Complex Analysis with Applications
Princeton University MAT330
Spring 2023 Final Exam

May 15, 2023

Note: the following is the final exam held on May 14th 2023 at 9am-noon in Fine 314. Please PRINT your first and last name in the box:

state the honor code pledge:

and sign it

Now please wait, without turning the page, until you are told to start the exam, at which point you shall have three hours.

Please write legibly and neatly. In the long questions part you are expected to justify your answer in full sentences. In the short questions part merely providing a correct succinct one-word answer will suffice.

1 Relevant formulas

In the following items, I remind you of relevant formulas but *not* of their scope of validity, which I expect you to know.

- The Cauchy-Riemann equations

$$\begin{aligned} \partial_x f_R &= \partial_y f_I \\ \partial_x f_I &= -\partial_y f_R \,. \end{aligned}$$

- Cauchy's integral formula:

$$f^{(n)}(z_0) = \frac{n!}{2\pi \mathrm{i}} \oint_\Gamma \frac{f(z)}{(z-z_0)^{n+1}} \mathrm{d}z \qquad (n \in \mathbb{N}_{\geq 0}) \,.$$

- Taylor's theorem:

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(z_0)(z-z_0)^n \,.$$

- The residue formula:

$$\mathrm{residue}_{z_0}(f) = \lim_{z\to z_0} \frac{1}{(n-1)!} \partial_z^{n-1} (z-z_0)^n f(z) \,.$$

- The argument principle:

$$\mathrm{index}_D(f) = \frac{1}{2\pi \mathrm{i}} \oint_\Gamma \frac{f'(z)}{f(z)} \mathrm{d}z \,.$$

- The Krammers-Kronig relation:

$$\lim_{\varepsilon \to 0^+} \frac{1}{x \pm \mathrm{i}\varepsilon} = \mp \mathrm{i}\pi\delta(x) + \mathscr{P}\left(\frac{1}{x}\right) \,.$$

- The Fourier series

$$(\mathscr{F}\psi)(n) \equiv \hat{\psi}(n) = \frac{1}{2\pi} \int_{\theta=0}^{2\pi} \mathrm{e}^{-\mathrm{i}n\theta} \psi(\theta) \mathrm{d}\theta$$

and

$$\left(\mathscr{F}^{-1}\hat{\psi}\right)(\theta) = \sum_{n\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}n\theta} \hat{\psi}(n) \,.$$

- The Fourier transform

$$(\mathscr{F}f)(\xi) \equiv \hat{f}(\xi) = \int_{x\in\mathbb{R}} \mathrm{e}^{-2\pi\mathrm{i}\xi x} f(x) \mathrm{d}x \,.$$

- Laplace and steepest descent asymptotics

$$\int_{x\in\mathbb{R}} \mathrm{e}^{-\lambda f(x)} g(x)\,\mathrm{d}x \overset{\lambda\to\infty}{\sim} \sqrt{\frac{2\pi}{\lambda \tilde{f}''(z_\star)}} \mathrm{e}^{-\lambda \tilde{f}(z_\star)} \tilde{g}(z_\star)\ .$$

- Jordan's inequality:

$$\frac{2}{\pi}\alpha \quad \leq \sin(\alpha) \leq \quad \alpha \qquad \left(\alpha \in \left[0, \frac{\pi}{2}\right]\right)\ ,$$

Kober's inequality

$$1 - \frac{2}{\pi}|x| \leq \cos(x) \leq 1 - \frac{x^2}{\pi} \qquad \left(x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right)\ ,$$

and the "big arc lemma":

$$\int_{\theta=0}^{\pi} \mathrm{e}^{-R\sin(\theta)} \mathrm{d}\theta \quad \lesssim \quad \frac{1}{R}\,.$$

2 Short questions [20 points]

In the following questions, *no* justification is necessary. Simply provide a short as possible *correct* response.

1. If f is analytic and injective, then f' can take on the value zero. Correct or incorrect?

2. If $F : \mathbb{R}^2 \to \mathbb{R}$ is a given harmonic function, provide an explicit expression for a harmonic conjugate $G : \mathbb{R}^2 \to \mathbb{R}$ to F at some $(x, y) \in \mathbb{R}^2$:

3. Provide an example $z \in \mathbb{C}$, $\alpha \in [0, 2\pi)$ for which $\mathbb{Im}\{\mathrm{Log}_\alpha(\overline{z})\} \neq -\mathbb{Im}\{\mathrm{Log}_\alpha(z)\}$, where Log_α is the complex logarithm with branch cut at α:

4. Let $\Omega \subseteq \mathbb{C}$ be open and bounded, and assume that $f : \Omega \to \mathbb{C}$ is analytic and extends continuously to $\overline{\Omega}$ such that $|f| \leq 1$ on $\partial\Omega$. What is $\sup_{z\in\Omega} |f(z)|$?

5. If an analytic function $f : \Omega \to \mathbb{C}$ with Ω open and connected vanishes on an open subset of Ω, does the function *have* to be the zero function?

3 Long questions [80 points]

In the following questions, you must justify your work and convince me that you not only know what the correct answer is, but also *why* it is so. You may freely invoke any result from the lecture notes, homework or various textbooks just so long as you properly cite and explain in what way you're invoking it. Note that if you're being asked about a result that appeared in the HW or lecture notes you can't verbatim invoke that very result: that would be silly.

6. Find a Taylor series at $z = 0$ for

$$f(z) = \frac{z^2}{(2+z)^2}$$

and indicate its domain of convergence.

7. Let $\gamma : [0,1] \to \mathbb{C}$ be a CCW contour such that $\Gamma \equiv \operatorname{im}(\gamma)$ is the *square* of side-length $2a > 0$ centered at the origin. Calculate $\oint_\Gamma \overline{z} \mathrm{d}z$ and $\oint_\Gamma \frac{\mathrm{e}^z}{z^2} \mathrm{d}z$.

8. Suppose p is a polynomial of degree $n \in \mathbb{N}_{\geq 0}$ and $R > 0$ such that $p \neq 0$ in $\mathbb{C} \setminus B_R(0)$. Calculate

$$\oint_{\partial B_R(0)} p, \oint_{\partial B_R(0)} \frac{1}{p}, \text{ and } \oint_{\partial B_R(0)} \frac{p'}{p}.$$

9. Calculate

$$\int_{x=0}^{\infty} \frac{\sin\left(x\right)}{x\left(1+x^{2}\right)} \mathrm{d}x\,.$$

10. Find a conformal equivalence $c : B_1(0) \cap \mathbb{H} \to \mathbb{H}$ (and prove it is so) where $\mathbb{H}$ is the open upper half plane and $B_1(0)$ the open unit disc.

11. Let $\alpha \in \mathbb{C}$ with $\mathbb{Im}\{\alpha\} > 0$. Calculate

$$-\!\!\!\!\!\!\int_{-\infty}^{\infty} \frac{1}{(x-1)\,(x-\alpha)^2}\,\mathrm{d}x\,.$$

12. Let $f : \mathbb{R} \to \mathbb{C}$ be given such that it extends to an analytic function $\tilde{f} : S_\varepsilon \to \mathbb{C}$ where S_ε is a horizontal strip of width 2ε about the real axis and such that

$$\sup_{y \in (-\varepsilon, \varepsilon)} \left| \tilde{f}(x + \mathrm{i}y) \right| \leq \frac{1}{1+x^2} \qquad (x \in \mathbb{R}) \ .$$

Show that there exists some $B \in (0, \infty)$ such that for any $\delta \in [0, \varepsilon)$,

$$|(\mathscr{F} f)(\xi)| \quad \leq \quad B \mathrm{e}^{-2\pi\delta|\xi|} \qquad (\xi \in \mathbb{R})$$

and give an explicit expression for B (i.e., if you find an integral, calculate it).

13. Calculate the leading order asymptotics as $\lambda \to \infty$ of

$$I(\lambda) = \int_{x \in \mathbb{R}} \exp(\mathrm{i}\lambda \cosh(x))\, \mathrm{d}x\,.$$