fixed S and d, the function $\delta \mapsto H^d_\delta(S)$ is monotone* decreasing.

If $\delta_1 < \delta_2$ then $H^d_{\delta_1}(S)$ is the infimum over a *subset* of the coverings in the infimum defining $H^d_{\delta_2}(S)$. So $H^d_{\delta_1}(S) \geq H^d_{\delta_1}(S)$.

Claim (b). *By part (a) and the fact that $H^d_\delta(S) \geq 0$, we know that*

$$H^d(S) := \lim_{\delta \to 0^+} H^d_\delta(S)$$

exists (in $[0, \infty]$) for any admissible S and d. For fixed d, the map $S \mapsto H^d(S)$ is an outer measure on X.

For fixed $\delta > 0$, let $\rho_\delta : \mathscr{P}(X) \to [0, \infty]$ by defined by $\rho_\delta(S) := (\operatorname{diam} S)^d$. Then by definition, $H^d_\delta(S) = \varphi_{\rho_\delta}$ in the sense of Proposition 2.65. It follows that $H^d_\delta(S)$ is an outer measure for fixed $\delta > 0$. By taking limits, criteria (1) and (2) in Definition 2.63 for H^d follow immediately from the corresponding properties of H^d_δ. As for countable subadditvity, suppose $S_1, S_2, \cdots \subseteq X$. Then

$$H^d\left(\bigcup_{i=1}^{\infty} S_i\right) = \lim_{\delta \to 0^+} H^d_\delta\left(\bigcup_{i=1}^{\infty} S_i\right) \leq \lim_{\delta \to 0^+} \sum_{i=1}^{\infty} H^d_\delta(S_i) = \sum_{i=1}^{\infty} \lim_{\delta \to 0^+} H^d_\delta(S_i) = \sum_{i=1}^{\infty} H^d(S_i)$$

The exchange of the infinite sum and the limit is justified by Lebesgue monotone convergence theorem for sums,[38] since $H^d_\delta(S)$ *increases* as δ decreases to 0.

Claim (c). *Let $\mathscr{B}(X)$ be the collection of Borel sets of X, i.e. the σ-algebra generated by the open subset of X. Then all sets in $\mathscr{B}(X)$ are H^d-measurable.*

To prove this, we use the following lemma:

Lemma 6. *Suppose $A, B \subseteq X$ satisfy $d(A, B) > 0$. Then*

$$H^d_\delta(A) + H^d_\delta(B) = H^d_\delta(A \sqcup B)$$

for all sufficiently small $\delta > 0$. It follows that $H^d(A) + H^d(B) = H^d(A \sqcup B)$.

[37] Note that for $d < 0$, the values are always infinite and the problem trivializes.

[38] This is Theorem 2.46 in the special case when $X = \mathbb{N}$ and μ is the counting measure on X.

Proof. Suppose $0 < \delta < d(A, B)$. Suppose $U_1, U_2, \dots \subseteq X$ form a cover of $A \cup B$ such that $\operatorname{diam} U_i < \delta$ for all $i \in \mathbb{N}$. If for some i we could find $a \in U_i \cap A$ and $b \in U_i \cap B$ then

$$\delta < d(A, B) \leq d(a, b) \leq \operatorname{diam} U_i < \delta$$

A contradiction. The sets $J_A := \{i \in \mathbb{N} \mid U_i \cap A \neq \varnothing\}$ and $J_B := \{i \in \mathbb{N} \mid U_i \cap B \neq \varnothing\}$ must therefore be disjoint. Since $\{U_i \mid i \in J_A\}$ and $\{U_i \mid i \in J_B\}$ form covers of A and B respectively, it follows that

$$\sum_{i=1}^{\infty} (\operatorname{diam} U_i)^d \geq \sum_{i \in J_A} (\operatorname{diam} U_i)^d + \sum_{i \in J_B} (\operatorname{diam} U_i)^d \geq H^d_\delta(A) + H^d_\delta(B)$$

Taking the infimum over all covers $U_1, U_2, \dots$ with $\operatorname{diam} U_i < \delta$ we get $H^d_\delta(A \cup B) \geq H^d_\delta(A) + H^d_\delta(B)$. The reverse inequality follows by subadditivity of H^d_δ. $\square$

As a corollary of this lemma, we get

Corollary. *Let $A_1 \subsetneq A_2 \subsetneq A_3 \subsetneq \dots$ be subsets of X with $A := \bigcup_{i \in \mathbb{N}} A_i$. Suppose that $d(A_i, A - A_{i+1}) > 0$ for each $i \in \mathbb{N}$. Then $H^d(A) = \sup_{i \in \mathbb{N}} H^d(A_i)$.*

Proof. The inequality $H^d(A) \geq \sup_{i \in \mathbb{N}} H^d(A_i)$ follows from the fact that H^d is an outer measure. To get the reverse inequality, we consider the infinite series $S := \sum_{i=1}^{\infty} H^d(A_{i+1} - A_i)$. On the one hand, from $A = \bigcup_{i=1}^{\infty} (A_{i+1} - A_i)$ we get $H^d(A) \leq S$ by countable subadditivity. On the other hand, we can use Lemma 6 inductively on the sets $A_2 - A_1, A_4 - A_3, \dots$ to get (for $n \in \mathbb{N}$)

$$\sum_{\substack{i=1 \\ i \text{ odd}}}^{n-1} H^d(A_{i+1} - A_i) = H^d \left(\bigcup_{\substack{i=1 \\ i \text{ odd}}}^{n-1} (A_{i+1} - A_i) \right) \leq H^d \left(\bigcup_{i=1}^{n-1} (A_{i+1} - A_i) \right) = H^d(A_n)$$

Taking the supremum over all $n \in \mathbb{N}$, we get

$$\sum_{\substack{i=1 \\ i \text{ odd}}}^{\infty} H^d(A_{i+1} - A_i) \leq \sup_n H^d(A_n)$$

By symmetry, we have a similar inequality for even i. We therefore get

$$S = \sum_{i=1}^{\infty} H^d(A_{i+1} - A_i) \leq \sum_{\substack{i=1 \\ i \text{ odd}}}^{\infty} H^d(A_{i+1} - A_i) + \sum_{\substack{i=1 \\ i \text{ even}}}^{\infty} H^d(A_{i+1} - A_i) \leq 2 \cdot \sup_n H^d(A_n)$$

In summary, we have

$$\frac{1}{2} S \leq \sup_n H^d(A_n) \leq H^d(A) \leq S$$

It follows that if one of these quantities of interest is infinite then they all are, in which case the desired inequality is trivial. We may therefore assume that all these quantities are finite. Then by countable subadditivity we have (for $n \in \mathbb{N}$)

$$H^d(A) = H^d\left(A_n \cup \bigcup_{i=n}^{\infty}(A_{i+1} - A_i)\right) \le H^d(A_n) + \sum_{i=n}^{\infty} H^d(A_{i+1} - A_i)$$

Taking the limit as $n \longrightarrow \infty$ we get

$$H^d(A) \le \lim_n H^d(A_n) + \lim_{n\to\infty} \sum_{i=n}^{\infty} H^d(A_{i+1} - A_i) = \sup_n H^d(A_n) + 0$$

where equality on the right follows from the convergence of S. $\square$

To deduce Claim (c), note that by Theorem 2.69, it suffices to show that all closed sets are H^d-measurable. For this purpose, let $C \subseteq X$ be closed and $A \subseteq X$ arbitrary. We will show that $H^d(A) \ge H^d(A \cap C) + H^d(A \cap C^{\mathsf{c}})$.

For each $n \in \mathbb{N}$, let $C_n := \{x \in X \mid d(x, C) \le \frac{1}{n}\}$. Then each C_n is closed and

$$\overline{C_n^{\mathsf{c}}} \cap \overline{C_{n+1}} \subseteq \left\{x \in X \;\middle|\; d(x, C) \ge \frac{1}{n}\right\} \cap \left\{x \in X \;\middle|\; d(x, C) \le \frac{1}{n+1}\right\} = \varnothing$$

Note that $\bigcap_{n\in\mathbb{N}} C_n = \overline{C} = C$. Letting $A_n := A - C_n$ and $A' := A \cap C^{\mathsf{c}}$, we get $A' = \bigcup_{n\in\mathbb{N}} A_n$. Also $A_n \subseteq C_n^{\mathsf{c}}$ and $A' - A_{n+1} \subseteq C_{n+1}$ imply

$$d(A_n, A' - A_{n+1}) \ge d(C_n^{\mathsf{c}}, C_{n+1}) > 0$$

So we may apply the corollary above to get $H^d(A') = \sup_n H^d(A_n)$. Let now $A'' := A \cap C$. Observe that for fixed $n \in \mathbb{N}$, we have $\overline{A_n} \cap \overline{A''} \subseteq C_n^{\mathsf{c}} \cap C = \varnothing$. So $d(A_n, A'') > 0$ and we may apply Lemma 6 to get

$$H^d(A_n) + H^d(A'') = H^d(A_n \sqcup A'') \le H^d(A)$$

Taking the supremum over $n \in \mathbb{N}$ we get

$$H^d(A) \ge H^d(A'') + \sup_n H^d(A_n) = H^d(A'') + H^d(A') = H^d(A \cap C) + H^d(A \cap C^{\mathsf{c}})$$

Claim (d)**.** *For fixed S, the function $d \mapsto H^d(S)$ is monotone decreasing on $[0, \infty)$. Its image may contain 0 and/or ∞ and/or at most one finite number $m \in (0, \infty)$.*[39] *If the last case holds, then there is a* unique *$d_* \in [0, \infty)$ such that $H^{d_*}(S) = m$. Furthermore, $H^d(S) = \infty$ for $d < d_*$ and $H^d(S) = 0$ for $d > d_*$.*

For δ restricted to the interval $(0, 1)$ and $0 \le d_1 < d_2$, it is clear that $H_\delta^{d_1}(S) \ge H_\delta^{d_2}(S)$ since we are taking sums over the same values of $\operatorname{diam} U_i \in [0, 1)$ but with exponent d_1 in one case and d_2 in the other. By taking the limit as $\delta \longrightarrow 0^+$, we get $H^{d_1}(S) \le H^{d_2}(S)$.

[39] All possible images are described schematically as: $\{0\}$, $\{\infty\}$, $\{0, \infty\}$, $\{0, m\}$, $\{0, \infty, m\}$.

To prove the uniqueness of d_* it suffices to show that if $H^d(S) < \infty$ for some $d \in [0, \infty)$ then $H^{d'}(S) = 0$ for all $d' > d$.[40] To see this, let $M := H^d(S)$ and let $\delta > 0$ be arbitrary. Then $H^d_\delta(S) \leq M$ implies that we can cover S by subsets $U_1, U_2, \cdots \subseteq X$ such that $\operatorname{diam} U_i < \delta$ for each $i \in \mathbb{N}$ and such that $\sum_{i=1}^\infty (\operatorname{diam} U_i)^d < M + 1$. Note that

$$(\operatorname{diam} U_i)^{d'} = (\operatorname{diam} U_i)^d (\operatorname{diam} U_i)^{d'-d} \leq (\operatorname{diam} U_i)^d \cdot \delta^{d'-d}$$

Hence

$$H^d_\delta(S) \leq \sum_{i=1}^\infty (\operatorname{diam} U_i)^d \leq \delta^{d'-d}(M+1)$$

Since δ was arbitrary, we get

$$H^d(S) = \lim_{\delta \to 0^+} H^d_\delta(S) \leq \lim_{\delta \to 0^+} \delta^{d'-d}(M+1) = 0$$

Claim (e)**.** *Suppose $d \in \mathbb{N}$ and $X = \mathbb{R}^d$. Then for $\lambda : \mathcal{B}(\mathbb{R}^d) \to [0, \infty]$ the d-dimensional Lebesgue measure we have*

$$\lambda(S) = \beta_d H^d(S)$$

for all $S \in \mathcal{B}(\mathbb{R}^d)$, where β_d is the volume of the d-ball of radius $1/2$.[41]

Proof. To show $\lambda \leq H^d$, we need the following result:

Theorem (Isodiametric inequality)**.** *For any subset $X \subset \mathbb{R}^d$, we have*

$$\lambda(X) \leq \beta_d (\operatorname{diam} X)^d$$

In other words, for a fixed diameter the ball achieves the maximum volume.[42][43]

Now let $S \subseteq \mathbb{R}^d$ be arbitrary and suppose S is covered by sets $U_1, U_2, \ldots$ of diameter $< \delta$. Then by the theorem

$$\beta_d \sum_{i=1}^\infty (\operatorname{diam} U_i)^d \geq \sum_{i=1}^\infty \lambda(U_i) \geq \lambda\left(\bigcup_{i=1}^\infty U_i\right) \geq \lambda(S)$$

It follows that $H^d_\delta(B^d) \geq \lambda(S)$ for all $\delta > 0$. Hence $H^d(S) \geq \lambda(S)$.

To get the opposite direction, we will need some preliminary work.

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[40] Those of you who have taken complex analysis may find it useful to compare this result with a similar result about the convergence of power series in the complex plane.

[41] For explicit formulae giving the volume of such a ball for arbitrary d, see `https://en.wikipedia.org/wiki/Volume_of_an_n-ball`.

[42] For a proof of this classical result, see Theorem 2.4 in *Measure Theory and Fine Properties of Functions* by Evans and Gariepy.

[43] Though the proof of the isodiametric inequality is not trivial, it is trivial to show that $\lambda(X) \leq \beta_d(2 \cdot \operatorname{diam} X)^d$. This much weaker inequality is still sufficient to recover the fact that λ and H^d are equal up to *some* scalar.

Lemma 7. *Fix a positive integer d. Then there exists a constant $c_d > 0$ such that each non-empty open set $U \subset \mathbb{R}^d$ of finite Lebesgue measure contains finitely many pairwise disjoint open balls $B_1, \ldots, B_m$ for which*

$$\lambda(B_1 \sqcup \cdots \sqcup B_m) \geq c_d \lambda(U)$$

Furthermore, at the risk of increasing m, the balls may be chosen to have radius $< \delta$ for any fixed $\delta > 0$.

Proof. We claim that $c_d := \beta_d/2$ will do. We have

$$\bigcap_{n \in \mathbb{N}} \left\{ x \in U \;\middle|\; d(x, U^{\mathsf{c}}) \leq \frac{1}{n} \right\} = U \cap \overline{U^{\mathsf{c}}} = \varnothing$$

Hence by Theorem 2.29(4), we can find $n \in \mathbb{N}$ such that

$$\lambda\left(\left\{ x \in U \;\middle|\; d(x, U^{\mathsf{c}}) \leq \frac{1}{n} \right\} \right) < \lambda(U)/2$$

Define $U' := \{x \in U \mid d(x, U^{\mathsf{c}}) > \frac{1}{2n}\}$ and $U'' := \{x \in U \mid d(x, U^{\mathsf{c}}) > \frac{1}{n}\}$. Note that the defining condition on n reads $\lambda(U'') > \lambda(U)/2$. Without loss of generality, we may also suppose $n > \delta^{-1}$. Let

$$I := U' \cap \left(\frac{1}{nd} \mathbb{Z}^n \right) = \{x \in U' \mid ndx \in \mathbb{Z}^n\}$$

For each $x \in I$, we let B_x be the open ball of diameter $\frac{1}{nd}$ centred at x. We also let C_x be the closed hypercube of sidelength $\frac{1}{nd}$ centred at x.[44] Then the balls B_x for $x \in I$ are pairwise-disjoint equally-sized open balls contained in U.[45] We claim that $U'' \subseteq \bigcup_{x \in I} C_x$. To see this, let $p \in U''$ be an arbitrary point. Let $q \in \frac{1}{nd} \mathbb{Z}^n$ have minimal distance to p. Then $d(p, q) \leq \frac{1}{nd} \cdot \frac{1}{2} \operatorname{diam}[0,1]^d = \frac{\sqrt{d}}{2nd} \leq \frac{1}{2n}$. Thus

$$d(q, U^{\mathsf{c}}) \geq d(p, U^{\mathsf{c}}) - d(p, q) > \frac{1}{n} - \frac{1}{2n} = \frac{1}{2n}$$

and so $q \in U'$, whence $q \in I$. So $p \in C_q$. Putting these things together, noting that $\lambda(B_x) = \beta_d \lambda(C_x)$, we get

$$\begin{aligned} \lambda\left(\bigsqcup_{x \in I} B_x \right) = \sum_{x \in I} \lambda(B_x) = \sum_{x \in I} \beta_d \lambda(C_x) &\geq \beta_d \lambda\left(\bigcup_{x \in I} C_x \right) \\ &\geq \beta_d \lambda(U'') > \beta_d \lambda(U)/2 = c_d \lambda(U) \end{aligned}$$

Since the balls B_x each have radius $\frac{1}{nd} < \delta^{-1}/d \leq \delta$, the last condition is also satisfied. ☐

[44] That is, $C_x := \{y \in \mathbb{R}^d \mid \|y - x\|_\infty \leq \frac{1}{2nd}\}$

[45] From this and the finiteness of $\lambda(U)$, it follows that I is finite.

From this lemma we deduce

Corollary 1. *Let $U \subseteq \mathbb{R}^d$ be an open set of finite Lebesgue measure and $\delta > 0$ arbitrary. Then there exists an (at most) countable collection of pairwise-disjoint open balls $B_1, B_2, \dots \subseteq U$ all of radius $< \delta$ such that $U - \bigcup_i B_i$ has measure 0.*

Proof. Define $U_0 := U$, $m_0 := 0$ and, having defined U_i and m_i, choose pairwise-disjoint open balls $B_{m_i+1}, B_{m_i+2}, \dots, B_{m_{i+1}} \subseteq U_i$ of radius $< \delta$ such that

$$\lambda(B_{m_i+1} \sqcup \cdots \sqcup B_{m_{i+1}}) \geq c_d \lambda(U_i)$$

and then define

$$U_{i+1} := U_i - \overline{(B_{m_i+1} \sqcup \cdots \sqcup B_{m_{i+1}})}$$

Then by construction the balls $B_1, B_2, \dots$ are pairwise-disjoint and for each $i \in \mathbb{N}$

$$\lambda(U - (B_1 \sqcup \cdots \sqcup B_{m_{i+1}})) \leq (1 - c_d)^i \lambda(U)$$

It follows from Theorem 2.29(4) that $\lambda(U - \bigsqcup_i B_i) = 0$. □

Lemma 8. *Suppose $S \subseteq \mathbb{R}^d$ satisfies $\lambda(S) = 0$. Then also $H^d(S) = 0$.*

Proof. Let $\delta > 0$ be arbitrary. The assumption on S implies[46] that for any $\epsilon > 0$ we can find closed hypercubes $C_1, C_2, \dots$ such that $S \subseteq \bigcup_{i=1}^{\infty} C_i$ and

$$\sum_{i=1}^{\infty} \lambda(C_i) < \epsilon$$

Let ℓ_i be the sidelength of C_i. Then $\lambda(C_i) = \ell_i^d$ and $\operatorname{diam} C_i = \sqrt{d}\ell_i$. If we choose ϵ small enough that $(\delta/\sqrt{d})^d > \epsilon$ then this will force (for each $i \in \mathbb{N}$)

$$\ell_i^d = \lambda(C_i) \leq \sum_{i=1}^{\infty} \lambda(C_i) < \epsilon < (\delta/\sqrt{d})^d$$

from which it follows that $\operatorname{diam} C_i < \delta$. Then

$$H_\delta^d(S) \leq \sum_{i=1}^{\infty} (\operatorname{diam} C_i)^d = \sum_{i=1}^{\infty} (\sqrt{d}\ell_i)^d = d^{d/2} \sum_{i=1}^{\infty} \lambda(C_i) < d^{d/2}\epsilon < \delta$$

It follows that $H^d(S) = 0$. □

Now let $U \subset \mathbb{R}^d$ be any open subset of finite measure. Fix $\delta > 0$. By Corollary 1 we can find pairwise-disjoint open balls $B_1, B_2, \dots$ of diameter $< \delta$ such that

$$\lambda\left(U - \bigcup_{i=1}^{\infty} B_i\right) = 0$$

[46] It is a formative exercise to go through the various definitions in the lecture notes to actually recover this fact.

Then using Lemma 8

$$\begin{aligned}
H_\delta^d(U) &\le H_\delta^d\left(\bigcup_{i=1}^\infty B_i\right) + H_\delta^d\left(U - \bigcup_{i=1}^\infty B_i\right)\\
&\le \sum_{i=1}^\infty (\operatorname{diam} B_i)^d + H^d\left(U - \bigcup_{i=1}^\infty B_i\right)\\
&= \beta_d^{-1}\sum_{i=1}^\infty \lambda(B_i) + 0\\
&= \beta_d^{-1}\lambda\left(\bigcup_{i=1}^\infty B_i\right) + \beta_d^{-1}\lambda\left(U - \bigcup_{i=1}^\infty B_i\right)\\
&= \beta_d^{-1}\lambda(U)
\end{aligned}$$

Taking the limit as $\delta \longrightarrow 0^+$, we get $\lambda(U) \ge \beta_d H^d(U)$. Combining this with the opposite inequality proved above, we get $\lambda(U) = \beta_d H^d(U)$.

The result that $\lambda = \beta_d H^d$ now follows from Proposition 3.11 since this result implies that both $\mu = \lambda$ and $\mu = \beta_d H^d$ are defined over *all* Borel sets by

$$\mu(S) = \inf_{\substack{U \supseteq S\\ U \text{ open}\\ \mu(U)<\infty}} \mu(U)$$

□