Complex Analysis with Applications
Princeton University MAT330
Practice Midterm

March 2, 2023

This is a practice midterm in preparation for the upcoming midterm (to be taken during class for 80 minutes on *March 8th 2023 at 11:00am*). I promise that the format of the actual midterm will be the same (i.e. five questions on a miscellanea of topics). I encourage you to set aside 80 minutes and try to solve the practice midterm quietly in one go in order to gauge your level as well as prepare yourself mentally for the day of the exam.

The grading will go as follows: Each question is worth 20 points. As usual, you will receive an additional 5+5 extra points if you write legibly and coherently, respectively, reaching to a maximum score of 110 points (to be truncated to max 100). In your solutions, you need to be correct but don't feel any obligation to follow the methods presented in class, the lecture notes, or the sample HW solutions. You do, however, need to provide full justification for your work.

1. (*Contour integration*) Show that $\int_0^\infty \sin\left(x^2\right) \mathrm{d}x = \frac{\sqrt{2\pi}}{4}$.
2. (*Polynomials*)
   (a) Find all polynomials $p, q : \mathbb{C} \to \mathbb{C}$ which obey the equation
   $$|p(z)| + |q(z)| \quad = \quad 1 + |z| \qquad (z \in \mathbb{C}) \,.$$
   (b) For a polynomial $p : \mathbb{C} \to \mathbb{C}$, express $\partial_z \partial_{\overline{z}} |p|^2$ in terms of p'.
3. (*The complex field* $\mathbb{C}$) Prove that
   $$\left|\frac{a-b}{1-\overline{a}b}\right| \quad < \quad 1 \qquad (a, b \in B_1(0))$$
   where $B_1(0) \equiv \{\, z \in \mathbb{C} \mid |z| < 1 \,\}$.
4. (*Holomorphicity*) Prove that if $f : \mathbb{C} \to \mathbb{C}$ is holomorphic and $\arg(f)$ is constant then f is constant.
5. (*Harmonic functions*) Show (by any means) that $F : \mathbb{R}^2 \setminus \{\, 0 \,\} \to \mathbb{R}$ given by
   $$F(x, y) = \log\left(x^2 + y^2\right)$$
   is harmonic.