1. Claim: \( \int_0^\infty \sin(x^2) \, dx = \frac{\sqrt{2\pi}}{4} \).

**Proof:** See Example 6.38 in the lecture notes.

**Proof.** Let us write

\[
\sin(x^2) = -\text{Im}\left\{e^{-ix^2}\right\}
\]

and define the function \( e^{-z^2} \) on the contour depicted in Figure 19. On its horizontal leg we have

\[
\int_0^R e^{-x^2} \, dx \to \int_0^\infty e^{-x^2} \, dx = \frac{1}{2} \int_{-\infty}^\infty e^{-x^2} \, dx = \frac{1}{2} \sqrt{\pi}.
\]

On its arc-like leg we have

\[
\int_0^\frac{\pi}{2} e^{-\left(Re^{i\theta}\right)^2} Rd\theta = iR\int_0^\frac{\pi}{2} e^{-R^2 \cos(2\theta)} e^{iR^2 \sin(2\theta)} \, d\theta
\]

Figure 19: The sector contour.
whose absolute value is bounded by
\[ \leq R \int_0^\frac{\pi}{2} e^{-R^2 \cos 2\theta} d\theta. \]

Now, \( \cos (2\theta) \geq 1 - 2\theta \) for all \( \theta \in \left[ 0, \frac{\pi}{4} \right] \). Indeed, by Taylor’s theorem with remainder we have
\[ \cos (2\theta) = 1 - 2 \cos (2t) \theta^2 \quad \forall t \in [0, \theta] \]
and so we have
\[ \cos (2\theta) \geq 1 - 2\theta^2 \]
\[ \frac{\partial}{\partial \theta} \geq 1 - 2\theta. \]

With this estimate we may carry out the (otherwise messy) integral to get
\[
\int_0^\frac{\pi}{2} e^{-R^2 \cos 2\theta} d\theta \leq \int_0^\frac{\pi}{4} e^{-R^2 (1 - 2\theta)} d\theta \\
= e^{-R^2} \int_0^\frac{\pi}{4} e^{-2R^2 \theta} d\theta \\
= e^{-R^2} \frac{1}{-2R^2} \left( e^{-2R^2 \frac{\pi}{4}} - 1 \right)
\]
so that this converges to zero very quickly as \( R \to \infty \).

On the radial leg we have
\[
\int_R^0 e^{-\left( re^{i\frac{\pi}{4}} \right)^2} e^{i\frac{\pi}{4}} dr = e^{i\frac{\pi}{4}} \int_R^0 e^{-ir^2} dr \\
= -e^{i\frac{\pi}{4}} \int_0^R e^{-ir^2} dr.
\]

Now, since \( z \mapsto e^{-z^2} \) is entire we find
\[
0 \xrightarrow{R \to \infty} \frac{1}{2\sqrt{\pi}} \left( -1 \right) e^{i\frac{\pi}{4}} \int_0^\infty e^{-ir^2} dr.
\]

Taking the imaginary part of this equation yields
\[
0 = \frac{1}{2\sqrt{\pi}} \sin \left( -\frac{\pi}{4} \right) - \text{Im} \left\{ \int_0^\infty e^{-ir^2} dr \right\}
\]
and so
\[
\int_0^\infty \sin \left( x^2 \right) dx = \frac{1}{2\sqrt{\pi}} \sin \left( \frac{\pi}{4} \right) \\
= \frac{1}{2\sqrt{\pi}} \frac{1}{\sqrt{2}} \\
= \frac{1}{2} \sqrt{\frac{\pi}{2}} \\
= \frac{\sqrt{2\pi}}{4}.
\]

Similarly taking the real part of the equation yields the cosine integral. \( \square \)
2(a) Claim: The only pair of poly. 
\[ p, q : \mathbb{C} \to \mathbb{C} \] which obey the eqn  
\[ |p(z)| + |q(z)| = 1 + |z| \quad (z \in \mathbb{C}) \]

are \( p(z) = a z, q(z) = b \) or vice versa

where \( a, b \in \mathbb{C}; |a| = |b| = 1 \).

Proof: Define \( g : \mathbb{C} \to [0, \infty) \) by

\[ g(z) := 1 + |z| \quad (z \in \mathbb{C}) \]

By the asymptotic behavior of \( g \) at zero and at \( \infty \) we may gain some information;

as \( R \to \infty \), \( g(R e^{i\theta}) = 1 + R \approx R \)

(linear growth). Hence \( \max(\{\deg(p), \deg(q)\}) = 1 \) necessarily since that will guarantee linear growth of \( |p(R e^{i\theta})| + |q(R e^{i\theta})| \) (see HW2Q4).

Next, consider the behavior of \( g \) at the origin
g is NOT diff. @ zero

(not even R-diff.)

\[ |p| + |q| \] cannot be diff @ zero.

\[ \Rightarrow \text{One of } p \text{ or } q \text{ must vanish at zero, since the map } \]
\[ C \ni z \rightarrow |a + b| z \in [0, \infty) \]

is diff. @ \( z = 0 \) if \( a \neq 0 \).

Say \( p(z) = az \) w/ \( a \neq 0 \) and \( q(z) = c + b \)

then, Again at zero we have:

\[ 1 = 1|q(z)| = |b| \]

Also, \( 1|q(z)| = 1 + |z| - |p(z)| \)
\[ = |1 - 2| - |\alpha| \]
\[ = 1 + (1 - |\alpha|) |z| \]

\[ \text{Case 1: } c \neq 0. \] Plug in \( z = -\frac{b}{c} \) (since \( c \neq 0 \)) to get

\[ 0 = 1 + (1 - |\alpha|) \frac{-b}{c} \]
\[ = 1 + (1 - |\alpha|) |c| \]

\[ \Leftrightarrow 0 = |c| + 1 - |\alpha| |c| \]
\[ \Leftrightarrow |\alpha| = 1 + |c| \]

But this is impossible since then

\[ |q(z)| = 1 - |\alpha| |c| \]

which becomes negative for large \( |z| > \frac{1}{|\alpha|} \).

\[ \Rightarrow \boxed{1} \]

\[ \text{Case 2: } c = 0. \] \( \Rightarrow \) \( |q(z)| = |b| = 1. \)

\[ \Rightarrow \boxed{1} \]
\[ \Rightarrow \boxed{1} = 1 + (1 - |\alpha|) |z| \]
\[ \Rightarrow \boxed{1} = (1 - |\alpha|) |z| \quad \forall \ z. \]

Hence \( |\alpha| = 1 \).

\[ \text{(b) Claim: For a poly. } p: \mathbb{C} \to \mathbb{C}, \]
\[ \partial_z \partial_{\bar{z}} |p|^2 = |p|^2 \]

Proof: \( p \) is poly. \( \Rightarrow \boxed{p} \) is holomorphic.
Similarly, one can show that if $f$ is holomorphic then $\overline{f}$ is "anti-holomorphic":

\[ \partial_{\overline{z}} f = 0 \quad \text{and} \quad \partial_{\overline{z}} \overline{f} = (\overline{f})'. \]

Hence

\[ \partial_{\overline{z}} \partial_{\overline{z}} |\overline{p}|^2 = \partial_{\overline{z}} \partial_{\overline{z}} \overline{\bar{p}} p = (\partial_{\overline{z}} \overline{p}) (\partial_{\overline{z}} \overline{\bar{p}}) = p' (\overline{\bar{p}})' = p \overline{(p')'} = 1|p'|^2. \]

3. **Claim**: \( \left| \frac{a-b}{1-\bar{a}b} \right| < 1 \) if \( |a|, |b| < 1. \)

**Proof**: \( \Leftarrow \) \( |a-b| < |1-\bar{a}b| \)

\( \Leftarrow \) \( |a-b|^2 < |1-\bar{a}b|^2 \)

\( \Leftarrow \) \( (a-b)(a-b) < (1-\bar{a}b)(1-\bar{a}b) \)
4. \textbf{Claim:} If $f : C \to C$ is holomorphic and \arg (f) is const then $f$ is const.

\textbf{Proof:} $f$ holomorphic $\Rightarrow \begin{cases} \partial_x f = \partial_y f_x \\ \partial_x f_x = -\partial_y f \\ \partial_x f = (\partial_x r) \cos(\theta) \text{ etc.} \end{cases}$

So CRE becomes: \begin{align*}
(\partial_x r) \cos(\theta) &= (\partial_y r) \sin(\theta) \\
(\partial_x r) \sin(\theta) &= - (\partial_y r) \cos(\theta)
\end{align*}

If partials are \textbf{NOT} zero, divide to get

\[ \frac{\partial y(\theta)}{\partial x(\theta)} = - \frac{1}{\partial y(\theta)} \]

\[ \partial y(\theta) + \frac{1}{\partial y(\theta)} = 0 \]

But in HW3Q8(b) we've seen...
\[ \left| \log(y) + \frac{1}{y} \right| \geq 2 \quad \forall \theta. \]

\[ \Rightarrow \quad \perp \quad \Rightarrow \quad \forall x, y \text{ must be zero.} \]

5. **Claim**: \( F : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R} \)

\( (x,y) \mapsto \log(x^2 + y^2) \)

is harmonic.

**Proof**: Calculate \( (\partial_x^2 + \partial_y^2) F \) explicitly (see Example 4.26)