MAR 8 2023

Midterm in Complex Analysis ~ Sample Sol-ns

Q1 $F : \mathbb{R}^2 \to \mathbb{R}$ given by

$$F(x,y) = x^5 - 10x^3y^2 + 5xy^4$$

is the real part of the holomorphic f^n $\mathbb{C} \ni z \mapsto z^5$ and is hence harmonic.

<u>Alt. sol-n:</u> Calculate $-\Delta F \equiv \partial_x^2 F + \partial_y^2 F$.

Q2 The f^n $\mathbb{C} \ni z \mapsto \exp(z^2)$ is holomorphic since it is the composition of two such f^n's: exp and $z \mapsto z^2$.

The chain rule $(f \circ g)' = (f' \circ g)\, g'$ implies the derivative is $z \mapsto 2z \exp(z^2)$.

Q3 By linearity of the int.,

$$\oint \left(\frac{1}{z} - 1\right) dz = \oint \frac{1}{z}\, dz - \oint dz .$$

Since $z \mapsto 1$ is entire, CIT states the 2^{nd} int. is zero.

By CIF, we have

$$\oint \frac{1}{z} dz = 2\pi i \underbrace{\frac{1}{2\pi i} \oint \frac{1}{z-0} dz}_{\substack{\text{the } f^n \ z \mapsto 1 \\ \text{evaluated @ } 0}} = 2\pi i.$$

$\boxed{Q5}$ The f^n $\mathbb{C} \ni z \mapsto \mathrm{Log}(e^{42} + \sin(z))$ is holomorphic in $B_1(0)$ (since sin is and Log is holomorphic away from negative real axis, and it is shifted suff. far by e^{42} so $\sin(z)$ cannot push its argument's real part to be negative for $|z| = 1$).

Hence by CIF,

$$\oint \frac{\mathrm{Log}(e^{42} + \sin(z))}{z} dz = 2\pi i \, \mathrm{Log}(e^{42} + \sin(0))$$

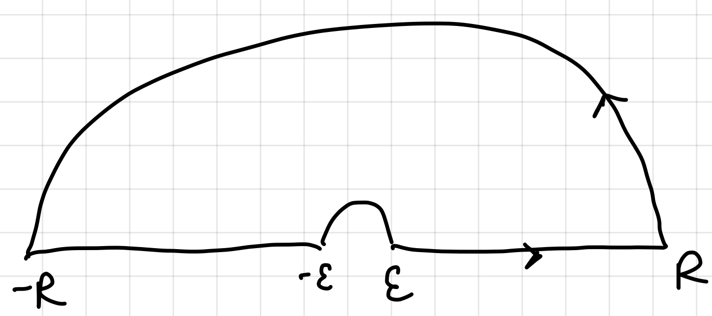
$$= 2\pi i \, \mathrm{Log}(e^{42})$$

$$= 2\pi i \, \log(e^{42}) = 84\pi i.$$

Q4 **Claim:** $\int_{-\infty}^{\infty} \frac{x - \sin(x)}{x^3}\,dx = \frac{\pi}{2}$,

Proof: Define $f(z) := \frac{z + i e^{iz}}{z^3}$ which is holomorphic away from zero.

Hence its int. on



is zero.

On the radius R semi-circle:

$$\left| \int_{\theta=0}^{\pi} \frac{Re^{i\theta} + i e^{iRe^{i\theta}}}{R^3 e^{3i\theta}} Re^{i\theta} i\,d\theta \right|$$

$$\leq \frac{\pi}{R} + \frac{1}{R^2} \underbrace{\int_{\theta=0}^{\pi} \underbrace{e^{-R\sin(\theta)}}_{\leq 1} d\theta}_{\leq \pi} \leq \frac{2\pi}{R} \xrightarrow{R\to\infty} 0.$$

On the radius ε semi-circle:

$$\int_{\theta=0}^{\pi} \frac{\varepsilon e^{i\theta} + i e^{i\varepsilon e^{i\theta}}}{\varepsilon^3 e^{3i\theta}} \varepsilon e^{i\theta} \, i \, d\theta = ?$$

To gain some insight we perform a Taylor expansion of the exponent,

$$e^{i\varepsilon e^{i\theta}} = 1 + i\varepsilon e^{i\theta} - \frac{1}{2}\varepsilon^2 e^{2i\theta} + O(\varepsilon^3)$$

whence we get

$$i \int_{\theta=0}^{\pi} \frac{i - \frac{i}{2}\varepsilon^2 e^{2i\theta} + O(\varepsilon^3)}{\varepsilon^3 e^{3i\theta}} \varepsilon e^{i\theta} \, d\theta$$

$$= -\frac{1}{\varepsilon^2} \underbrace{\int_0^{\pi} e^{-2i\theta} \, d\theta}_{=0} + \frac{1}{2}\int_{\theta=0}^{\pi} d\theta + O(\varepsilon)$$

$$= \frac{\pi}{2}.$$

On the other hand, the two real legs converge to what we want (its real part). ∎