MAR 4 2023

MAT330 — Practice Midterm Sample Sol-ns

1. **Claim:** $\displaystyle\int_0^\infty \sin(x^2)\,dx = \frac{\sqrt{2\pi}}{4}$.

Proof: See Example 6.38 in the lecture notes:

Proof. Let us write

$$\sin\left(x^2\right) = -\mathbb{Im}\left\{e^{-ix^2}\right\}$$

and define the function Use e^{-z^2} on the contour depicted in Figure 19. On its horizontal leg we have

$$\begin{aligned}\int_0^R e^{-x^2}dx &\to \int_0^\infty e^{-x^2}dx\\ &= \frac{1}{2}\int_{-\infty}^\infty e^{-x^2}dx\\ &= \frac{1}{2}\sqrt{\pi}\,.\end{aligned}$$

On its arc-like leg we have

$$\int_0^{\frac{\pi}{4}} e^{-\left(Re^{i\theta}\right)^2} Re^{i\theta} i d\theta = iR\int_0^{\frac{\pi}{4}} e^{-R^2\cos(2\theta)} e^{i\left[\theta - R^2\sin(2\theta)\right]} d\theta$$

57

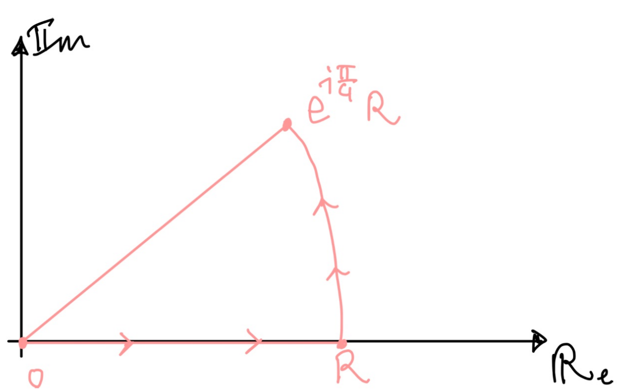


Figure 19: The sector contour.

whose absolute value is bounded by

$$\leq \quad R\int_0^{\frac{\pi}{4}} \mathrm{e}^{-R^2\cos(2\theta)}\mathrm{d}\theta\,.$$

Now, $\cos(2\theta) \geq 1-2\theta$ for all $\theta \in \left[0,\frac{\pi}{4}\right]$. Indeed, by Taylor's theorem with remainder we have

$$\cos(2\theta) \quad = \quad 1-2\cos(2t)\,\theta^2 \exists t\in[0,\theta]$$

and so we have

$$\begin{aligned}\cos(2\theta) \quad &\geq \quad 1-2\theta^2\\ &\overset{\theta\leq 1}{\geq} \quad 1-2\theta\,.\end{aligned}$$

With this estimate we may carry out the (otherwise messy) integral to get

$$\begin{aligned}\int_0^{\frac{\pi}{4}} \mathrm{e}^{-R^2\cos(2\theta)}\mathrm{d}\theta \quad &\leq \quad \int_0^{\frac{\pi}{4}} \mathrm{e}^{-R^2(1-2\theta)}\mathrm{d}\theta\\ &= \quad \mathrm{e}^{-R^2}\int_0^{\frac{\pi}{4}} \mathrm{e}^{-2R^2\theta}\mathrm{d}\theta\\ &= \quad \mathrm{e}^{-R^2}\frac{1}{-2R^2}\left(\mathrm{e}^{-2R^2\frac{\pi}{4}}-1\right)\end{aligned}$$

so that this converges to zero very quickly as $R\to\infty$.

On the radial leg we have

$$\begin{aligned}\int_R^0 \mathrm{e}^{-\left(r\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}\right)^2}\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}\mathrm{d}r \quad &= \quad \mathrm{e}^{\mathrm{i}\frac{\pi}{4}}\int_R^0 \mathrm{e}^{-\mathrm{i}r^2}\mathrm{d}r\\ &= \quad -\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}\int_0^R \mathrm{e}^{-\mathrm{i}r^2}\mathrm{d}r\,.\end{aligned}$$

Now, since $z\mapsto \mathrm{e}^{-z^2}$ is entire we find

$$0 \quad \overset{R\to\infty}{=} \quad \frac{1}{2}\sqrt{\pi}-\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}\int_0^\infty \mathrm{e}^{-\mathrm{i}r^2}\mathrm{d}r\,.$$

Taking the imaginary part of this equation yields

$$0 \quad = \quad \frac{1}{2}\sqrt{\pi}\sin\left(-\frac{\pi}{4}\right)-\mathbb{Im}\left\{\int_0^\infty \mathrm{e}^{-\mathrm{i}r^2}\mathrm{d}r\right\}$$

and so

$$\begin{aligned}\int_0^\infty \sin\left(x^2\right)\mathrm{d}x \quad &= \quad \frac{1}{2}\sqrt{\pi}\sin\left(\frac{\pi}{4}\right)\\ &= \quad \frac{1}{2}\sqrt{\pi}\frac{1}{\sqrt{2}}\\ &= \quad \frac{1}{2}\sqrt{\frac{\pi}{2}}\\ &= \quad \frac{\sqrt{2\pi}}{4}\,.\end{aligned}$$

Similarly taking the real part of the equation yields the cosine integral. $\square$

2.(a) <u>Claim</u>: The only pair of poly.

$p, q : \mathbb{C} \to \mathbb{C}$ which obey the eq-n

$$|p(z)| + |q(z)| \overset{\circledast}{=} 1 + |z| \qquad (z \in \mathbb{C})$$

are $p(z) = az$, $q(z) = b$ or vice versa

where $a, b \in \mathbb{C}$: $|a| = |b| = 1$.

<u>Proof</u>: Define $g : \mathbb{C} \to [0, \infty)$ by

$$g(z) := 1 + |z| \qquad (z \in \mathbb{C}).$$

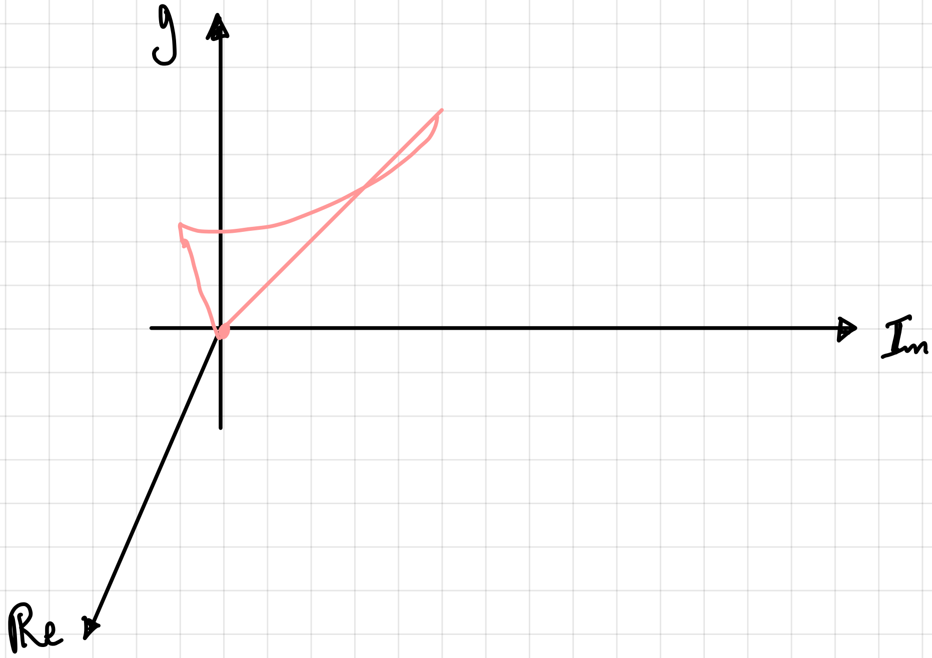
By the asymptotic behavior of g at zero and at ∞ we may gain some information:

as $R \to \infty$, $g(Re^{i\theta}) = 1 + R \approx R$

(linear growth).

Hence $\max(\{\deg(p), \deg(q)\}) = 1$ necessarily since that will guarantee linear growth of $|p(Re^{i\theta})| + |q(Re^{i\theta})|$ (see HW2Q4).

Next, consider the behavior of g at the origin



g is NOT diff. @ zero

(not even $\mathbb{R}$-diff.)

$\Rightarrow$ $|p|+|q|$ cannot be diff @ zero.

$\Rightarrow$ One of p or q must vanish at zero, since the map

$$\mathbb{C} \ni z \mapsto |a+bz| \in [0,\infty)$$

is diff. @ $z=0$ if $a \neq 0$.

Say $p(z)=az$ w/ $a\neq 0$ and $q(z)=cz+b$ then. Again at zero we have:

$$1 = |q(z)| = |b|.$$

Also, $|q(z)| = 1+|z|-|p(z)|$

$$= 1 + |z| - |az|$$
$$= 1 + (1-|a|)|z|.$$

Case 1: $c \neq 0$. Plug in $z = -b/c$ $(q(-b/c) = 0)$ to get

$$0 = 1 + (1-|a|)|-b/c|$$
$$= 1 + (1-|a|)/|c| \qquad |b| = 1$$

$$\Leftrightarrow \quad 0 = |c| + 1 - |a|$$
$$\Leftrightarrow \quad |a| = 1 + |c|$$

But this is impossible since then
$$|q(z)| = 1 - |c||z|$$
which becomes negative for large $|z| > \frac{1}{|c|}$.

$$\Rightarrow \boxed{\bot}$$

Case 2: $c = 0$. $\Rightarrow$ $|q(z)| = |b| = 1$.

$$\Rightarrow \quad \cancel{1} = \cancel{1} + (1-|a|)|z|$$
$$0 = (1-|a|)|z| \qquad \forall z.$$

Hence $|a| = 1$. ∎

(b) Claim: For a poly. $p : \mathbb{C} \to \mathbb{C}$,
$$\partial_z \partial_{\bar z} |p|^2 = |p'|^2$$

Proof: p is poly. $\Rightarrow$ p is holomorphic

$$\Longrightarrow \quad \partial_{\bar z} p = 0 \quad \text{and}$$

$$\partial_z p \equiv \tfrac{1}{2}(\partial_x - i\partial_y)(p_R + i p_I)$$
$$= \tfrac{1}{2}(\partial_x p_R + \partial_y p_I + i\partial_x p_I - i\partial_y p_R)$$
$$\overset{\text{CRE}}{=} \partial_x p_R - i\partial_y p_R$$
$$= p'$$

Eq-n (4.3) in lecture notes

Similarly, one can show that if f is holomorphic then $\bar f$ is "anti-holomorphic":

$$\partial_z \bar f = 0 \quad \text{and} \quad \partial_{\bar z} \bar f = (\bar f)'.$$

Hence
$$\partial_z \partial_{\bar z} |p|^2 = \partial_z \partial_{\bar z} \bar p p$$
$$= (\partial_z p)(\partial_{\bar z} \bar p)$$
$$= p' (\bar p)'$$
$$= p' \overline{(p')}$$
$$= |p'|^2 .$$

$\blacksquare$

3\. Claim: $\left|\dfrac{a-b}{1-\bar a b}\right| < 1$ if $|a|, |b| < 1$.

Proof:
$$\Leftrightarrow |a-b| < |1-\bar a b|$$
$$\Leftrightarrow |a-b|^2 < |1-\bar a b|^2$$
$$\Leftrightarrow (a-b)\overline{(a-b)} < (1-\bar a b)\overline{(1-\bar a b)}$$

$$\Leftrightarrow \underbrace{(a-b)(\bar a-\bar b)}_{|a|^2+|b|^2-\cancel{a\bar b}-\cancel{\bar a b}} < (1-\bar a b)(1-a\bar b)$$

$$|a|^2+|b|^2-\cancel{a\bar b}-\cancel{\bar a b} < 1-\cancel{\bar a b}-\cancel{a\bar b}+|a|^2|b|^2$$

$$0 < \underbrace{1-|a|^2-|b|^2+|a|^2|b|^2}_{(1-|a|^2)(1-|b|^2)}$$

which is indeed always true when $|a|,|b|<1$. $\blacksquare$

4. Claim: If $f: \mathbb{C}\to\mathbb{C}$ is holomorphic and $\arg(f)$ is const then f is const.

Proof: f holomorphic $\Rightarrow \begin{cases} \partial_x f_R = \partial_y f_I \\ \partial_x f_I = -\partial_y f_R \end{cases}$

Write $f = re^{i\theta}$ $\exists$ $r: \mathbb{C}\to[0,\infty)$ and θ constant.

$\Rightarrow \partial_x f_R = (\partial_x r)\cos(\theta)$ etc.

So CRE becomes: $\begin{cases} (\partial_x r)\cos(\theta) = (\partial_y r)\sin(\theta) \\ (\partial_x r)\sin(\theta) = -(\partial_y r)\cos(\theta) \end{cases}$

If partials are NOT zero, divide to get

$$tg(\theta) = -\frac{1}{tg(\theta)}$$

$$tg(\theta) + \frac{1}{tg(\theta)} = 0$$

But in HW3Q8(b) we've seen

$$\left|tg(\theta) + \frac{1}{tg(\theta)}\right| \geq 2 \qquad \forall \quad \theta.$$

$\Longrightarrow \quad \perp \quad \Longrightarrow \quad \partial_x r, \ r_y r$ must be zero.

5. **Claim:** $F : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$

$$(x,y) \longmapsto \log(x^2+y^2)$$

is harmonic.

Proof: Calculate $(\partial_x^2 + \partial_y^2) F$ explicitly

(see Example 4.26)

$\blacksquare$