Complex Analysis with Applications
Princeton University MAT330
Practice Final Exam

May 5, 2023

Note: the following is a *practice* final exam in preparation of the final to be held on May 14th 2023 at 9am-noon in Fine 314. The *format* of the final should agree with what you're seeing here; the selection of topics and emphasis of various themes might be differently chosen. The general difficulty should be about the same (though this is of course subjective).

1 Relevant formulas

In the following items, I remind you of relevant formulas but *not* of their scope of validity, which I expect you to know.

- The Cauchy-Riemann equations

$$\begin{aligned} \partial_x f_R &= \partial_y f_I \\ \partial_x f_I &= -\partial_y f_R \,. \end{aligned}$$

- Cauchy's integral formula:

$$f^{(n)}(z_0) = \frac{n!}{2\pi \mathrm{i}} \oint_\Gamma \frac{f(z)}{(z-z_0)^{n+1}} \mathrm{d}z \qquad (n \in \mathbb{N}_{\geq 0}) \,.$$

- Taylor's theorem:

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(z_0)(z-z_0)^n \,.$$

- The residue formula:

$$\mathrm{residue}_{z_0}(f) = \lim_{z \to z_0} \frac{1}{(n-1)!} \partial_z^{n-1} (z-z_0)^n f(z) \,.$$

- The argument principle:

$$\mathrm{index}_D\left(f\right) = \frac{1}{2\pi\mathrm{i}}\oint_\Gamma \frac{f'\left(z\right)}{f\left(z\right)}\mathrm{d}z\,.$$

- The Krammers-Kronig relation:

$$\lim_{\varepsilon\to0^+}\frac{1}{x\pm\mathrm{i}\varepsilon} = \mp\mathrm{i}\pi\delta\left(x\right)+\mathscr{P}\left(\frac{1}{x}\right)\,.$$

- The Fourier series

$$\left(\mathscr{F}\psi\right)\left(n\right)\equiv\hat{\psi}\left(n\right) = \frac{1}{2\pi}\int_{\theta=0}^{2\pi}\mathrm{e}^{-\mathrm{i}n\theta}\psi\left(\theta\right)\mathrm{d}\theta$$

and

$$\left(\mathscr{F}^{-1}\hat{\psi}\right)\left(\theta\right) \quad = \quad \sum_{n\in\mathbb{Z}}\mathrm{e}^{\mathrm{i}n\theta}\hat{\psi}\left(n\right)\,.$$

- The Fourier transform

$$\left(\mathscr{F}f\right)\left(\xi\right)\equiv\hat{f}\left(\xi\right) = \int_{x\in\mathbb{R}}\mathrm{e}^{-2\pi\mathrm{i}\xi x}f\left(x\right)\mathrm{d}x\,.$$

- Laplace and steepest descent asymptotics

$$\int_{x\in\mathbb{R}}\mathrm{e}^{-\lambda f(x)}g\left(x\right)\mathrm{d}x \overset{\lambda\to\infty}{\sim} \sqrt{\frac{2\pi}{\lambda\tilde{f}''\left(z_\star\right)}}\mathrm{e}^{-\lambda\tilde{f}(z_\star)}\tilde{g}\left(z_\star\right)\,.$$

- Jordan's inequality:

$$\frac{2}{\pi}\alpha \quad \leq\sin\left(\alpha\right)\leq \quad \alpha \qquad \left(\alpha\in\left[0,\frac{\pi}{2}\right]\right)\,,$$

Kober's inequality

$$1-\frac{2}{\pi}\left|x\right|\leq\cos\left(x\right)\leq1-\frac{x^2}{\pi} \qquad \left(x\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\right)\,,$$

and the "big arc lemma":

$$\int_{\theta=0}^{\pi}\mathrm{e}^{-R\sin(\theta)}\mathrm{d}\theta \quad \lesssim \quad \frac{1}{R}\,.$$

2 Short questions [20 points]

In the following questions, *no* justification is necessary. Simply provide a short as possible *correct* response.

1. Provide an example of a meromorphic function with a pole of order 6 at some $z_0 \in \mathbb{C}$:

$$z \longmapsto \frac{1}{(z-z_0)^6}$$

2. Provide an example of a function with an essential singularity at $z_0 = \mathrm{i}$:

$$z \longmapsto \exp\left(\frac{1}{(z-i)}\right)$$

Example 7.37

3. Provide an example of an $\mathbb{R}^2$-differentiable function which is not $\mathbb{C}$-differentiable:

$$z \longmapsto |z|^2$$

Example 4.11

4. Provide an example of a connected but not simply-connected set:

$$\partial B_1(0) \equiv \mathbb{S}^1$$

Example 4.22

5. For a function $f : \mathbb{C} \to \mathbb{C}$, if $\partial_x f_R$ and $\partial_y f_R$ are provided, write an expression for f':

$$f' = \partial_x f_R - i\, \partial_y f_R$$

see eq-n (4.5)

3 Long questions [80 points]

In the following questions, you must justify your work and convince me that you not only know what the correct answer is, but also *why* it is so. You may freely invoke any result from the lecture notes, homework or various textbooks just so long as you properly cite and explain in what way you're invoking it.

6. Express
$$\sum_{n\in\mathbb{Z}} (z-n)^{-4}$$
in closed form using trigonometric functions.

In HW8Q2 we have seen:

$$f(z) := \underbrace{\sum_{n\in\mathbb{Z}} (z+n)^{-2}}_{\sum_{n\in\mathbb{Z}} (z-n)^{-2}} = \frac{\pi^2}{\sin(\pi z)^2} \qquad (z \in \mathbb{C}\setminus\mathbb{Z})$$

Note f is meromorphic and hence the series above converges uniformly in cpt. s/sets of $\mathbb{C}\setminus\mathbb{Z}$. Hence we may exchange differentiation w/ series to get:

$$f'(z) = \sum_{n\in\mathbb{Z}} (z-n)^{-3}(-2) = \frac{(-2)\pi^2}{\sin(\pi z)^3}\cos(\pi z)\pi$$

$$= \frac{-2\pi^3\cos(\pi z)}{\sin(\pi z)^3}$$

$$f''(z) = \sum_{n\in\mathbb{Z}} 6(z-n)^{-4} = \frac{+2\pi^3\sin(\pi z)\pi}{\sin(\pi z)^3} + \frac{6\pi^4\cos(\pi z)^2}{\sin(\pi z)^4}$$

$$\Longrightarrow \boxed{\sum_{n\in\mathbb{Z}} (z-n)^{-4} = \frac{\pi^4}{\sin(\pi z)^2}\left[\frac{1}{3} + \operatorname{ctg}(\pi z)^2\right]}$$

7. Let

$$U := \{ z \in B_1(0) \mid z \neq 0 \wedge 0 < \operatorname{Arg}(z) < \alpha \}$$

for some $\alpha \in (0, 2\pi)$. Find a conformal equivalence $c : U \to \mathbb{H}$ where $\mathbb{H}$ is the open upper half plane.

Define $c(z) := -z^{\pi/\alpha} - z^{-\pi/\alpha} \quad \forall \; z \in \mathcal{U}.$

This is the desired equivalence because:

① $z \mapsto z^{\pi/\alpha}$ is a conf. equiv.:

$$re^{i\theta} \mapsto \underbrace{r^{\frac{\pi}{\alpha}}}_{\substack{\cap \\ (0,1)}} e^{i\frac{\pi\theta}{\alpha}}$$

$$\frac{\pi\theta}{\alpha} \in \frac{\pi}{\alpha}(0,\alpha) = \pi(0,1) = (0,\pi)$$

Hence it is an equiv. $\mathcal{U} \to B_1(0) \cap \mathbb{H}$.

② $z \mapsto -z - \frac{1}{z}$ is an equivalence

$B_1(0) \cap \mathbb{H} \to \mathbb{H}$

(see HW8Q5 sample sol-ns).

8. Calculate

$$\int_{x\in\mathbb{R}} \frac{x^6}{(1+x^4)^2}\,dx\,.$$

$$f(z) := \frac{z^6}{(1+z^4)^2}$$

poles @ $z^4 = -1 \Rightarrow z \in \left\{ e^{i\frac{\pi}{4}}, e^{i\left(\frac{\pi}{4}+\frac{\pi}{2}\right)}, e^{i\left(\frac{\pi}{4}+\pi\right)}, e^{i\left(\frac{\pi}{4}+\frac{3\pi}{2}\right)} \right\}$

of order 2

$$f(z) = \frac{z^6}{\left[(z-\exp(i\pi/4))(z-\exp(i\frac{3\pi}{4}))(z-\exp(i\frac{5\pi}{4}))(z-\exp(i\frac{7\pi}{4}))\right]^2}$$

If we close the contour up w/ a semicircle (can do either up or down as $\frac{R^6 \cdot R}{(R^4-1)^2} \xrightarrow{R\to\infty} 0$) we find

$$\int_{x\in\mathbb{R}} f(x)dx = 2\pi i \left(\operatorname{res}_{e^{i\frac{\pi}{4}}}(f) + \operatorname{res}_{e^{i\frac{3\pi}{4}}}(f) \right)$$

The residues are:

$$\operatorname{res}_{e^{i\frac{\pi}{4}}}(f) = \lim_{z\to e^{i\frac{\pi}{4}}} \partial_z \frac{z^6}{\left[(z-\exp(i\frac{3\pi}{4}))(z-\exp(i\frac{5\pi}{4}))(z-\exp(i\frac{7\pi}{4}))\right]^2}$$

$$= \dots = -\frac{3}{16} e^{i\frac{3\pi}{4}}$$

$$\operatorname{res}_{e^{i\frac{3\pi}{4}}}(f) = \dots = -\frac{3}{16} e^{i\frac{\pi}{4}}$$

$$\Rightarrow 2\pi i\left(-\frac{3}{16} e^{i\frac{3\pi}{4}} - \frac{3}{16} e^{i\frac{\pi}{4}}\right) = \frac{3\pi}{4\sqrt{2}} .$$

$$\Rightarrow \boxed{\int_{x\in\mathbb{R}} \frac{x^6}{(1+x^4)^2}\,dx = \frac{3\pi}{4\sqrt{2}}} .$$

9. Let $f : \mathbb{C} \to \mathbb{C}$ be entire and $U \in \text{Open}(\mathbb{C})$ be bounded. Show that if $|f|$ is constant on ∂U then f is either a constant or has a zero in U.

Assume f has no zero in $\mathcal{U}$.

$M := |f(z)|$ for some $z \in \partial \mathcal{U}$ (it is const.)

$|f| \leq M$ on $\mathcal{U}$ by max princip. and

$f \neq 0$ on $\mathcal{U} \Rightarrow M$ is strictly positive.
(rather than $M \geq 0$).

Since $f \neq 0$, $\frac{1}{f}$ is analytic and hence also obeys max principle.

But $\frac{1}{|f(z)|}$ is const. on $\partial \mathcal{U}$ so

$$\frac{1}{|f|} = \frac{1}{M} \quad \text{on} \quad \partial \mathcal{U}$$

$$\overset{\text{max princip.}}{\Longrightarrow} \frac{1}{|f|} \leq \frac{1}{M} \quad \text{in} \quad \mathcal{U}.$$

$$\Rightarrow M \leq |f| \leq M \quad \text{in} \quad \mathcal{U} \Leftrightarrow |f| \text{ is const in } \mathcal{U}.$$

But we have seen in the open mapping Thm. 7.49 that if f is analytic and non-const. then f is open $\Rightarrow |f|$ is open $\Rightarrow |f|(\mathcal{U}) \in \text{Open}$

But $|f| = M$ const. $\Rightarrow |f|(\mathcal{U}) \in \text{Closed} \Rightarrow \perp$.

We conclude f is const. too. $\blacksquare$

10. Calculate

$$\int_{x=0}^{\infty} \frac{\cos(x) - 1}{x^2} \mathrm{d}x \, .$$

See Example 6.37:

$$\int_{x=0}^{\infty} \frac{\cos(x) - 1}{x^2} \, dx = -\frac{\pi}{2} \, .$$

11. Let $f : \mathbb{C} \to \mathbb{C}$ be entire so that

$$f(z) = f\left(\frac{1}{z}\right) \qquad (z \in \mathbb{C} \setminus \{0\}) .$$

Show that f is a constant.

$$f(z) = f\left(\tfrac{1}{z}\right) \implies f_R(z) = f_R\left(\tfrac{1}{z}\right)$$

$\implies f_R$ is bounded. Indeed, for any $z \in \mathbb{C}$ with $|z| \geq 1$ we may bound

$$|f_R(z)| = \left|f_R\left(\tfrac{1}{z}\right)\right| \leq \max_{w \in B_1(0)} |f_R(w)| \underset{\text{continuity}}{<} \infty$$

Thus, via HW7Q4, f is a const.

12. Calculate the leading order asymptotics of

$$I(\lambda) = \int_{x\in\mathbb{R}} e^{-\lambda x^2} \cos(\lambda x) \exp(x^2)\, dx\,.$$

See HW10Q16:

$$I(\lambda) \overset{\lambda\to\infty}{\sim} \sqrt{\frac{\pi}{\lambda}} \exp\left(-\frac{\lambda}{4}\right) \exp\left(-\frac{1}{4}\right).$$

13. Calculate

$$\int_{x=0}^{\infty} \frac{1}{\sqrt{x}\,(1+x^2)}\,\mathrm{d}x\,.$$

See HW7Q1(h):

$$\int_{x=0}^{\infty} \frac{1}{\sqrt{x}(1+x^2)}\,dx = \frac{\pi}{\sqrt{2}}\,.$$