Complex Analysis with Applications
Princeton University MAT330
Midterm
March 8th 2023 11:00am-12:20pm

March 9, 2023

Please clearly print and box your full name (in all capital letters) at the top right of the first page.

The grading is as follows: Each question is worth 20 points. As usual, you will receive an additional 5+5 extra points if you write legibly and coherently, respectively, reaching to a maximum score of 110 points (to be truncated to max 100). In your solutions, you need to be correct but don't feel any obligation to follow the methods presented in class, the lecture notes, or the sample HW solutions. Please don't re-invent the wheel by proving results we have in the lecture. Just please provide reasonable justification for your work and *cite the theorems you invoke*:

- Do: "By Cauchy's integral theorem, ..."
- Do: "By the theorem which states that a closed contour integral is zero if the integrand is holomorphic in the loop's interior, ..."
- Don't: "...".

And now, without further ado, the questions:

1. Show (by any means) that $F : \mathbb{R}^2 \to \mathbb{R}$ given by
$$F(x,y) = x^5 - 10x^3y^2 + 5xy^4$$
is harmonic.
2. Show that $\mathbb{C} \ni z \mapsto \exp\left(z^2\right)$ is holomorphic and find its derivative.
3. Calculate $\oint_\Gamma \left(\frac{1}{z} - 1\right) \mathrm{d}z$ where Γ is a CCW square of side length 1 centered at the origin.
4. Calculate (by any means) $\oint_\Gamma \frac{\mathrm{Log}\left(\mathrm{e}^{42}+\sin(z)\right)}{z} \mathrm{d}z$ where Γ is the counter-clockwise circle $\partial B_1(0) \equiv \{\, z \in \mathbb{C} \mid |z| = 1 \,\}$. Note that $\sin : \mathbb{C} \to \mathbb{C}$ is the function defined via $\sin(z) \equiv \frac{1}{2\mathrm{i}}\left[\mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z}\right]$.
5. Prove that $\int_{-\infty}^{\infty} \frac{x - \sin(x)}{x^3} \mathrm{d}x = \frac{\pi}{2}$. *Hint*: $f(z) = \frac{z + \mathrm{i}\mathrm{e}^{\mathrm{i}z}}{z^3}$ is holomorphic and Figure 1 contains a possibly relevant contour. You may expand a Taylor series of an exponential within the integral without further justification.

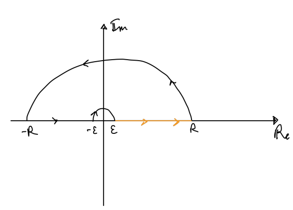

Figure 1: The indented semi-circle.