Complex Analysis with Applications
Princeton University MAT330
Lecture Notes

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Abstract

These lecture notes correspond to a course given in the Spring semester of 2023 in the math department of Princeton University.

Contents

Syllabus

- Main source of material for the lectures: this very document (to be published and weekly updated on the course website–please do not print before the course is finished and the label "final version" appears at the top).
- Official course textbook: Brown and Churchill *Complex Variables and Applications* [BC13] 9th edition.
- Other books: One may also consult more mathematically oriented sources [Ahl21, SS03].
- Two lectures per week: MW 11am-12:20pm in Jadwin A09. There will be a weekly review session run by the undergraduate course assistants (time/place TBD).
- People involved:
  - Instructor: Jacob Shapiro jacobShapiro@princeton.edu
    Office hours: Fine 603, Mondays 5:30pm-7:30pm (starting January 30th 2023), or, by appointment.
  - Graduate assistants: Otte Heinaevaara, Xiao Ma

  - Undergrad assistants: Frank Lu, Ollie Thakar

- HW to be submitted weekly on Friday evening on Gradescope (first HW due Feb 10th). Submission by Sunday evening will not harm your grade but is not recommended. HW may be worked together in groups but needs to be written down and submitted separately for each student. 10% automatic extra credit on HW (up to a maximum of 100%) if you write *legibly* and *coherently*. You may use LaTeX or LyX to submit if you like.

- Grade: 30% HW, 30% Midterm, 40% Final.

- Attendance policy: attendance is recommended but not policed.

- Anonymous Ed discussion enabled. Use it to ask questions or to raise issues (technical or academic) with the course.

- If you alert me about typos and mistakes in this manuscript (unrelated to the sections marked [todo]) I'll grant you 1 extra point to your final grade (whose maximum is 100 points $= A+$). The total maximal extra credits due to finding typos is 5 points. In doing so, please refer to a version of the document by the date of typesetting.

  - Thanks goes to: Cutter ($\times 5$), Laney ($\times 3$), Icey ($\times 6$), Olivia, Adrian, Vincent ($\times 5$), Jake ($\times 2$), Samuel ($\times 4$), Nicholas, John, Michael ($\times 5$), Lyla.

1 The complex field $\mathbb{C}$

The complex plane $\mathbb{C}$ is a set, which may be identified with the set of points on the plane $\mathbb{R}^2$; see Figure 1. In multivariate calculus we usually refer to any point in $\mathbb{R}^2$ with two components as a column $\begin{bmatrix} x \\ y \end{bmatrix}$, a row (x, y), or using unit vectors $x\hat{i}+y\hat{j}$ etc (there are *many* different and even contrasting conventions). In complex analysis we think of the pair x, y together as *one* complex number z and write it as $z = x + \mathrm{i}y$, which is very similar to the unit vector notation of multivariate calculus, except that the real component does not get a special unit vector and (confusingly) what is in multivariate calculus $\hat{j}$ is now in complex analysis i (unless you are an engineer in which case you can just remove the hat and write $x + \mathrm{j}y$). We call x the real part and y the imaginary part of $z = x + \mathrm{i}y$. We also have special symbols for this:

$$\begin{aligned} \mathbb{Re}\{x+\mathrm{i}y\} &\equiv x \\ \mathbb{Im}\{x+\mathrm{i}y\} &\equiv y\,. \end{aligned}$$

The terminology comes from the identification $\mathrm{i} \equiv \sqrt{-1}$ (which is "imaginary" if using the $\mathbb{R}$ number system). While we keep the geometric picture from multivariate calculus (that $x + \mathrm{i}y$ denotes a point on the plane whose projection to the horizontal axis is x and to the vertical axis is y), we *add* on top of it *algebraic* structure which will play a crucial role in this subject:

Definition 1.1 (Addition and multiplication)**.** Addition of complex numbers is done component wise (just like addition of vectors):

$$(x+\mathrm{i}y)+(\tilde{x}+\mathrm{i}\tilde{y}) \quad := \quad x+\tilde{x}+\mathrm{i}\,(y+\tilde{y})\ .$$

However, multiplication of complex numbers is a new structure (compared with vectors) and follows the rule $\mathrm{i}^2 = -1$, i.e.,

$$(x+\mathrm{i}y)\cdot(\tilde{x}+\mathrm{i}\tilde{y}) \quad := \quad x\tilde{x}-y\tilde{y}+\mathrm{i}\,(x\tilde{y}+y\tilde{x})\ .$$

With these two definitions, we may conclude that arithmetic operations (such as distribution, association, etc) carry over to the complex field using the rule $\mathrm{i}^2 = -1$. In particular, addition and multiplication are commutative.

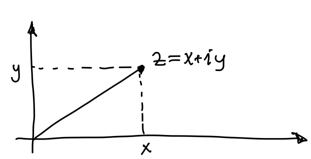

Figure 1: The geometric position of a complex number.

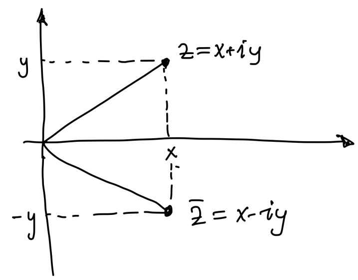

Figure 2: The geometric interpretation of complex conjugation.

Definition 1.2 (Division)**.** For any $z \neq 0$, we define $\frac{1}{z}$ as the unique number such that $\frac{1}{z} \cdot z = 1$.

To verify that for any $z \neq 0$ such a number exists:

$$\begin{aligned} \frac{1}{z} &= \frac{1}{x+\mathrm{i}y} \\ &= \frac{1}{x+\mathrm{i}y} \cdot \frac{x-\mathrm{i}y}{x-\mathrm{i}y} \\ &= \frac{x-\mathrm{i}y}{(x+\mathrm{i}y)\cdot(x-\mathrm{i}y)} \\ &= \frac{x-\mathrm{i}y}{x^2+y^2} \\ &= \frac{x}{x^2+y^2} - \mathrm{i}\frac{y}{x^2+y^2}\,. \end{aligned}$$

Definition 1.3 (Complex conjugation)**.** An additional algebraic operation on complex numbers is the *complex conjugate*: $z \mapsto \overline{z}$, which is defined as

$$\overline{x+\mathrm{i}y} \quad = \quad x-\mathrm{i}y\,.$$

Geometrically it is reflection through the horizontal axis, see Figure 2.

Definition 1.4 (Euclidean distance)**.** Distance in $\mathbb{C}$ is measured in the same Euclidean way as in $\mathbb{R}^2$:

$$|x+\mathrm{i}y| \quad \equiv \quad \sqrt{x^2+y^2}\,.$$

We call $|z|$ the modulus of z, or the absolute value of z.

It is useful to remember the some basic relations:

1. $|\mathbb{Re}\left\{z\right\}| \leq |z|$ and $|\mathbb{Im}\left\{z\right\}| \leq |z|$.
2. The triangle inequality: $|z+w| \leq |z|+|w|$.
3. The reverse triangle inequality: $|z+w| \geq ||z|-|w||$.
4. Multiplicativity: $|zw| = |z|\,|w|$.

Definition 1.5 (Exponential)**.** We will make heavy use of the exponential function of a complex number: $\exp : \mathbb{C} \to \mathbb{C}$. We will actually define it properly later on, but recall that $\mathrm{e} \approx 2.718\dots$, $\exp(x) \equiv \mathrm{e}^x$ for real x, and using the familiar rules of exponents,

$$\exp(x + \mathrm{i}y) \quad \equiv \quad \exp(x)\exp(\mathrm{i}y)$$

and we understand the second factor via *Euler's formula*

$$\exp(\mathrm{i}\theta) \quad = \quad \cos(\theta) + \mathrm{i}\sin(\theta)\ .$$

Hence

$$\exp(x + \mathrm{i}y) \equiv \mathrm{e}^x \cos(y) + \mathrm{i}\mathrm{e}^x \sin(y)\ . \tag{1.1}$$

If you are curious about Euler's formula and how to understand it: one way to define the exponential function is using a power series $\exp(\alpha) \equiv \sum_{n=0}^{\infty} \frac{\alpha^n}{n!}$, in which case we get:

$$\exp(\mathrm{i}\theta) \quad = \quad \sum_{n=0}^{\infty} \frac{(\mathrm{i}\theta)^n}{n!}\ .$$

but now, i^n follows a pattern: $\mathrm{i}^0 = 1$, $\mathrm{i}^1 = \mathrm{i}$, $\mathrm{i}^2 \equiv -1$, $\mathrm{i}^3 = -\mathrm{i}$and $\mathrm{i}^4 = 1$. Succinctly this may be summarized as $\mathrm{i}^{2n} = (-1)^n$.

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(\mathrm{i}\theta)^n}{n!} &= \sum_{n=0}^{\infty} \frac{(\mathrm{i}\theta)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(\mathrm{i}\theta)^{2n+1}}{(2n+1)!} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{\theta^{2n}}{(2n)!} + \mathrm{i}\sum_{n=0}^{\infty} (-1)^n \frac{\theta^{2n+1}}{(2n+1)!}\end{aligned}$$

and we identify the last two terms as the power series for cosine and sine respectively.

Definition 1.6 (Polar form)**.** (1.1) leads us to the polar form of any complex number, which is just a representation in polar coordinates of the vector $\begin{bmatrix} x \\ y \end{bmatrix}$(written in *Cartesian form* as $x + \mathrm{i}y$) using magnitude $\sqrt{x^2 + y^2}$ and angle $\operatorname{atan2}(y, x) \in (-\pi, \pi]$ (see Appendix C below for a discussion of the difference between atan2 and arctan). We write this representation usually using Euler's formula, so the polar to Cartesian dictionary is

$$\begin{aligned} x + \mathrm{i}y &\quad \to \quad \sqrt{x^2 + y^2}\mathrm{e}^{\mathrm{i}\operatorname{atan2}(y,x)} \\ r[\cos(\theta) + \mathrm{i}\sin(\theta)] &\quad \leftarrow \quad r\mathrm{e}^{\mathrm{i}\theta}\ .\end{aligned}$$

See Figure 3. Even though atan2 is defined so that $\operatorname{atan2}(y, x) \in (-\pi, \pi]$, geometrically, there is a 2π-ambiguity. However, as you notice it doesn't matter for us since it always goes inside the exponent–we'll have more to say about this issue later on. We give a special symbol to this *multivalued-angle, the argument of a complex number*

$$\arg(x + \mathrm{i}y) \quad \equiv \quad \operatorname{atan2}(y, x) + 2\pi n \qquad (n \in \mathbb{Z})\ .$$

The *principal value* of the argument (the principal argument) is the choice of angle that lines within $(-\pi, \pi]$, denoted by

$$\operatorname{Arg}(z) := \operatorname{atan2}(y, x) \quad \in \quad (-\pi, \pi]\ .$$

We will have more to say about this later.

Multiplication and division is especially simple in polar form:

$$\begin{aligned} z_1 z_2 &= r_1 \mathrm{e}^{\mathrm{i}\theta_1} r_2 \mathrm{e}^{\mathrm{i}\theta_2} \\ &= r_1 r_2 \mathrm{e}^{\mathrm{i}(\theta_1 + \theta_2)}\end{aligned}$$

and

$$\begin{aligned} \frac{1}{z} &= \frac{1}{r\mathrm{e}^{\mathrm{i}\theta}} \\ &= \frac{1}{r}\mathrm{e}^{-\mathrm{i}\theta}\ .\end{aligned}$$

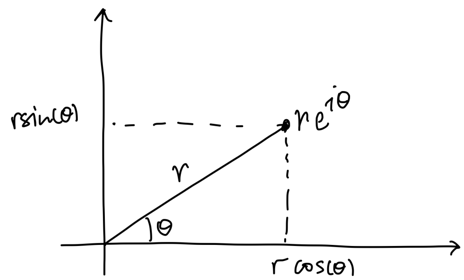

Figure 3: The polar form of a complex number.

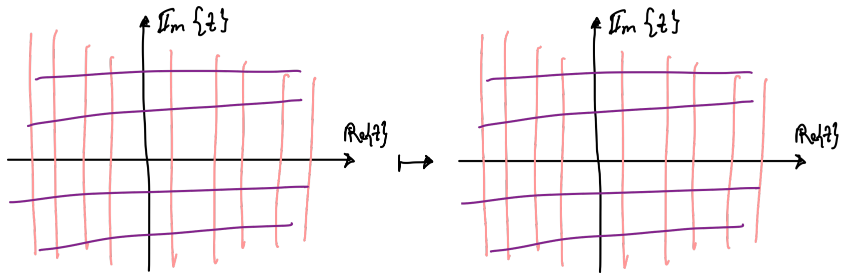

Figure 4: The identity map.

Using these we can raise a number to an integer power:

$$z^n \quad = \quad r^n \mathrm{e}^{\mathrm{i}n\theta} \qquad (n \in \mathbb{Z})$$

with the convention $z^0 \equiv 1$.

2 Functions as mappings of $\mathbb{C}$

We may define functions $f : \mathbb{C} \to \mathbb{C}$ via compositions of these basic algebraic operations, and study their geometric pictures. To each function $f : \mathbb{C} \to \mathbb{C}$ there are associated two real-valued functions on the plane $f_R, f_I : \mathbb{C} \to \mathbb{R}$ given by

$$\begin{aligned} f_R(z) &:= \mathbb{R}\mathrm{e}\{f(z)\} \\ f_I(z) &:= \mathbb{I}\mathrm{m}\{f(z)\} \end{aligned}$$

and vice versa, to any two functions $u, v : \mathbb{C} \to \mathbb{R}$ we may associated $f : \mathbb{C} \to \mathbb{C}$ as

$$f(z) \quad := \quad u(z) + \mathrm{i}v(z) \; .$$

If we wanted to plot this function, we would have had to plot a map $\mathbb{R}^2 \to \mathbb{R}^2$ which is impossible (we only have three dimensions to play with). So instead we must rely on other representational tools. For example, asking what do certain special curves (e.g., straight lines) get mapped to.

Example 2.1. The identity map $f : \mathbb{C} \to \mathbb{C}$, $z \mapsto z$ does not change the geometry of the plane. See Figure 4.

Example 2.2. $\mathbb{C} \ni z \mapsto az + b \in \mathbb{C}$ for two *fixed* numbers $a, b \in \mathbb{C}$. If $a = 1$ we call this a *shift by b:* see *Figure 5*.
If $b = 0$ and $a = r\cos(\theta) + \mathrm{i}r\sin(\theta)$ for $r > 0$ and $\theta \in \mathbb{R}$, we get a scaling by r and rotation by θ: see Figure 6.

Example 2.3. $\mathbb{C} \ni z \mapsto z^2 = x^2 - y^2 + 2\mathrm{i}xy \in \mathbb{C}$. Plug a vertical line $\mathbb{R} \ni t \mapsto x_0 + \mathrm{i}t$ into it to get

$$t \mapsto x_0^2 - t^2 + 2\mathrm{i}x_0 t$$

which is a parabola: see Figure 7. Try also a horizontal line $\mathbb{R} \ni t \mapsto t + \mathrm{i}y_0$ (another parabola).

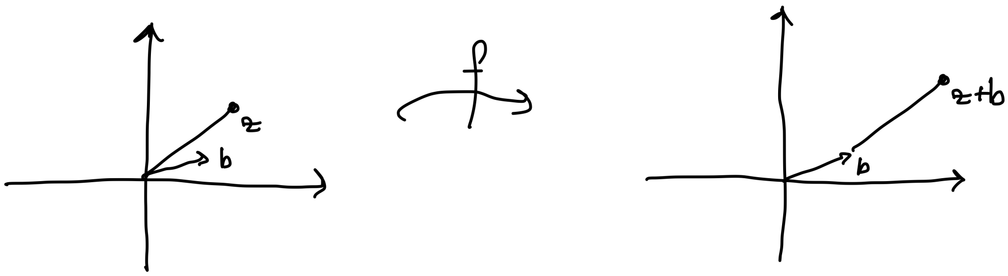

Figure 5: The map $z \mapsto z + b$ corresponds to a linear shift.

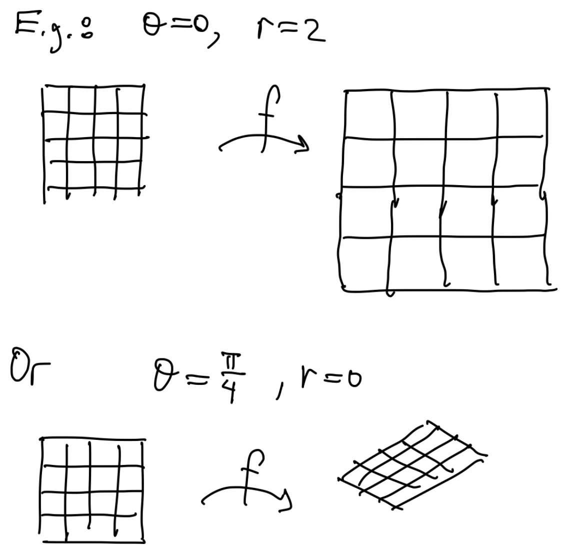

Figure 6: The map $z \mapsto r\mathrm{e}^{\mathrm{i}\theta} z$ corresponds to scaling by r and rotating by θ.

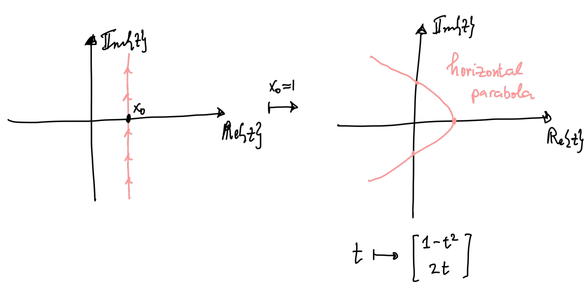

Figure 7: A depiction of how the map $z \mapsto z^2$ transforms a vertical line.

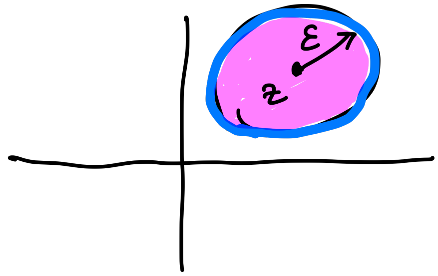

Figure 8: An open ball does *not* include its boundary (blue) circle.

Example 2.4. The exponential map $\mathbb{C} \ni z \mapsto \exp(z) \in \mathbb{C}$. To understand it we use

$$\begin{aligned} \exp(z) &\equiv \exp(x + \mathrm{i}y) \\ &= \exp(x)\exp(\mathrm{i}y) \\ &\overset{\text{Euler's formula}}{=} \exp(x)\left[\cos(y) + \mathrm{i}\sin(y)\right] . \end{aligned}$$

Then vertical lines $\mathbb{R} \ni t \mapsto x_0 + \mathrm{i}t$ get mapped to $t \mapsto \mathrm{e}^{x_0}\cos(t) + \mathrm{i}\mathrm{e}^{x_0}\sin(t)$ which is a circle of radius e^{x_0}. On the other hand horizontal lines get mapped to straight lines (verify).

3 Limits, continuity and topology in $\mathbb{C}$

In this section we review some material from calculus (or analysis) briefly before emphasizing what is honestly *new* in analysis in complex analysis versus analysis in $\mathbb{R}^n$: the notion of differentiability.

3.1 Limits

It will be useful for us to define circular neighborhoods of points in the plane:

Definition 3.1 (Open ball)**.** An *open ball* is the set of all points surrounding a point of given radius, excluding the boundary. More precisely, the open ball of radius $\varepsilon > 0$ about $z \in \mathbb{C}$ is denoted

$$B_\varepsilon(z) \quad := \quad \{\, w \in \mathbb{C} \mid |z - w| < \varepsilon \,\} \, .$$

See Figure 8.

Definition 3.2 (Limit)**.** Let $f : \mathbb{C} \to \mathbb{C}$. The limit $\lim_{z \to z_0} f(z)$ exists and equals $C \in \mathbb{C}$, denoted as

$$\lim_{z \to z_0} f(z) \quad = \quad C$$

iff $\forall\, \varepsilon > 0 \;\exists\, \delta_\varepsilon(z_0) > 0$ such that $\forall\, z \in B_{\delta_\varepsilon(z_0)}(z_0) \setminus \{\, z_0 \,\}$, $f(z) \in B_\varepsilon(C)$. Equivalently (in more human language), iff for any $\varepsilon > 0$ there exists some $\delta > 0$ (which in principle depends on $z_0 \in \mathbb{C}$ and $\varepsilon > 0$) such that

$$\text{If } z \text{ is such that } 0 < |z - z_0| < \delta \text{ then } |f(z) - C| < \varepsilon \, . \tag{3.1}$$

Remark 3.3. Note this definition is built so that we actually could also have f *undefined* at z_0 (think of $z \mapsto \exp\left(-\frac{1}{|z|}\right)$ which converges to zero as $z \to 0$). Also of importance is the fact that since we are working on $\mathbb{C} \cong \mathbb{R}^2$, the limit definition stipulates *any* approach (i.e., from any direction) towards z_0 yields the same result (cf. one-variable calculus the notion of right-hand and left-hand limit).

Remark 3.4. To prove that limit exist (using the $\varepsilon - \delta$ definition above), it is often useful to work on (3.1) backwards. I.e., start from

$$|f(z) - C|$$

and apply inequalities and estimates on it (e.g. the triangle inequality) until we get constraints on z itself. This will often be useful when f involves basic operations.

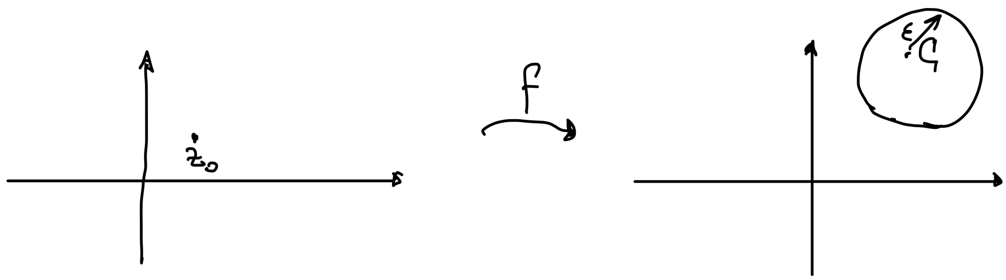

Figure 9: First step of using a limit: prescribe desired precision level $\varepsilon > 0$ around $f(z_0) = C$ in the codomain.

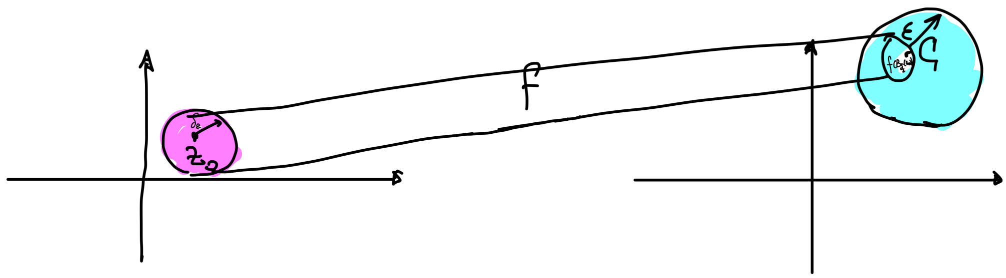

Figure 10: Second step of using a limit: the limit guarantees the existence of a precision level around z_0 in the domain to reach the prescribed target in the codomain.

Here is a step by step description of how to the limit definition works. The initial data is that we have some function $f : \mathbb{C} \to \mathbb{C}$ which maps $\mathbb{C} \ni z_0 \mapsto f(z_0) =: C \in \mathbb{C}$. Then the fact the limit exists means we can follow these steps:

1. Choose any $\varepsilon > 0$ we like around C (this will be the precision accuracy in the codomain). See Figure 9.
2. Now, the existence of the limit guarantees the existence of some $\delta_\varepsilon > 0$ (which in principle depends on z_0) such that the function f maps points in $B_{\delta_\varepsilon}(z_0)$ *into* $B_\varepsilon(C)$. This is a constraint on how close to z_0 we must go in the domain to have f map within the prescribed ε-ball from the previous step. See Figure 10.

How could the existence of the limit fail? Being unbounded, oscillating, or converging to different limits depending on the path taken.

Example 3.5 (Unboundedness)**.** The limit of $z \mapsto \frac{1}{|z|}$ at the origin fails to exist because the function *diverges to infinity*.

Example 3.6 (Oscillations)**.** Consider $f(z) = 5 + \mathrm{i}\sin\left(\frac{1}{|z|}\right)$. It has no limit near the origin. See Figure 11 for a plot of f_I as a function of $|z|$.

A more interesting way in which the existence of the limit can fail is if its value depends on how the point z_0 is approached:

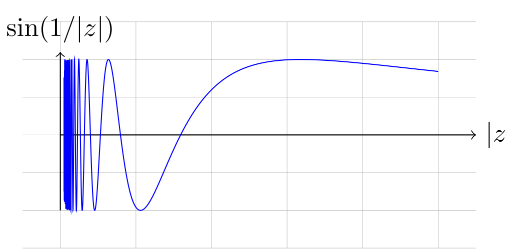

Figure 11: The topologist's sine curve.

Example 3.7 (Dependence of approach path)**.** Consider the following pathological example: $f:\mathbb{C}\to\mathbb{C}$ defined via

$$f(z) \quad := \quad \begin{cases}\frac{2x^2y}{x^4+y^2} & z\neq 0\\ 0 & z=0\end{cases}\,.$$

We claim this function has no limit as $z\to 0$. How to see this?

$$\frac{2x^2y}{x^4+y^2} \quad = \quad \frac{2\frac{y}{x^2}}{1+\left(\frac{y}{x^2}\right)^2}$$

so that any approach to the origin along the line $\frac{y}{x^2}=k$ for some fixed k will yield the limit $\frac{2k}{1+k^2}$. Since we can pick k arbitrarily, the limit cannot exist. Another classical example is

$$f(z) \quad = \quad \frac{2xy}{x^2+y^2}=\frac{2\left(\frac{y}{x}\right)}{1+\left(\frac{y}{x}\right)^2}\,.$$

Finally, let us see how to prove a limit exists when it does in a very simple case:

Example 3.8. $f(z):=\frac{i}{2}\overline{z}$ has a limit at $z=1$:

$$\lim_{z\to 1}f(z) \quad = \quad \frac{\mathrm{i}}{2}\,.$$

Proof. Following the advice of Remark 3.4, we estimate

$$\begin{aligned}\left|f(z)-\frac{\mathrm{i}}{2}\right| \quad &= \quad \frac{1}{2}\left|\overline{z}-1\right|\\ &= \quad \frac{1}{2}\left|z-1\right|\end{aligned}$$

whence it becomes obvious: pick $\delta_\varepsilon:=2\varepsilon$. □

Example 3.9 (The polar part of a complex number)**.** Consider $f:\mathbb{C}\setminus\{0\}\to\mathbb{C}$ given by $f(z):=\frac{z}{|z|}$. We have $\lim_{z\to 0}f(z)$ does *not* exist. How to understand that? We write in polar form

$$\frac{z}{|z|} \quad = \quad \mathrm{e}^{\mathrm{i}\theta}$$

so all points on the plane are projected to the unit sphere. Then, as we approach $z\to 0$, say on the horizontal axis from left or from right, we get mapped to -1 or 1 respectively, although the points in the domain get arbitrarily close: If $\delta>0$ and small, $\delta\mapsto 1$, $-\delta\mapsto -1$.

Proof. If one *insists* on a formal proof (not really necessary for us, but if you've always been curious):

$$\begin{gathered}![\exists C\in\mathbb{C}:\forall\varepsilon>0\exists\delta_\varepsilon>0:\forall z\in B_{\delta_\varepsilon}(0)\,,f(z)\in B_\varepsilon(C)]\\ \Updownarrow\\ \forall C\in\mathbb{C},\exists\varepsilon_C>0:\forall\delta>0,\exists z\in B_\delta(0):f(z)\notin B_{\varepsilon_C}(C)\,.\end{gathered}$$

So let $C\in\mathbb{C}$, written in polar form as $C=a\mathrm{e}^{\mathrm{i}\alpha}$. If $a=0$ any $\varepsilon<1$ would do since then $|f(z)-0|=1>\varepsilon$. If, on the other hand, $a\neq 0$, let $\varepsilon=1$. Then for any $\delta>0$, take z antipodal to C: $z=a\mathrm{e}^{\mathrm{i}(\alpha+\pi)}=-a\mathrm{e}^{\mathrm{i}\alpha}$. Then

$$\begin{aligned}\left|f(z)-C\right|^2 &= \left|\mathrm{e}^{\mathrm{i}(\alpha+\pi)}-a\mathrm{e}^{\mathrm{i}\alpha}\right|^2\\ &= \left|\mathrm{e}^{\mathrm{i}\pi}-a\right|^2\\ &= a^2+1-2a\cos(\pi)\\ &= a^2+1+2a\\ &= (a+1)^2\,.\end{aligned}$$

But $a+1>1$ by definition! □

Similarly to the theorems of multivariate calculus which stipulate that if the two component functions have limits then so does the vector-valued function, we have

Claim 3.10. Let $f:\mathbb{C}\to\mathbb{C}$ be given and $f_R:=\mathbb{Re}\{f\}$, $f_I:=\mathbb{Im}\{f\}$. Let $C\in\mathbb{C}$. The following are equivalent:

1. $\lim_{z\to z_0}f(z)=C$.
2. $\lim_{z\to z_0}f_R(z)=\mathbb{Re}\{C\}$ and $\lim_{z\to z_0}f_I(z)=\mathbb{Im}\{C\}$.

Proof. We show that the first condition implies the second two: Let $\varepsilon>0$ be given. We seek some $\delta_\varepsilon>0$ such that, e.g., $f_R(B_{\delta_\varepsilon}(z_0))\subseteq B_\varepsilon(\mathbb{Re}\{C\})$. By the first condition we know there is some $\tilde{\delta}_\varepsilon>0$ such that $f\left(B_{\tilde{\delta}_\varepsilon}(z_0)\right)\subseteq B_\varepsilon(C)$, which means that for any $z\in B_{\tilde{\delta}_\varepsilon}(z_0)$, $|f(z)-C|<\varepsilon$. Now,

$$\left|f(z)-C\right|^2 = (f_R(z)-\mathbb{Re}\{C\})^2+(f_I(z)-\mathbb{Im}\{C\})^2\,.$$

But of course, if $(f_R(z)-\mathbb{Re}\{C\})^2+(f_I(z)-\mathbb{Im}\{C\})^2<\varepsilon^2$ implies $(f_R(z)-\mathbb{Re}\{C\})^2<\varepsilon^2$. So just pick $\tilde{\delta}_\varepsilon=\delta_\varepsilon$. Same for f_I.

To show the converse, pick $\delta_\varepsilon:=\min\left(\left\{\delta^R_{\tilde{\varepsilon}},\delta^I_{\tilde{\varepsilon}}\right\}\right)$. So that if $|z-z_0|<\delta_\varepsilon$, we have (using $|\mathbb{Re}\{z\}-\mathbb{Re}\{z_0\}|<|z-z_0|$)

$$\begin{aligned}\left|\mathbb{Re}\{z\}-\mathbb{Re}\{z_0\}\right| &< \tilde{\varepsilon}\\ \left|\mathbb{Im}\{z\}-\mathbb{Im}\{z_0\}\right| &< \tilde{\varepsilon}\end{aligned}$$

so that

$$\begin{aligned}\left|f(z)-C\right|^2 &\equiv (f_R(z)-\mathbb{Re}\{C\})^2+(f_I(z)-\mathbb{Im}\{C\})^2\\ &\leq 2\tilde{\varepsilon}^2\end{aligned}$$

so pick $\tilde{\varepsilon}$ so that $2\tilde{\varepsilon}^2=\varepsilon$. □

The following two lemmas should be familiar to anyone who has taken calculus:

Lemma 3.11. *Let $f,g:\mathbb{C}\to\mathbb{C}$ be given. Then we may form two new functions, $(f+g):\mathbb{C}\to\mathbb{C}$ given by $(f+g)(z)\equiv f(z)+g(z)$ and $(fg):\mathbb{C}\to\mathbb{C}$ given by $(fg)(z)\equiv f(z)g(z)$. Let $z_0\in\mathbb{C}$. Then, if $\lim_{z\to z_0}f(z)$ and $\lim_{z\to z_0}g(z)$ exist, so do $\lim_{z\to z_0}(fg)(z)$ and $\lim_{z\to z_0}(f+g)(z)$.*

Lemma 3.12. *Let $f,g:\mathbb{C}\to\mathbb{C}$ and $z_0\in\mathbb{C}$ be given. Assume that both $\lim_{z\to z_0}g(z)=L_g$ and $\lim_{w\to L_g}f(w)=L_f$ exist. Then the composition $f\circ g:\mathbb{C}\to\mathbb{C}$ defined by $(f\circ g)(z)\equiv f(g(z))$ also has a limit $\lim_{z\to z_0}f(g(z))=L_f$.*

3.2 Continuity

While continuity is a fundamental property of functions which may be phrased without talking about limits, let us start by defining continuity via limits, as one would in the first calculus class:

Definition 3.13 (Continuity)**.** A function $f:\mathbb{C}\to\mathbb{C}$ is continuous at $z_0\in\mathbb{C}$ iff its limit there exists and equals $f(z_0)$:

$$\lim_{z\to z_0} f(z) \quad = \quad f(z_0)\,.$$

This definition immediately implies (via Lemma 3.11 and Lemma 3.12) that sums, products and compositions of continuous functions are continuous.

Here are two immediate useful consequences of continuity (just to get our hands dirty a little bit):

Claim 3.14. If $f(z_0)\neq 0$ and f is continuous at z_0 then $\exists B_\varepsilon(z_0)$ on which f is non-zero.

Proof. By continuity we can find a ball of z's in which $|f(z)-f(z_0)|<\varepsilon$. Using the reverse triangle inequality then we get

$$\begin{aligned}|f(z_0)|-|f(z)| &\quad<\quad \varepsilon\\ &\Updownarrow\\ |f(z)| &\quad>\quad |f(z_0)|-\varepsilon\,.\end{aligned}$$

So pick $\varepsilon:=\frac{1}{2}|f(z_0)|$. $\square$

3.3 A bit of topology

It is useful for us to re-phrase continuity in somewhat more abstract terms, using the field of mathematics called topology. Let us start with some of its basic notions:

Definition 3.15 (Openness, closedness, interior and closure)**.** A set $A\subseteq\mathbb{C}$ is *open* iff any point in it may be covered by a sufficiently small open ball which lies entirely within the set. In symbols, if $\forall a\in A$, $\exists\varepsilon_{a,A}>0$ such that

$$B_{\varepsilon_{a,A}}(a) \quad\subseteq\quad A\,.$$

$A\subseteq\mathbb{C}$ is *closed* iff $A^c\equiv\mathbb{C}\setminus A$ is open. The set of all open sets is denoted $\mathrm{Open}(\mathbb{C})$, and the set of all closed sets is denoted by $\mathrm{Closed}(\mathbb{C})$. The *interior* of a set is the *largest* (in the sense of set inclusion) open set contained in it, denoted by $\mathrm{int}(\mathrm{A})$, and the *closure* of a set is the smallest closed set containing it, denoted $\mathrm{clo}(\mathrm{A})$. That is, if $B\subseteq A$ for some open B then $B\subseteq\mathrm{int}(A)$, and if $A\subseteq B$ for some closed B, then $\mathrm{clo}(\mathrm{A})\subseteq\mathrm{B}$. Next, the boundary of a set A, denoted ∂A is the set of points in the closure but not in the interior:

$$\partial A:=\mathrm{clo}(\mathrm{A})\setminus\mathrm{int}(\mathrm{A})\,. \tag{3.2}$$

The words "open" and "closed" here do not necessarily correspond to what they would in our daily life (see Remark 3.24 below), but roughly speaking, if our set is a "nice" geometric shape (such as a disc), then closed means it includes its boundary and open means it does not. If it includes only part of its boundary, then it is neither. This starts to get funny when we have weird sub-dimensional sets (which are dots and lines) essentially and that's when one has to be careful and follow the definitions.

Lemma 3.16. *A set is open if and only if it equals its interior. A set is closed if and only if it equals its closure.*

Proof. Assume that a set $A\subseteq\mathbb{C}$ is open. Since $A\subseteq A$, it is itself the largest open set contained inside of it and hence $A=\mathrm{int}(\mathrm{A})$. To prove the converse, assume that $A=\mathrm{int}(\mathrm{A})$. Since $\mathrm{int}(\mathrm{A})$ is defined to be the largest *open* set contained within A, it is in particular open, and so A is too. The statement about closed sets and the closure follows similarly. $\square$

Lemma 3.17 (Connection between interior and closure)**.** *For any set* $A\subseteq\mathbb{C}$, $\mathrm{clo}(A)=\mathrm{int}(\mathrm{A}^c)^c$.

Proof. Let B be a closed set containing A. Then B^c is an open set contained within A^c. Clearly then the smallest closed set containing A is the largest open set contained within A^c. □

Lemma 3.18. $\operatorname{clo}(A)$ *is the set of points* $z \in \mathbb{C}$ *such that for all* $\varepsilon > 0$, $B_\varepsilon(z) \cap A \neq \varnothing$.

Proof. We seek to prove

$$\operatorname{clo}(A) = \{ z \in \mathbb{C} \mid \forall \varepsilon > 0, B_\varepsilon(z) \cap A \neq \varnothing \} . \tag{3.3}$$

To show the equivalence of two sets, we may show both $\subseteq$ and $\supseteq$.

Let us start with $\subseteq$. Let $z \in \operatorname{clo}(A)$. If $z \in A$ we are finished, so assume otherwise. Assume further that $\exists \varepsilon_0 > 0$ such that $B_{\varepsilon_0}(z) \cap A = \varnothing$. The set $B_{\varepsilon_0}(z)$ is open and hence its complement $B_{\varepsilon_0}(z)^c$ is closed, and $B_{\varepsilon_0}(z) \cap A = \varnothing$ is equivalent to $B_{\varepsilon_0}(z)^c \supseteq A$. Now since intersections of closed sets are closed, $\operatorname{clo}(A) \cap B_{\varepsilon_0}(z)^c$ is now a closed set which contains A and is *strictly* smaller than $\operatorname{clo}(A)$. But this is a contradiction, since $\operatorname{clo}(A)$ is supposed to be the smallest closed set which contains A.

Now we do the converse, $\supseteq$. Take any point $z \in \mathbb{C}$ on the right hand side set in (3.3). We want to show $z \in \operatorname{clo}(A)$. Via Lemma 3.17 we alternatively want to show that $z \in \operatorname{int}(A^c)^c$, i.e., that $z \notin \operatorname{int}(A^c)$. This is clearly true since $z \in \operatorname{int}(A^c)$ implies z is in the largest open set contained in A^c, but if for any $\varepsilon > 0$ we have $B_\varepsilon(z) \cap A \neq \varnothing$, z cannot belong to any open set contained in A^c. □

Lemma 3.19 (Abstract characterization of "open")**.** *The following properties hold:*

1. *$\varnothing$ and $\mathbb{C}$ are open.*
2. *Any union of open sets is open.*
3. *An intersection* of finitely many *open sets is open.*

In fact, these three properties may be taken as the definition of what an "open" means in any *topology which yields a coherent system of definitions. The particular choice of open sets as those generated by the open balls is called* The Standard Euclidean Topology*. There is an analogous characterization of "closed".*

Definition 3.20 (Accumulation point)**.** A point $z \in \mathbb{C}$ is *an accumulation point* (or *limit point*) of a set $A \subseteq \mathbb{C}$ iff for any $\varepsilon > 0$,

$$(B_\varepsilon(z) \cap A) \setminus \{ z \} \neq \varnothing .$$

In words, if any open ball around the point z contains a point of the set A which is not z itself. We emphasize an accumulation point of a set does not have to actually belong to the set.

Example 3.21. The set $\left\{ \frac{1}{n} \,\middle|\, n \in \mathbb{N}_{\geq 1} \right\}$ has a limit point 0 which does not actually lie within the set.

Lemma 3.22. *A point $z \in \mathbb{C}$ is a limit point of $A \subseteq \mathbb{C}$ iff it is in the closure of $A \setminus \{ z \}$. In fact, the closure of A is the set of points in A union with its set of limit points.*

Example 3.23. We list a few basic entries

1. As mentioned above, $\mathbb{C}$ and $\varnothing$ are open.
2. $\{ z \}$ is *closed* for any $z \in \mathbb{C}$. This is also true for any finite union of points. Contrast with $\left\{ \frac{1}{n} \,\middle|\, n \in \mathbb{N}_{\geq 1} \right\}$ which is not closed.
3. $\mathbb{R}$ is *not* open in $\mathbb{C}$. It is however closed in $\mathbb{C}$.
4. $B_\varepsilon(z_0)$ is itself open (hence the name, *open ball*) for any $\varepsilon > 0$ and any $z_0 \in \mathbb{C}$.

5. $\text{clo}\left(B_\varepsilon\left(z_0\right)\right) \equiv \left\{\, z \in \mathbb{C} \mid |z - z_0| \leq \varepsilon \,\right\}$ is closed, as it contains its outer boundary rim.

6. In both cases,
$$\partial B_\varepsilon\left(z_0\right) = \partial\text{clo}\left(B_\varepsilon\left(z_0\right)\right) = \left\{\, z \in \mathbb{C} \mid |z - z_0| = \varepsilon \,\right\}\,.$$
Is the boundary, itself also a set, open or closed? Think back to its definition (3.2).

Remark 3.24 (Counter-intuition). Contrary to how this works in human language, a set may be both simultaneously closed and open (in which case we call it clopen) or it could be neither. For example, $\mathbb{C}$ is clopen, and $B_1(0) \cup \{\, 5 \,\}$ is neither.

Definition 3.25 (Pre-image)**.** Let $f : A \to B$ be a given function (with $A, B \subseteq \mathbb{C}$) The pre-image of $S \subseteq B$ under f, denoted by $f^{-1}(S)$ is defined as the set of all points of A which map into S under f:
$$f^{-1}(S) \quad \equiv \quad \{\, z \in A \mid f(z) \in S \,\}\,.$$

This should be contrasted with *the image of a set*: If $T \subseteq A$ then the image of T under f is the set of all points *to which T maps:*
$$f(T) \quad \equiv \quad \{\, f(z) \in B \mid z \in T \,\}\,.$$

Example 3.26. Let $f : \mathbb{C} \to \mathbb{C}$ be given by $f(z) = |z|$. Then
$$f(\mathbb{C}) = \{\, r \in \mathbb{R} \mid r \geq 0 \,\} \subseteq \mathbb{C}\,.$$
And, degenerately,
$$\begin{aligned} f^{-1}(\{\, -5 \,\}) &\equiv \{\, z \in \mathbb{C} \mid |z| = -5 \,\} \\ &= \varnothing \end{aligned}$$
and more interestingly,
$$f^{-1}([1,2]) \quad = \quad \{\, z \in \mathbb{C} \mid |z| \in [1,2] \,\}$$
is an annulus between radii 1 and 2.

Lemma 3.27. *Pre-images "commute" with unions and intersections:*
$$\begin{aligned} f^{-1}(A \cup B) &= f^{-1}(A) \cup f^{-1}(B) \\ f^{-1}(A \cap B) &= f^{-1}(A) \cap f^{-1}(B) \end{aligned}$$
but the image does not. For unions we still have
$$f(A \cup B) \quad = \quad f(A) \cup f(B)$$
but *for intersections we only have*
$$f(A \cap B) \quad \subseteq \quad f(A) \cap f(B)\,.$$

Proof. We omit the proof of all the equalities above but provide an example for when the last inclusion is indeed strict. Consider the function $f : \mathbb{Z} \to \mathbb{Z}$ given by $n \mapsto n^2$. Consider the choice $A = \mathbb{N}_{\geq 0} = \{\, 0, 1, 2, 3, \dots \,\}$ and $B = \mathbb{Z}_{\leq 0} = \{\, \dots, -3, -2, -1, 0 \,\}$. The intersection is $A \cap B = \{\, 0 \,\}$. For the images, we have
$$f(A \cap B) = f(\{\, 0 \,\}) \quad = \quad \{\, 0 \,\}$$
and yet
$$f(A) = f(B) = \{\, 0, 1, 4, 9, 16, \dots \,\} \Longrightarrow f(A) \cap f(B) = \{\, 0, 1, 4, 9, 16, \dots \,\}$$

so we see the inclusion here is indeed strict: $\{ 0 \} \subsetneq \{ 0, 1, 4, 9, 16, \dots \}$. ☐

Now that we have some language from topology we may revisit the definition of continuity from Definition 3.13 in a more intrinsic way, rather than relying on limits:

Lemma 3.28 (Continuity via openness)**.** *$f : \mathbb{C} \to \mathbb{C}$ is continuous at $z_0 \in \mathbb{C}$ iff*

$$\forall U \in \mathrm{Open}(\mathbb{C}) : f(z_0) \in U, \quad f^{-1}(U) \text{ contains an open subset which contains } z_0 \,.$$

Globally instead of pointwise, we say that $f : \mathbb{C} \to \mathbb{C}$ is continuous iff

$$\forall U \in \mathrm{Open}(\mathbb{C}) \,,\; f^{-1}(U) \in \mathrm{Open}(\mathbb{C}) \;.$$

An analogous statement can be made about pre-images of closed sets under continuous maps.

Since this is an if and only if statement, it could serve as an alternative definition of continuity, and indeed in more general settings it is the starting point for continuity.

Proof of Lemma 3.28. To establish the claim, we must show that the present criterion is equivalent to the one given in Definition 3.13. Let $f : \mathbb{C} \to \mathbb{C}$ and $z_0 \in \mathbb{C}$ be given.

In one direction: let us assume f is continuous at z_0 according to Definition 3.13. We try to establish the present criterion, so let $U \in \mathrm{Open}(\mathbb{C})$ such that $f(z_0) \in U$. We need to show that $f^{-1}(U) \in \mathrm{Open}(\mathbb{C})$. Since U is open and $f(z_0) \in U$, following Definition 3.15, we know that $\exists \varepsilon > 0$ such that $B_\varepsilon(f(z_0)) \subseteq U$. Since f is continuous at z_0, $\lim_{z \to z_0} f(z) = f(z_0)$, so for that particular $\varepsilon > 0$ there's some $\delta_\varepsilon > 0$ such that

$$z \in B_{\delta_\varepsilon}(z_0) \Longrightarrow f(z) \in B_\varepsilon(f(z_0)) \;.$$

This condition is equivalent to $B_{\delta_\varepsilon}(z_0) \subseteq f^{-1}(B_\varepsilon(f(z_0)))$ and since $B_\varepsilon(f(z_0)) \subseteq U$, we have $B_{\delta_\varepsilon}(z_0) \subseteq f^{-1}(U)$, so $B_{\delta_\varepsilon}(z_0)$ is the desired open subset which contains z_0.

In the other direction, assume the present criterion for f and z_0 and try to establish $\lim_{z \to z_0} f(z) = f(z_0)$. So let $\varepsilon > 0$ be given. We know that $B_\varepsilon(f(z_0)) \in \mathrm{Open}(\mathbb{C})$ and $f(z_0) \in B_\varepsilon(f(z_0))$, so we apply the present criterion with $U = B_\varepsilon(f(z_0))$ to get that $f^{-1}(B_\varepsilon(f(z_0)))$ contains an open subset which contains z_0. Call it V. That means, by Definition 3.15, that there is some $\delta > 0$, such that $z_0 \in B_\delta(z_0) \subseteq V \subseteq f^{-1}(B_\varepsilon(f(z_0)))$ which is precisely Definition 3.2. ☐

Remark 3.29. One might ask why continuity is defined here using the inverse image rather than the image. The answer is as follows: continuous functions are functions which preserve the topological structure of a space (the structure of openness), much like group homomorphisms preserve the group structure. This topological structure, abstractly defined in Lemma 3.19, involves unions and intersections. But Lemma 3.27 shows us that it is the pre-image, rather than the image, which has any hope to "factorize" through the basic set operations of unions and intersections.

Remark 3.30. For metric spaces ($\mathbb{C}$ is a metric space with metric $|\cdot - \cdot|$; if you find this confusing you may safely ignore this remark), it suffices to work with open balls instead of general open sets in Lemma 3.28 since "the open balls form a basis for the metric topology".

Claim 3.31 (Continuity via open balls implies continuity via openness). $f : \mathbb{C} \to \mathbb{C}$ is continuous at $z_0 \in \mathbb{C}$ iff $\forall \varepsilon > 0$,

$$f^{-1}(B_\varepsilon(f(z_0))) \quad \text{contains an open subset which contains } z_0 \,.$$

Again we also have the global statement: $f : \mathbb{C} \to \mathbb{C}$ is continuous iff for any $\varepsilon > 0$ and any $z \in \mathbb{C}$,

$$f^{-1}(B_\varepsilon(z)) \quad \in \quad \mathrm{Open}(\mathbb{C}) \;.$$

The proof of this claim is clear from glancing at the proof of Lemma 3.28.

Remark 3.32. Of course we also have the sequential definition of continuity: f is continuous at z_0 iff for any sequence $\{ w_n \}_n \subseteq \mathbb{C}$ converging to z_0, the sequence $\{ f(w_n) \}_n$ converges to $f(z_0)$. All these notions are equivalent when talking about functions $\mathbb{C} \to \mathbb{C}$.

Definition 3.33 (Bounded sets)**.** A set $A \subseteq \mathbb{C}$ is *bounded* iff there exists some $R > 0$ such that $A \subseteq B_R(0)$.

Example 3.34. Clearly $\mathbb{C}$ is not bounded, and neither are $\mathbb{R}$ or $\mathbb{N}$ (considered as subsets of $\mathbb{C}$). The set

$$\left\{ \frac{1}{n} \,\middle|\, n \in \mathbb{N}_{\geq 1} \right\}$$

is bounded (using, e.g., $B_1(0)$).

Definition 3.35 (Bounded functions)**.** A function $f : A \to \mathbb{C}$ (with some domain $A \subseteq \mathbb{C}$) is *bounded* iff its infinity-norm is finite

$$\|f\|_\infty \quad := \quad \sup_{z \in A} |f(z)| < \infty\,.$$

Example 3.36. The function $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = z$ is not bounded. We can change its domain to be, say, the unit square, whence it becomes bounded (with $\|f\|_\infty = \sqrt{2}$). On the other hand, the function $g : \mathbb{C} \to \mathbb{C}$ given by $g(z) = \frac{1}{\sqrt{1+|z|^2}}$ is bounded throughout the entire complex plane with $\|g\|_\infty = 1$.

Definition 3.37 (Compact sets)**.** A set $A \subseteq \mathbb{C}$ is *compact* iff any open cover, i.e., any collection $\{ U_i \}_{i \in \mathcal{I}}$ of open subsets (with emphasis on *any*, in particular, not necessarily finite or countable) such that $\bigcup_{i \in \mathcal{I}} U_i \supseteq A$ contains a finite sub-cover, i.e., some finite set $F \subseteq \mathcal{I}$ so that $\bigcup_{i \in F} U_i \supseteq A$.

Theorem 3.38 (Heine-Borel)**.** *$A \subseteq \mathbb{C}$ is compact iff it is closed and bounded.*

This theorem works in any metric space that has the so-called *Heine-Borel property,* which is precisely spaces where closed and bounded is equivalent to compact.

Theorem 3.39 (Bolzano-Weierstraß)**.** *If $\{ z_n \}_n \subseteq \mathbb{C}$ is bounded then it has a convergent subsequence.*

Claim 3.40 (convergent sequence in closed set has limit inside)*.* If $\{ z_n \}_n \subseteq K \subseteq \mathbb{C}$ converges to some $z \in \mathbb{C}$ and K is closed then the limit is actually also in K: $z \in K$.

Proof. Assume otherwise. That means that $z \in \mathbb{C} \setminus K$. By Definition 3.15, $\mathbb{C} \setminus K$ is open, so that there exists some $\varepsilon > 0$ such that $B_\varepsilon(z) \subseteq \mathbb{C} \setminus K$. Since $z_n \to z$, for that same $\varepsilon > 0$, there is some $N_\varepsilon \in \mathbb{N}$ such that if $n \geq N_\varepsilon$ then $|z - z_n| < \varepsilon$, i.e., so that $z_n \in B_\varepsilon(z)$. But this is a contradiction since $z_n \in K$ and $B_\varepsilon(z) \subseteq \mathbb{C} \setminus K$! ☐

Claim 3.41*.* If $K \subseteq \mathbb{C}$ is closed and bounded and $f : K \to \mathbb{C}$ is continuous then $\|f\|_\infty < \infty$.

Proof. Proceed by contradiction. If $\|f\|_\infty = \infty$, then necessarily, for any $n \in \mathbb{N}$ there is $z_n \in K$ such that $|f(z_n)| \geq n$. Since K is bounded, the sequence $\{ z_n \}_n \subseteq K$ either converges, or oscillates, but cannot diverge to infinity. If it oscillates, take a subsequence which does converge (by Bolzano-Weierstraß above this is always possible). But then this convergent subsequence $\{ w_n \}_n \subseteq K$ has a limit which lies *inside* K, since K is closed, by Claim 3.40. So $w_n \to w \in K$. But then since f is continuous $f(w_n) \to f(w)$, in particular, $\{ f(w_n) \}_n$ is bounded, which is a contradiction to the assumed $|f(w_n)| \to \infty$. ☐

Theorem 3.42. *A continuous function $f : K \to \mathbb{C}$ on a compact set K attains its maximum and minimum on K.*

Proposition 3.43. *If* $\{ \Omega_n \}_{n=1}^{\infty} \subseteq \mathbb{C}$ *is a nested sequence of non-empty compact sets, in the sense that*

$$\Omega_n \quad \subseteq \quad \Omega_{n-1} \qquad (n \geq 2)$$

satisfying the property that

$$\lim_{n\to\infty} \operatorname{diam}(\Omega_n) = 0 \tag{3.4}$$

then there exists a unique point

$$w \in \cap_{n=1}^{\infty} \Omega_n \,.$$

Proof. Since for each n, Ω_n is non-empty, choose some $z_n \in \Omega_n$ for each n. The condition (3.4) implies that the sequence $\{ z_n \}_n$ is Cauchy. Since $\mathbb{C}$ is a complete metric space, $\{ z_n \}_n$ converges to some limit in $\mathbb{C}$, call it $w \in \mathbb{C}$. Now due to the nesting property, for each n, $\{ z_m \}_{m\geq n} \subseteq \Omega_n$ and it is a convergent subsequence so that (using the fact Ω_n is closed since it is compact and bounded) its limit w must lie within it. Hence $w \in \Omega_n$ for each n. Now, if there were some $\tilde{w} \in \mathbb{C} \setminus \{ w \}$ satisfying the same property, we would have $|w - \tilde{w}| > 0$ and hence the condition (3.4) would be violated. □

3.4 More examples for continuity

Let us resume now the more concrete discussion of continuity with a few more examples.

Example 3.44. $f(z) = z^2$ is continuous on $\mathbb{C}$.

Proof. We have

$$\begin{aligned}
|f(z) - f(z_0)| &= \left|z^2 - z_0^2\right| = |(z - z_0)(z + z_0)| \\
&= |z + z_0|\,|z - z_0| \\
&= |z - z_0 + 2z_0|\,|z - z_0| \\
&\leq 2|z_0|\,|z - z_0| + |z - z_0|^2 \\
&\overset{\text{Assume } |z-z_0|\leq 1}{\leq} (2|z_0| + 1)\,|z - z_0|
\end{aligned}$$

so we find

$$(2|z_0| + 1)\,\delta \quad \overset{!}{<} \quad \varepsilon$$

which leads to

$$\delta_\varepsilon \quad := \quad \min\left(\left\{ 1, \frac{\varepsilon}{2|z_0| + 1} \right\}\right).$$

Coincidentally, this shows (heuristically, but this is not just a defect of the proof) the continuity is *not uniform* in z_0. □

Example 3.45. The function $f(z) := \begin{cases} \frac{z}{|z|} & z \neq 0 \\ 0 & z = 0 \end{cases}$ is not continuous at zero but is continuous everywhere else.

Proof. Away from zero, we have

$$\begin{aligned}
|f(z) - f(z_0)|^2 &= \left|\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{\mathrm{i}\theta_0}\right|^2 \\
&= 2(1 - \cos(\theta - \theta_0))
\end{aligned}$$

whereas

$$\begin{aligned}
|z-z_0|^2 &= \left|re^{i\theta}-r_0e^{i\theta_0}\right|^2 \\
&= r^2+r_0^2-2rr_0\cos(\theta-\theta_0) \\
&= (r-r_0)^2+2rr_0\left[1-\cos(\theta-\theta_0)\right] \\
&\overset{(r-r_0)^2\geq 0}{\geq} 2rr_0\left[1-\cos(\theta-\theta_0)\right] \\
&= 2rr_0\,|f(z)-f(z_0)|^2 \\
&= r_0^2\,|f(z)-f(z_0)|^2+r_0(r-r_0)\,|f(z)-f(z_0)|^2 \\
&\geq r_0^2\,|f(z)-f(z_0)|^2-r_0\,|r-r_0|\,|f(z)-f(z_0)|^2 \\
&= r_0(r_0-|r-r_0|)\,|f(z)-f(z_0)|^2 \\
&\overset{|r-r_0|\leq|z-z_0|}{\geq} r_0(r_0-|z-z_0|)\,|f(z)-f(z_0)|^2 \\
&\overset{\text{Take } |z-z_0|<\frac{3}{4}r_0}{\geq} \frac{1}{4}r_0^2\,|f(z)-f(z_0)|^2\,.
\end{aligned}$$

i.e., we have found

$$|f(z)-f(z_0)| \leq \frac{2}{r_0}|z-z_0|$$

So pick

$$\delta_\varepsilon := \min\left(\left\{\frac{3}{4}r_0,\frac{r_0}{2}\varepsilon\right\}\right)=\frac{r_0}{2}\min\left(\left\{\frac{3}{2},\varepsilon\right\}\right)\,.$$

On the other hand, at zero we have $|f(z)-f(z_0)|=|f(z)|=1$ so pick any $\varepsilon\in(0,1)$ to show continuity will fail. □

4 $\mathbb{C}$-Differentiability

We are finally ready to jump into the main novelty of complex analysis compared with multivariable calculus: complex differentiability. Here differentiability will turn out to be *more stringent* than differentiability on $\mathbb{R}^n$.

Definition 4.1 ($\mathbb{C}$-differentiability)**.** A function $f:\mathbb{C}\to\mathbb{C}$ is $\mathbb{C}$-*differentiable* at $z_0\in\mathbb{C}$ iff the following limit exists

$$\lim_{z\to0}\frac{f(z_0+z)-f(z_0)}{z}\,.$$

In case the limit indeed exists, we define the derivative of f at z_0, denoted conveniently by $f'(z_0)$, as the result of that limit:

$$f'(z_0) = \lim_{z\to0}\frac{f(z_0+z)-f(z_0)}{z}\,.$$

Special to complex analysis is the terminology about differentiable functions:

Definition 4.2. Another common name for $\mathbb{C}$-differentiable is *holomorphic*. In fact, later on we will see (but we already preempt this now) that also *analytic* is synonymous, due to an important theorem that guarantees that a holomorphic function has derivatives of all orders (and is hence smooth), but moreover, its power series converges at every point and is hence analytic in the sense of real analysis. There is also a technicality about holomorphicity on sets: a function is called holomorphic on a set $S\subseteq\mathbb{C}$ if it is holomorphic for every $z_0\in U$ where $S\subseteq U\in\mathrm{Open}(\mathbb{C})$.

Let us begin our study with some examples:

Example 4.3. Consider $f(z) = z^2$. We claim it is everywhere $\mathbb{C}$-differentiable:

$$\begin{aligned}\lim_{z\to 0}\frac{(z_0+z)^2 - z_0^2}{z} &= \lim_{z\to 0}\frac{2z_0 z + z^2}{z} \\ &= \lim_{z\to 0}(2z_0+z) \\ &= \left(\lim_{z\to 0} 2z_0\right) + \left(\lim_{z\to 0} z\right) \\ &= 2z_0\,.\end{aligned}$$

So the limit indeed exists everywhere and so we conclude $f'(z_0) = 2z_0$, in line with the familiar formula from single variable real calculus.

The same rules we are familiar with apply:

1. $\text{const}' = 0$.
2. $(z^n)' = nz^{n-1}$ if $n \in \mathbb{Z} \setminus \{-1\}$.
3. $(f+g)' = f' + g'$ if both f, g are $\mathbb{C}$-diff.
4. $(fg)' = f'g + fg'$ if both f, g are $\mathbb{C}$-diff.
5. $(f \circ g)' = (f' \circ g)\, g'$.

Let us, just for fun, prove the fourth rule:

Lemma 4.4. *If f, g are both $\mathbb{C}$-differentiable then fg (the product function) is also $\mathbb{C}$-differentiable and $(fg)' = f'g + fg'$.*

Proof. If fg is $\mathbb{C}$-differentiable at $z_0 \in \mathbb{C}$, then the limit

$$\lim_{z\to 0}\frac{(fg)(z_0+z) - (fg)(z_0)}{z}$$

should exist. But

$$\begin{aligned}\frac{(fg)(z_0+z) - (fg)(z_0)}{z} &= \frac{f(z_0+z)\,g(z_0+z) - f(z_0)\,g(z_0)}{z} \\ &= \frac{f(z_0+z)\,g(z_0+z) - f(z_0)\,g(z_0+z) + f(z_0)\,g(z_0+z) - f(z_0)\,g(z_0)}{z} \\ &= \frac{[f(z_0+z) - f(z_0)]\,g(z_0+z) + f(z_0)\,[g(z_0+z) - g(z_0)]}{z} \\ &= \left(\frac{f(z_0+z) - f(z_0)}{z} g(z_0+z)\right) + \left(f(z_0)\,\frac{g(z_0+z) - g(z_0)}{z}\right).\end{aligned}$$

Hence it seems like the result will be implied using the facts that:

1. The limit of the a sum is the sum of limits, if the two respective limits exist.
2. The limit of a product is the product of limits, if the two respective limits exist.
3. f and g are $\mathbb{C}$-differentiable at z_0.
4. g is is continuous at z_0.

This last point is guaranteed by the following lemma. □

Claim 4.5. If f is $\mathbb{C}$-differentiable at $z_0 \in \mathbb{C}$ then it is continuous at z_0.

Proof. Let $\varepsilon > 0$. We are seeking some $\delta_\varepsilon > 0$ such that

$$z \in B_{\delta_\varepsilon}(z_0) \quad\Longrightarrow\quad f(z) \in B_\varepsilon(f(z_0)) \ .$$

Let us again using Remark 3.4 start backwards:

$$\begin{aligned} |f(z) - f(z_0)| &= |f(z) - f(z_0) - f'(z_0)(z - z_0) + f'(z_0)(z - z_0)| \\ &\le |f(z) - f(z_0) - f'(z_0)(z - z_0)| + |f'(z_0)| |z - z_0| && \text{(Triangle ineq.)} \\ &= \left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| |z - z_0| + |f'(z_0)| |z - z_0| \ . \end{aligned}$$

But now, since

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \quad = \quad f'(z_0)$$

we have some $\tilde{\delta}_\varepsilon > 0$ such that if

$$z \in B_{\tilde{\delta}_\varepsilon}(z_0) \Longrightarrow \left(\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right) \in B_\varepsilon(0) \ .$$

So pick

$$\delta_\varepsilon \quad := \quad \min\left(\left\{ \tilde{\delta}_{\frac{\varepsilon}{2}}, \frac{\varepsilon}{2(|f'(z_0)| + 1)} \right\} \right) \ .$$

This does the job, because then

$$\begin{aligned} |f(z) - f(z_0)| \quad &\le \quad \frac{\varepsilon}{2} \frac{\varepsilon}{2(|f'(z_0)| + 1)} + |f'(z_0)| \frac{\varepsilon}{2(|f'(z_0)| + 1)} \\ &\le \quad \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \quad \varepsilon \ . \end{aligned}$$

In the penultimate estimate, we have used

1. $\frac{1}{|f'(z_0)|+1} \le 1$.
2. $\frac{\varepsilon^2}{2} \le \varepsilon$ for $\varepsilon \le 2$ (if ε is large the proof is not interesting).
3. $\frac{|f'(z_0)|}{|f'(z_0)|+1} \le 1$.

$\square$

The general intuition about $\mathbb{C}$-differentiability should be the following $f : \mathbb{C} \to \mathbb{C}$ is differentiable if $f(z)$ may be written entirely in terms of z but not using $\overline{z}$: so $|z|^2$ is actually *not* $\mathbb{C}$-differentiable!

Example 4.6 (Non-$\mathbb{C}$-differentiability)**.** $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = \overline{z}$ is *not* $\mathbb{C}$-differentiable.

Proof. To show this we need to prove the limit $\lim_{z \to 0} \frac{f(z_0+z) - f(z_0)}{z}$ does not exist for any z_0. The pre-limit is

$$\begin{aligned} \frac{f(z_0 + z) - f(z_0)}{z} &= \frac{\overline{z_0 + z} - \overline{z_0}}{z} \\ &= \frac{\overline{z}}{z} && \text{(complex conj. respects addition)} \\ &= \frac{\mathrm{e}^{-\mathrm{i}\theta}}{\mathrm{e}^{\mathrm{i}\theta}} \\ &= \mathrm{e}^{-2\mathrm{i}\theta} \ . \end{aligned}$$

But we've already seen in Example 3.9 that this limit does not exist because it depends on the angle with which we approach the origin! ☐

This example seems very weird because $z \mapsto \overline{z}$ is, geometrically, merely reflection about the vertical axis. Why isn't this most basic operation not $\mathbb{C}$-differentiable? To really understand this we must dig deeper into what differentiability means, which is as the possibility for linear approximations [Car17].

4.1 Fréchet differentiability

Definition 4.7 (Differentiability for normed vector spaces)**.** Let V, W be two normed vector spaces. Then $f : V \to W$ is differentiable at $x \in V$ iff $\exists$ bounded linear map $L : V \to W$ such that

$$\lim_{h \in V : \|h\|_V \to 0} \frac{\|f(x+h) - f(x) - Lh\|_W}{\|h\|_V} \quad = \quad 0\,.$$

This notion of differentiability is sometimes also called Fréchet differentiability, and it is equivalent to the one we learn in multivariable calculus if we take $V = \mathbb{R}^n$, $W = \mathbb{R}^m$ with the Euclidean norm. Indeed, in that setting, if the function is indeed Fréchet differentiable, the linear map $L \equiv \mathscr{D}f$ would be the Jacobian matrix of all possible partial derivatives $\partial_j f_k$. Then, as we learn in Taylor's theorem, we have

$$f(x+y) \quad \approx \quad f(x) + \sum_{k=1}^{m} e_k \sum_{j=1}^{n} (\partial_j f_k)(x)\, y_j + \dots \qquad (x, y \in \mathbb{R}^n)\ .$$

Remark 4.8. Note that f having partial derivatives is actually not enough for it to be Fréchet differentiable. A sufficient condition is that $\partial_j f_k$ exist and are continuous in an open neighborhood.

4.2 The Cauchy-Riemann equations via the Frechet-derivative being $\mathbb{C}$-linear

What happens if we re-interpret a function $f : \mathbb{C} \to \mathbb{C}$ as a function $F : \mathbb{R}^2 \to \mathbb{R}^2$ and study its Fréchet differentiability? To do so, we re-write

$$f(x + \mathrm{i}y) \quad = \quad f_R(x + \mathrm{i}y) + \mathrm{i} f_I(x + \mathrm{i}y)$$

so that $f_R, f_I : \mathbb{C} \to \mathbb{R}$. Then

$$F(x,y) \quad = \quad \begin{bmatrix} F_1(x,y) \\ F_2(x,y) \end{bmatrix} := \begin{bmatrix} f_R(x + \mathrm{i}y) \\ f_I(x + \mathrm{i}y) \end{bmatrix}\,.$$

It appears all we need is to make sure that the matrix

$$\mathscr{D}F = \begin{bmatrix} \partial_x F_1 & \partial_y F_1 \\ \partial_x F_2 & \partial_y F_2 \end{bmatrix} = \begin{bmatrix} \partial_x f_R & \partial_y f_R \\ \partial_x f_I & \partial_y f_I \end{bmatrix} \tag{4.1}$$

exists and is continuous in some open neighborhood. But we have been extremely cavalier in doing so. We have implicitly applied the notion of what a linear map is from $\mathbb{R}^2 \to \mathbb{R}^2$. However, the map we started with is $f : \mathbb{C} \to \mathbb{C}$ and $\mathbb{C}$ is actually a normed vector space *which has its own, different linear structure than* $\mathbb{R}^2$*!* So what would happen if we tried to apply the definition of Fréchet differentiability, but now insisting that the derivative be a $\mathbb{C}$-linear map instead of an $\mathbb{R}^2$-linear map? Let us study this deeper.

What does $\mathbb{C}$-linearity mean? Instead of multiplication by a 2×2 matrix for $\mathbb{R}^2$, in $\mathbb{C}$ it will simply be multiplication by one complex number, i.e.,

$$z \quad \mapsto \quad wz$$

for some fixed $w \in \mathbb{C}$. Rewriting $w = a + \mathrm{i}b$ and $z = x + \mathrm{i}y$ we get

$$\begin{aligned} wz \quad &= \quad (a + \mathrm{i}b)(x + \mathrm{i}y) \\ &= \quad ax - by + \mathrm{i}(ay + bx)\ . \end{aligned}$$

If we insisted to lift the action of multiplication by w and write it as a 2×2 matrix acting on the 2 vector $\begin{bmatrix}x\\y\end{bmatrix}$ we would have to write it as

$$\begin{bmatrix}a & -b\\ b & a\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}ax-by\\bx+ay\end{bmatrix}.$$

Hence we see that we could actually keep the structure of $\mathbb{C}$-multiplication at the level of the isomorphism $\mathbb{C}\cong\mathbb{R}^2$, i.e., thinking of matrices acting on vectors. *However*, we must restrict our attention to a very special form of matrices in the set

$$\mathcal{B}(\mathbb{C})\cong\left\{\begin{bmatrix}a & -b\\ b & a\end{bmatrix}\in\mathrm{Mat}_{2\times 2}(\mathbb{R})\;\middle|\; a,b\in\mathbb{R}\right\}. \tag{4.2}$$

We now have a more concrete way to connect $\mathbb{C}$-differentiability (in a way that produces a $\mathbb{C}$-linear approximation) with general Fréchet differentiability as follows:

Theorem 4.9. *A map $f:\mathbb{C}\to\mathbb{C}$ is $\mathbb{C}$-differentiable iff the associated map $F:\mathbb{R}^2\to\mathbb{R}^2$ is Frechet differentiable as in Definition 4.7 but with the further restriction that the linear map L lies in (4.2), i.e., that L is $\mathbb{C}$-linear. Furthermore, one has the relation*

$$\det(\mathcal{D}F) = (\partial_x f_R)^2+(\partial_x f_I)^2 = |f'|^2.$$

To be explicit, beyond the pathologies that might happen (that existence of partial derivatives do not imply existence of the Frechet derivative), the constraint would be that

$$\begin{bmatrix}\partial_x f_R & \partial_y f_R\\ \partial_x f_I & \partial_y f_I\end{bmatrix}=\begin{bmatrix}a & -b\\ b & a\end{bmatrix} \tag{4.3}$$

for some $a,b\in\mathbb{R}$, which really translates to the set of two equations

$$\boxed{\begin{cases}\partial_x f_R & =\partial_y f_I\\ \partial_x f_I & =-\partial_y f_R\end{cases}}. \tag{4.4}$$

These two equations are called the *Cauchy-Riemann equations.*

Let us return to the example of the non-differentiable function $z\mapsto\overline{z}$ and understand using this new found Cauchy-Riemann structure why it is not differentiable:

Example 4.10 ($z\mapsto\overline{z}$ re-examined)**.** Let us re-write $z\mapsto\overline{z}$ as

$$\mathbb{R}^2\ni\begin{bmatrix}x\\y\end{bmatrix}\mapsto\begin{bmatrix}x\\-y\end{bmatrix}\in\mathbb{R}^2.$$

It is easy to calculate the total derivative of this function:

$$\begin{bmatrix}\partial_x x & \partial_y x\\ \partial_x(-y) & \partial_y(-y)\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}.$$

Comparing this matrix with (4.2) we see that it is *clearly* not $\mathbb{C}$-linear because $1\neq -1$.

Example 4.11. Let us see why $z\mapsto|z|^2$ is not $\mathbb{C}$-differentiable. We have this map on 2 vectors as

$$\begin{bmatrix}x\\y\end{bmatrix}\mapsto\begin{bmatrix}x^2+y^2\\0\end{bmatrix}$$

so that its matrix of partial derivatives is

$$\begin{bmatrix}\partial_x\left(x^2+y^2\right) & \partial_y\left(x^2+y^2\right)\\ \partial_x(0) & \partial_y(0)\end{bmatrix} = \begin{bmatrix}2x & 2y\\ 0 & 0\end{bmatrix}$$

which clearly does not belong to (4.2). On the other hand, $z \mapsto z^2$ will certainly be differentiable since

$$\begin{bmatrix} x \\ y \end{bmatrix} \quad \mapsto \quad \begin{bmatrix} x^2 - y^2 \\ 2xy \end{bmatrix}$$

has derivative

$$\begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix}$$

which lies in (4.2).

4.3 The Cauchy-Riemann equations

Let us gain further perspective on the Cauchy-Riemann equations by deriving them in another way. We have the derivative given by the limit

$$f'(z_0) \quad \equiv \quad \lim_{z\to 0} \frac{f(z_0+z) - f(z_0)}{z}\,.$$

Now if we approach the origin along the positive real axis, i.e., $z = \varepsilon > 0$ (which we are certainly *allowed* to do), and write $f = f_R + \mathrm{i} f_I$, we have

$$\begin{aligned} f'(z_0) \quad &= \quad \lim_{\varepsilon\to 0^+} \frac{f_R(z_0+\varepsilon) + \mathrm{i} f_I(z_0+\varepsilon) - f_R(z_0) - \mathrm{i} f_I(z_0)}{\varepsilon} \\ &= \quad \lim_{\varepsilon\to 0^+} \frac{f_R(z_0+\varepsilon) - f_R(z_0) + \mathrm{i}\left[f_I(z_0+\varepsilon) - f_I(z_0)\right]}{\varepsilon} \end{aligned}$$

and we find that if f' exists, it equals

$$f'(z_0) \quad = \quad (\partial_x f_R)(z_0) + \mathrm{i}\,(\partial_x f_I)(z_0)\,.$$

On the other hand, since we have the freedom to take any path we like, we must also have (approaching along the positive imaginary axis)

$$\begin{aligned} f'(z_0) \quad &= \quad \lim_{\varepsilon\to 0^+} \frac{f(z_0+\mathrm{i}\varepsilon) - f(z_0)}{\mathrm{i}\varepsilon} \\ &= \quad -\mathrm{i}\,(\partial_y f_R)(z_0) + (\partial_y f_I)(z_0)\,. \end{aligned}$$

Since the two must equal, we find

$$(\partial_x f_R)(z_0) + \mathrm{i}\,(\partial_x f_I)(z_0) \quad = \quad -\mathrm{i}\,(\partial_y f_R)(z_0) + (\partial_y f_I)(z_0)$$

which is equivalent to (4.4).

This actually immediately implies that if f is differentiable, then it suffices to know only the derivative of f_R or of f_I to calculate it, because

$$\boxed{f' = \partial_x f_R - \mathrm{i}\partial_y f_R = \partial_y f_I + \mathrm{i}\partial_x f_I}\,. \tag{4.5}$$

Above we have seen that the Cauchy-Riemann equations (together with existence of the partial derivatives) actually guarantees the existence of f'.

Wirtinger derivatives In studying the above expression for f', we may define (purely as notation)

$$\partial_{\overline{z}} \quad := \quad \frac{1}{2}\left(\partial_x + \mathrm{i}\partial_y\right)$$

and

$$\partial_z \quad = \quad \frac{1}{2}\left(\partial_x - \mathrm{i}\partial_y\right)\,.$$

Then,

$$\begin{aligned} \partial_{\overline{z}} f &= \frac{1}{2}\left(\partial_x + \mathrm{i}\partial_y\right)\left(f_R + \mathrm{i} f_I\right) \\ &= \frac{1}{2}\left(\partial_x f_R - \partial_y f_I\right) + \frac{\mathrm{i}}{2}\left(\partial_x f_I + \partial_y f_R\right) \\ &= 0\,. && \text{(Using the Cauchy-Riemann equations)} \end{aligned}$$

With this notation we have

Lemma 4.12 (Differentiability via Wirtinger derivatives)**.** *The function $f:\mathbb{C}\to\mathbb{C}$ is $\mathbb{C}$-differentiable iff $f_R, f_I : \mathbb{R}^2\to\mathbb{R}$ are differentiable and if $\partial_{\overline{z}} f = 0$.*

The Wirtinger derivative makes it abundantly clear why certain functions are not differentiable. Compare how we have proved $z\mapsto |z|^2$ is not differentiable in Example 4.11 versus the following argument. $|z|^2 = \overline{z}z$ and

$$\partial_{\overline{z}}(\overline{z}z) \quad = \quad z\neq 0!$$

What about the function $z\mapsto \mathbb{Re}\{z\}$? Re-write $\mathbb{Re}\{z\} = \frac{1}{2}(z+\overline{z})$ and conclude.

The following example illustrates that the Cauchy-Riemann equations are necessary but not sufficient for differentiability. The function still has to be differentiable in the multi-variable sense!

Example 4.13 (Non-differentiability despite CRE)**.** Consider the function $f:\mathbb{C}\to\mathbb{C}$ given by

$$f(z) \quad := \quad \begin{cases} \frac{z}{|z|^2}\mathbb{Re}\{z\}\,\mathbb{Im}\{z\} & z\neq 0 \\ 0 & z=0\,. \end{cases}$$

We claim it is not differentiable at zero.

Proof. If the derivative existed, the following limit exists:

$$\begin{aligned} \lim_{z\to 0}\frac{f(z)-f(0)}{z} &= \lim_{z\to 0}\frac{\frac{z}{|z|^2}\mathbb{Re}\{z\}\,\mathbb{Im}\{z\}}{z} \\ &= \lim_{z\to 0}\frac{\mathbb{Re}\{z\}\,\mathbb{Im}\{z\}}{|z|^2} \\ &= \lim_{(x,y)\to(0,0)}\frac{xy}{x^2+y^2}\,. \end{aligned}$$

The latter pre-limit, however, does *not* exist. Indeed, along $y=0, x>0$ it equals zero. However, along $x=y=t$ we get

$$\frac{t^2}{2t^2} \quad = \quad \frac{1}{2}$$

so the limit clearly exists and equals $\frac{1}{2}$. Since the result depends on the path of approach, the limit cannot exist.

Curiously, the function actually does obey (4.4) at the origin $(x,y)=(0,0)$. Indeed, we have $f_R(x,y) = \frac{x^2y}{x^2+y^2}$ and $f_I(x,y) = \frac{xy^2}{x^2+y^2}$. Then

$$(\partial_x f_R)(x,y) \quad = \quad \frac{2xy}{x^2+y^2} - \frac{x^2y}{(x^2+y^2)^2}2x$$

and

$$(\partial_y f_I)(x,y) \quad = \quad \frac{2xy}{x^2+y^2} - \frac{xy^2}{(x^2+y^2)^2}2y\,.$$

Similarly for the other two derivatives. □

Another perspective on the Cauchy-Riemann equations in terms of differentials Let us write

$$f(z) \quad := \quad F(x,y) + \mathrm{i}G(x,y)$$

for $F,G:\mathbb{R}^2\to\mathbb{R}$. Then for $\varepsilon_R,\varepsilon_I>0$ we have

$$\begin{aligned} f(z+\varepsilon_R+\mathrm{i}\varepsilon_I) - f(z) &= F(x+\varepsilon_R, y+\varepsilon_I) + \mathrm{i}G(x+\varepsilon_R,y+\varepsilon_I) - F(x,y) - \mathrm{i}G(x,y) \\ &= (\partial_x F)(z)\,\varepsilon_R + (\partial_y F)(z)\,\varepsilon_I + \mathrm{i}(\partial_x G)(z)\,\varepsilon_R + \mathrm{i}(\partial_y G)(z)\,\varepsilon_I + \mathcal{O}\left(\varepsilon_R^2,\varepsilon_I^2\right) \\ &= (\partial_x f)\,\varepsilon_R + (\partial_y f)\,\varepsilon_I + \mathcal{O}\left(\varepsilon_R^2,\varepsilon_I^2\right)\,. \end{aligned}$$

If we now define

$$\begin{aligned}\Delta z &:= \varepsilon_R + \mathrm{i}\varepsilon_I\\ \Delta\overline{z} &:= \varepsilon_R - \mathrm{i}\varepsilon_I\end{aligned}$$

we get

$$\begin{aligned}\varepsilon_R &= \frac{1}{2}\left(\Delta z + \Delta\overline{z}\right)\\ \varepsilon_I &= \frac{1}{2\mathrm{i}}\left(\Delta z - \Delta\overline{z}\right)\end{aligned}$$

with which we get

$$\begin{aligned}\Delta f &\equiv f\left(z + \varepsilon_R + \mathrm{i}\varepsilon_I\right) - f\left(z\right)\\ &= \left(\partial_x f\right)\varepsilon_R + \left(\partial_y f\right)\varepsilon_I + \mathcal{O}\left(\varepsilon_R^2, \varepsilon_I^2\right)\\ &= \left(\partial_x f\right)\frac{1}{2}\left(\Delta z + \Delta\overline{z}\right) + \left(\partial_y f\right)\frac{1}{2\mathrm{i}}\left(\Delta z - \Delta\overline{z}\right) + \mathcal{O}\left(\varepsilon_R^2, \varepsilon_I^2\right)\\ &= \left[\left(\partial_x - \mathrm{i}\partial_y\right) f\right]\Delta z + \left[\left(\partial_x + \mathrm{i}\partial_y\right) f\right]\Delta\overline{z}\\ &= 2\left(\partial_z f\right)\Delta z + 2\left(\partial_{\overline{z}} f\right)\Delta\overline{z}\,.\end{aligned}$$

In conclusion, we see that f is $\mathbb{C}$-differentiable if the change in Δf "comes" only from Δz but not from $\Delta\overline{z}$.

4.4 Harmonic functions and conjugate pairs

The Laplacian $-\Delta$ is a well-known operator on functions (i.e. it *linearly* maps functions into functions) defined as follows. Let $F : \mathbb{R}^2 \to \mathbb{R}$ be given. We denote its arguments as x, y as usual and define a new function, $(-\Delta F) : \mathbb{R}^2 \to \mathbb{R}$ via

$$\left(-\Delta F\right)(x, y) \quad := \quad \left(\partial_x^2 F\right)(x, y) + \left(\partial_y^2 F\right)(x, y) \qquad \left(x, y \in \mathbb{R}^2\right)\,.$$

This definition at least a-priori requires F to have second partial derivatives though this can be weakened by the use of so-called weak derivatives. We avoid doing so. In fact, there is an entire discussion to be had about just what kind of functions f the Laplacian $-\Delta$ is allowed to act on, but it belongs to a class on functional analysis. Instead, we go straight ahead to the

Definition 4.14 (Harmonic function)**.** A function $F : \mathbb{R}^2 \to \mathbb{R}$ is said to be *harmonic* iff

$$-\Delta F = 0.$$

Proposition 4.15. *$F : \mathbb{R}^2 \to \mathbb{R}$ is harmonic iff $\exists f : \mathbb{C} \to \mathbb{C}$ which is $\mathbb{C}$-differentiable such that $F = f_R$ or $F = f_I$.*

Proof. Let $f : \mathbb{C} \to \mathbb{C}$ be $\mathbb{C}$-differentiable, which implies it obeys the Cauchy-Riemann equations:

$$\begin{cases}\partial_x f_R & = \partial_y f_I\\ \partial_x f_I & = -\partial_y f_R\end{cases}.$$

Applying ∂_x on the first equation, and ∂_y on the second equation we find

$$\begin{cases}\partial_x^2 f_R & = \partial_x\partial_y f_I\\ \partial_y\partial_x f_I & = -\partial_y^2 f_R\end{cases}$$

and adding the two equations we get

$$-\Delta f_R \quad = \quad \left(\partial_x\partial_y - \partial_y\partial_x\right) f_I\,.$$

Now according to Schwarz's (or Clairaut's) theorem ([Rud76, Theorem 9.41]) if f_I has continuous second partial derivatives then $\left(\partial_x\partial_y - \partial_y\partial_x\right) f_I = 0$ so that we find f_R is harmonic. Later on [TODO: cite] we will see that in fact any $\mathbb{C}$-differentiable function f is *smooth* so that in particular Schwarz's condition is satisfied. In a similar manner we can prove that $-\Delta f_I = 0$ as well.

Conversely, let $F : \mathbb{R}^2 \to \mathbb{R}$ be harmonic. We are seeking to *construct* some $\mathbb{C}$-differentiable $f : \mathbb{C} \to \mathbb{C}$ so that $F = f_R$. Since we are defining $f_R := F$, we are really asking for some differentiable $f_I : \mathbb{C} \to \mathbb{R}$ such that the

Cauchy-Riemann equations are satisfied. I.e., in terms of multivariable calculus, given harmonic $F:\mathbb{R}^2\to\mathbb{R}$, we are seeking some $G:\mathbb{R}^2\to\mathbb{R}$ such that

$$\begin{cases}\partial_x F &= \partial_y G\\ \partial_x G &= -\partial_y F\end{cases}.$$

Rewriting this in vector form these equations are equivalent to

$$\operatorname{grad}(G) = \begin{bmatrix}-\partial_y F\\ \partial_x F\end{bmatrix} =: V \tag{4.6}$$

where we have defined a new vector field $V:\mathbb{R}^2\to\mathbb{R}^2$. When is there a solution G to the partial differential equation (4.6)? Precisely when the vector field V is conservative, which is defined as the property that the line integral of V over some path $\gamma:[0,1]\to\mathbb{R}^2$,

$$I(V,\gamma) := \int_0^1 \langle V(\gamma(t)), \dot\gamma(t)\rangle_{\mathbb{R}^2}\,\mathrm{d}t$$

is independent of the path γ except through its end points $\gamma(0),\gamma(1)$. This line integral would precisely define G: (1) it induces a well defined function $\mathbb{R}^2\ni\gamma(1)\mapsto I(V,\gamma)\in\mathbb{R}$ up to the choice of $\gamma(0)$; we define G to be that function; and (2) it satisfies (4.6). To show that, we must take the gradient of G and find it equals to V. This amounts to the fundamental theorem of line integrals. We need to differentiate $I(V,\gamma)$ with respect to the ending point $\gamma(1)$. Since we are free to choose any path we want (V is conservative), consider then a new path $\tilde\gamma:[0,1+\varepsilon]$ which ends at $\tilde\gamma(1+\varepsilon):=\gamma(1)+\varepsilon e_i$, goes from $\gamma(1)$ to its end point on a straight line, and otherwise has $\tilde\gamma(t)=\gamma(t)$ for $t\in[0,1]$. We then have by the definition of the partial derivative:

$$\begin{aligned}
(\partial_i G)(\gamma(1)) &= \lim_{\varepsilon\to0^+}\frac{I(V,\tilde\gamma)-I(V,\gamma)}{\varepsilon}\\
&= \lim_{\varepsilon\to0^+}\frac{\int_0^{1+\varepsilon}\left\langle V(\tilde\gamma(t)),\dot{\tilde\gamma}(t)\right\rangle\mathrm{d}t-\int_0^1\langle V(\gamma(t)),\dot\gamma(t)\rangle\,\mathrm{d}t}{\varepsilon}\\
&= \lim_{\varepsilon\to0^+}\frac{\int_1^{1+\varepsilon}\langle V(\gamma(1)+te_i),e_i\rangle\,\mathrm{d}t}{\varepsilon}\\
&= \lim_{\varepsilon\to0^+}\frac{\int_1^{1+\varepsilon}\left\langle V(\gamma(1))+(\mathscr{D}V)(\gamma(1))\,te_i+\mathcal{O}\left(t^2\right),e_i\right\rangle\mathrm{d}t}{\varepsilon}\\
&= \lim_{\varepsilon\to0^+}\left[V_i(\gamma(1))+(\mathscr{D}V)(\gamma(1))_{ii}\,\varepsilon+\mathcal{O}\left(\varepsilon^2\right)\right]\\
&= V_i(\gamma(1))
\end{aligned}$$

which is equivalent to $\operatorname{grad}(G)=V$. Note that we have used the estimate $\int_1^{1+\varepsilon}g(t)\,\mathrm{d}t\approx\varepsilon g(1)+\mathcal{O}\left(\varepsilon^2\right)$ appropriate for continuous g.

In conclusion, to define G at $z\in\mathbb{C}$ when V is conservative, we pick some reference point $z_0\in\mathbb{C}$ and *any* path $\gamma:[0,1]\to\mathbb{C}$ which obeys $\gamma(0)=z_0$ and $\gamma(1)=z$ to get

$$\begin{aligned}
G(z) &= G(z_0)+\int_0^1\langle V(\gamma(t)),\dot\gamma(t)\rangle\,\mathrm{d}t\\
&= G(z_0)+\int_0^1\left\langle\begin{bmatrix}-\partial_y F\\ \partial_x F\end{bmatrix},\dot\gamma(t)\right\rangle\mathrm{d}t\\
&= G(z_0)+\int_0^1\left((\partial_x F)(\gamma(t))\,\dot\gamma_y(t)-(\partial_y F)(\gamma(t))\,\dot\gamma_x(t)\right)\mathrm{d}t\,.
\end{aligned}$$

We note that this definition is only unique up to a constant, $G(z_0)$.

Next, we ask why should our particular V be conservative? One useful criterion is presented in Lemma 4.23 below. It says that if $\operatorname{curl}(V)=0$ then V is conservative. Let us calculate the curl of V:

$$\begin{aligned}
\operatorname{curl}(V) &= \partial_x V_y-\partial_y V_x\\
&= \partial_x^2 F+\partial_y^2 F\\
&= 0\,. && \text{(-}\Delta\text{F=0 by assumption of harmonic function)}
\end{aligned}$$

So it seems that harmonic scalar fields F induce vector fields $V \equiv \begin{bmatrix} -\partial_y F \\ \partial_x F \end{bmatrix}$ which are curl-free. ☐

Hence to complete the proof we must explain why $\operatorname{curl}(V) = 0$ implies that V is conservative, which is what the rest of this chapter is devoted to.

Given a harmonic function $F : \mathbb{R}^2 \to \mathbb{R}$, the harmonic function $G : \mathbb{R}^2 \to \mathbb{R}$ which we have defined as

$$G(x, y) := \int_0^1 \left(\left(\partial_x F\right)(\gamma(t))\, \dot{\gamma}_y(t) - \left(\partial_y F\right)(\gamma(t))\, \dot{\gamma}_x(t) \right) \mathrm{d}t \tag{4.7}$$

for *any* differentiable path $\gamma : [0, 1] \to \mathbb{R}^2$ such that $\gamma(1) = (x, y)$ is called *a harmonic conjugate* of F. It is unique up to an additive constant (and in particular independent of γ except through its end points).

4.5 Connectedness and path-connectedness

To complete the proof of Proposition 4.15 we need *even more* topology. We phrase most of the statements in terms of the ambient space $\mathbb{R}^2$ with its Euclidean (open ball) standard topology, but how to make more general statements should be clear.

Definition 4.16 (path-connectedness)**.** A subset $\Omega \subseteq \mathbb{R}^2$ is *path-connected* iff for any two points $x, y \in \Omega$, there is a path between them passing in Ω, i.e., there is a continuous map $\gamma : [0, 1] \to \Omega$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

This notion should be compared with the usual notion in topology of connectedness:

Definition 4.17 (connectedness)**.** A subset $\Omega \subseteq \mathbb{R}^2$ is connected iff it is *not* the union of two non-empty disjoint open sets.

Lemma 4.18. *A subset $\Omega \subseteq \mathbb{R}^2$ is connected iff*

$$\operatorname{Clopen}(\Omega) \quad = \quad \{ \Omega, \varnothing \}$$

iff any continuous map $\gamma : \Omega \to \{ 0, 1 \}$ is constant.

Lemma 4.19. *Path-connectedness implies connectedness but the converse is in principle false. The standard counter-example is the topologist's sine curve, see Figure 11.*

We omit the proofs which may be found in standard topology textbooks such as [Mun00]. We mention in passing that there is a notion called *locally-path-connected.* A space which is locally-path-connected and connected is in fact also path-connected. These distinctions are *well* beyond our scope and in fact, *we can just pretend that path-connected and connected are the same for us unless otherwise specified*!

Example 4.20. Here are a few examples:

1. The open ball $B_R(0)$ is path-connected (and hence connected) for any $R > 0$.
2. The union $B_1(0) \cup B_1(5)$ is not connected (and hence not path-connected).
3. The whole space $\mathbb{R}^2$ is path-connected.

Definition 4.21. A subset $\Omega \subseteq \mathbb{R}^2$ is called *simply-connected* iff it is path-connected and, in addition, if $\gamma, \tilde{\gamma} : [0, 1] \to \Omega$ are two continuous paths with $\gamma(t) = \tilde{\gamma}(t)$ for $t = 0, 1$, then there is a continuous deformation from γ to $\tilde{\gamma}$: there is a continuous map (a *homotopy*) $\Gamma : [0, 1]^2 \to \Omega$ (with arguments $(t, s) \in [0, 1]^2$, t being the path parameter and s being the homotopy parameter) such that:

1. Start from γ: $\Gamma(\cdot, s = 0) = \gamma$.
2. End with $\tilde{\gamma}$: $\Gamma(\cdot, s = 1) = \tilde{\gamma}$.
3. End points are fixed: $\Gamma(t, s) = \gamma(t)$ for all $t = 0, 1$ and for all $s \in [0, 1]$.

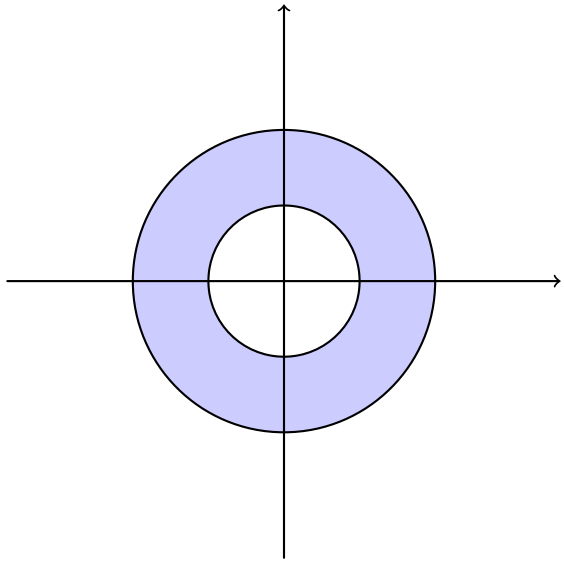

Figure 12: The annulus is *not* simply connected because it has a hole. Loops that wind around the hole cannot be deformed to loops that do not.

Intuitively we should think of simply-connectedness in two-dimensions as the absence of holes in our space. Any hole allows for a loop around it, and loops that go around the hole and those which do not cannot be equivalent. One might ask what is the relationship between equivalence classes of loops and the existence of a homotopy between any two given *paths*. The idea is that since any two given paths with the same end points form a loop, the existence of a homotopy between them means that the associated loop is "trivial", or in technical language, contractible.

Example 4.22. The whole space $\mathbb{R}^2$ is simply-connected. The ball $B_R(0)$ is simply-connected. The annulus

$$B_2(0) \setminus B_1(0)$$

is *not* simply connected; see Figure 12. The punctured plane $\mathbb{R}^2 \setminus \{ 0 \}$ is *not* simply-connected. Though we've been talking about subsets of $\mathbb{R}^2$, try to think whether $\mathbb{R}^3 \setminus \{ 0 \}$ is simply-connected based on classification of loops.

4.6 Poincaré's lemma in 2D

Lemma 4.23 (Poincaré's lemma for 1-forms in 2D)**.** *Let* $\Omega \subseteq \mathbb{R}^2$ *be a* simply-connected *subset and assume that a vector field* $V : \Omega \to \mathbb{R}^2$ *obeys*

$$\operatorname{curl}(V) \equiv \partial_x V_y - \partial_y V_x = 0\,.$$

Then V *is conservative, in the sense that* $V = \operatorname{grad}(G)$ *for some scalar field* $G : \mathbb{R}^2 \to \mathbb{R}$*, or, equivalently, the line integral of* V *is independent of the path chosen.*

More generally, this claim is related to the Helmholtz decomposition in $\mathbb{R}^3$, to the Hodge decomposition in any dimension and to the general Poincaré's lemma ("any closed form is exact in a simply connected manifold").

Proof of Lemma 4.23. The proof essentially comprises of proving that our earlier definition

$$G(z) \quad = \quad G(z_0) + \int_0^1 \langle V(\gamma(t)), \dot{\gamma}(t) \rangle \, \mathrm{d}t$$

is independent of the path $\gamma : [0,1] \to \Omega$ as long as it has $\gamma(0) = z_0$ and $\gamma(1) = z$. Hence, let $\gamma, \tilde{\gamma}$ be two such paths and compare G_γ and $G_{\tilde{\gamma}}$ (defined in an obvious way). Then

$$G_{\tilde{\gamma}}(z) - G_\gamma(z) \quad = \quad \int_0^1 \left\langle V \circ \tilde{\gamma}, \dot{\tilde{\gamma}} \right\rangle - \int_0^1 \langle V \circ \gamma, \dot{\gamma} \rangle \, \mathrm{d}t\,.$$

Now, since Ω is simply-connected, we are guaranteed the existence of a *continuous* map $\Gamma : [0,1]^2 \to \Omega$ (a homotopy) such that $\Gamma(\cdot,0) = \gamma$ and $\Gamma(\cdot,1) = \tilde{\gamma}$. In principle now there should be a song and dance about how given such a Γ we could always find a twice-differentiable homotopy based on Γ. Let us sweep that detail under the rug and march on. Let us denote by t the usual path parameter and by s the homotopy parameter. Hence

$$\begin{aligned}
G_{\tilde{\gamma}}(z) - G_{\gamma}(z) &= \int_0^1 \langle V(\Gamma(t,1)), (\partial_t\Gamma)(t,1)\rangle\, \mathrm{d}t - \int_0^1 \langle V(\Gamma(t,0)), (\partial_t\Gamma)(t,0)\rangle\, \mathrm{d}t \\
&= \int_0^1 \partial_s \left[\int_0^1 \langle V(\Gamma(t,s)), (\partial_t\Gamma)(t,s)\rangle\, \mathrm{d}t\right] \mathrm{d}s \\
&\overset{\text{Fubini and Leibniz}}{=} \int_0^1 \int_0^1 \partial_s \langle V(\Gamma(t,s)), (\partial_t\Gamma)(t,s)\rangle\, \mathrm{d}t\mathrm{d}s\,.
\end{aligned}$$

Here by Leibniz we mean the Leibniz integral rule.

Let us carry out the derivative in the integrand using the chain rule:

$$\begin{aligned}
\partial_s \langle V(\Gamma(t,s)), (\partial_t\Gamma)(t,s)\rangle &\equiv \partial_s \sum_{i=1}^{2} V_i(\Gamma(t,s))(\partial_t\Gamma_i)(t,s) \\
&= \sum_{i=1}^{2} V_i(\Gamma(t,s))(\partial_s\partial_t\Gamma_i)(t,s) + \sum_{i=1}^{2}\sum_{j=1}^{2} (\partial_j V_i)(\Gamma(t,s))(\partial_s\Gamma_j)(t,s)(\partial_t\Gamma_i)(t,s) \\
&= \sum_{i=1}^{2} V_i(\Gamma(t,s))(\partial_s\partial_t\Gamma_i)(t,s) + \sum_{i=1}^{2}\sum_{j=1}^{2} \mathrm{Jacobian}(V)_{ij}(\Gamma(t,s))(\partial_s\Gamma_j)(t,s)(\partial_t\Gamma_i)(t,s) \\
&= \langle V(\Gamma(t,s)), (\partial_s\partial_t\Gamma)(t,s)\rangle + \langle(\partial_t\Gamma)(t,s), \mathrm{Jacobian}(V)(\Gamma(t,s))(\partial_s\Gamma)(t,s)\rangle
\end{aligned}$$

where we have identified

$$\mathrm{Jacobian}(V) \equiv \begin{bmatrix} \partial_1 V_1 & \partial_1 V_2 \\ \partial_2 V_1 & \partial_2 V_2 \end{bmatrix}.$$

Running the same calculation with switching the roles of $t \leftrightarrow s$ is simple and yields

$$\partial_t \langle V(\Gamma(t,s)), (\partial_s\Gamma)(t,s)\rangle = \langle V(\Gamma(t,s)), (\partial_t\partial_s\Gamma)(t,s)\rangle + \langle(\partial_s\Gamma)(t,s), \mathrm{Jacobian}(V)(\Gamma(t,s))(\partial_t\Gamma)(t,s)\rangle\,.$$

Now, we note two important points:

1. $\partial_s\partial_t\Gamma = \partial_t\partial_s\Gamma$ for any reasonable Γ we would pick.
2. $\mathrm{Jacobian}(V)$ is a symmetric 2×2 matrix because of the zero curl condition. Indeed, the zero curl condition is equivalent to $\partial_x V_y = \partial_y V_x$ which is the symmetry of the Jacobian matrix.

These two points imply the following important consequence:

$$\partial_s \langle V(\Gamma(t,s)), (\partial_t\Gamma)(t,s)\rangle = \partial_t \langle V(\Gamma(t,s)), (\partial_s\Gamma)(t,s)\rangle\,. \tag{4.8}$$

Plugging this back into $G_{\tilde{\gamma}}(z) - G_{\gamma}(z)$ we find

$$\begin{aligned}
G_{\tilde{\gamma}}(z) - G_{\gamma}(z) &\overset{(4.8)}{=} \int_0^1 \int_0^1 \partial_t \langle V(\Gamma(t,s)), (\partial_s\Gamma)(t,s)\rangle\, \mathrm{d}t\mathrm{d}s \\
&\overset{\text{fund. thm. of calc. on dt}}{=} \int_0^1 \langle V(\Gamma(1,s)), (\partial_s\Gamma)(1,s)\rangle\, \mathrm{d}s - \int_0^1 \langle V(\Gamma(0,s)), (\partial_s\Gamma)(0,s)\rangle\, \mathrm{d}s\,.
\end{aligned}$$

These last two integrals, however, are manifestly zero, because the homotopy *must* keep the starting point and ending points fixed throughout the homotopy, which implies $(\partial_s\Gamma)(1,s) = (\partial_s\Gamma)(0,s) = 0$. This completes the proof that G is well-defined if V is curl-free and Ω is simply-connected. □

4.7 Some examples of harmonic conjugates

We conclude this chapter with a few more examples of harmonic functions.

Example 4.24. The function $F:\mathbb{R}^2\to\mathbb{R}$ given by $F(x,y)=x^2-y^2$ is harmonic. Its harmonic conjugates are

$$G(x,y) = 2xy+C$$

where C is any real constant.

Proof. We may calculate $-\Delta F=0$ easily:

$$\begin{aligned}(\partial_x F)(x,y) &= 2x\\ (\partial_y F)(x,y) &= -2y\,.\end{aligned}$$

Now we have the definition from (4.7) given by

$$\begin{aligned}G(x,y) &= \int_0^1 \left((\partial_x F)(\gamma(t))\,\dot\gamma_y(t)-(\partial_y F)(\gamma(t))\,\dot\gamma_x(t)\right)\mathrm{d}t\\ &= 2\int_0^1\left(\gamma_x(t)\,\dot\gamma_y(t)+\gamma_y(t)\,\dot\gamma_x(t)\right)\mathrm{d}t\,.\end{aligned}$$

For example, let us take a path $\gamma:[0,1]\to\mathbb{R}^2$ which goes first horizontally and then vertically from (x_0,y_0) to (x,y). This yields

$$\begin{aligned}\gamma(t) &= \begin{bmatrix}2\left(\frac{1}{2}-t\right)x_0+2tx\\ y_0\end{bmatrix}\chi_{[0,\frac{1}{2}]}(t)+\begin{bmatrix}x\\ 2(1-t)\,y_0+2\left(t-\frac{1}{2}\right)y\end{bmatrix}\chi_{[\frac{1}{2},1]}(t)\\ \dot\gamma(t) &= 2\begin{bmatrix}x-x_0\\ 0\end{bmatrix}\chi_{[0,\frac{1}{2}]}(t)+2\begin{bmatrix}0\\ y-y_0\end{bmatrix}\chi_{[\frac{1}{2},1]}(t)\,.\end{aligned}$$

Plugging this into G we find

$$\begin{aligned}G(x,y) &= 2\int_{\frac{1}{2}}^1 x\cdot 2(y-y_0)\,\mathrm{d}t+2\int_0^{\frac{1}{2}}\left(y_0\cdot 2(x-x_0)\right)\mathrm{d}t\\ &= 2x(y-y_0)+2y_0(x-x_0)\\ &= 2xy-2xy_0+2y_0x-2y_0x_0\\ &= 2xy-2x_0y_0\,.\end{aligned}$$

One may verify that $-\Delta G=0$. The corresponding analytic function is

$$\begin{aligned}f(x+\mathrm{i}y) &= x^2-y^2+2\mathrm{i}xy\\ &= (x+\mathrm{i}y)^2\\ &= z^2\,.\end{aligned}$$

□

Example 4.25. The function $F(x,y)=x^3-3xy^2$ is harmonic on $\mathbb{C}$ and one of its harmonic conjugates is $G(x,y)=3x^2y-y^3$.

Proof. Consider the function $f(z)=z^3$ and its real and imaginary parts. □

Example 4.26. The function $F(x,y)=\log\left(\sqrt{x^2+y^2}\right)$ is harmonic on $\{\,z\in\mathbb{C}\mid x>0\,\}$, and one of its harmonic conjugates is $G(x,y)=\arctan(y/x)$. If one were to use the domain $\mathbb{C}\setminus\{\,0\,\}$, there would not be a harmonic conjugate, which implies that there isn't an analytic function whose real part is F defined on $\mathbb{C}\setminus\{\,0\,\}$.

Proof. Rewriting $F(x,y) = \frac{1}{2}\log\left(x^2+y^2\right)$ we have

$$\begin{aligned}(\partial_x F)(x,y) &= \frac{1}{2}\frac{2x}{x^2+y^2}\\ (\partial_y F)(x,y) &= \frac{1}{2}\frac{2y}{x^2+y^2}\end{aligned}$$

so that

$$\begin{aligned}\left(\partial_x^2 F\right)(x,y) &= \frac{1}{x^2+y^2} - \frac{x}{\left(x^2+y^2\right)^2}2x = \frac{y^2-x^2}{\left(x^2+y^2\right)^2}\\ \left(\partial_y^2 F\right)(x,y) &= \frac{1}{x^2+y^2} - \frac{y}{\left(x^2+y^2\right)^2}2y = \frac{x^2-y^2}{\left(x^2+y^2\right)^2}\end{aligned}$$

so that $-\Delta F = 0$ indeed. To find the harmonic conjugate we use again (4.7) to get

$$\begin{aligned}G(x,y) &= \int_0^1 \left(\left(\partial_x F\right)(\gamma(t))\,\dot{\gamma}_y(t) - \left(\partial_y F\right)(\gamma(t))\,\dot{\gamma}_x(t)\right)\mathrm{d}t\\ &= \int_0^1 \left(\frac{\gamma_x(t)}{\|\gamma(t)\|^2}\dot{\gamma}_y(t) - \frac{\gamma_y(t)}{\|\gamma(t)\|^2}\dot{\gamma}_x(t)\right)\mathrm{d}t\,.\end{aligned}$$

At this point it would be easier to switch the path to polar coordinates:

$$\gamma(t) = \gamma_r(t)\,\mathrm{e}^{\mathrm{i}\gamma_\theta(t)}$$

so that $\|\gamma\|^2 = {\gamma_r}^2$ and so

$$\gamma_x\dot{\gamma}_y = \gamma_r^2\cos(\gamma_\theta)^2\,\dot{\gamma}_\theta$$

yet

$$\gamma_y\dot{\gamma}_x = -\gamma_r^2\sin(\gamma_\theta)^2\,\dot{\gamma}_\theta$$

with which we find

$$\begin{aligned}G(x,y) &= \int_0^1 \dot{\gamma}_\theta\\ &= \gamma_\theta(1) - \gamma_\theta(0)\\ &= \arctan\left(\frac{y}{x}\right) - \arctan\left(\frac{y_0}{x_0}\right)\\ &= \arg(z) - \arg(z_0)\ .\end{aligned}$$

Note that since we are working in the half-plane $\{\, z\in\mathbb{C} \mid x>0 \,\}$ (which is simply-connected) this calculation was kosher to begin with. Furthermore, $\arg(z)$ is only defined up to multiples of 2π, and the harmonic conjugate is only defined up to an additive constant (in this case $2\pi n$ for $n\in\mathbb{Z}$). Conversely, $\mathbb{C}\setminus\{\,0\,\}$ is *not* simply connected, and this is made abundantly clear in this particular example: there *is* a dependence on the winding of the path in the expression $\gamma_\theta(1)-\gamma_\theta(0)$.

We will see below that the corresponding analytic function is

$$f(z) = \log(|z|) + \mathrm{i}\,\mathrm{Arg}(z)$$

as long as we are working on simply-connected domains within $\mathbb{C}$. □

Claim 4.27. Let $\Omega \subseteq \mathbb{C}$ be simply-connected and $f : \Omega \to \mathbb{C}$ be $\mathbb{C}$-differentiable. If $f(\Omega)$ is contained within a straight line, i.e., if there exist $a, b, c \in \mathbb{R}$ with not both $a, b = 0$ such that

$$a f_R(x, y) + b f_I(x, y) = c \qquad (x, y \in \Omega)$$

then f is the constant function.

5 Basic functions

5.1 The logarithm

Recall the exponential function we discussed in Definition 1.5:

$$\exp(z) \quad \equiv \quad \mathrm{e}^x (\cos(y) + \mathrm{i}\sin(y)) \ .$$

It enjoys the following properties which may be derived from Definition 1.5:

1. It is $\mathbb{C}$-differentiable and its derivative is itself: $\exp' = \exp$.
2. It obeys the exponential law: $\exp(z_1 + z_2) = \exp(z_1)\exp(z_2)$ for all $z_1, z_2 \in \mathbb{C}$.
3. $|\exp(z)| = \mathrm{e}^x$ for $z = x + \mathrm{i}y$.
4. It is periodic in vertical strips of the complex plane: $\exp(z + 2\pi\mathrm{i}) = \exp(z)$ for all $z \in \mathbb{C}$.

Let us prove just the derivative claim:

Claim 5.1. $\exp : \mathbb{C} \to \mathbb{C}$ is holomorphic.

Proof. Clearly the real and imaginary parts of exp are real differentiable, so we only verify that the CRE hold. To that end,

$$\begin{aligned}(\partial_x f_R)(x, y) \quad &= \quad \mathrm{e}^x \cos(y) \\ (\partial_y f_I)(x, y) \quad &= \quad \mathrm{e}^x \cos(y)\end{aligned}$$

and similarly for the other equation. Now that we know that the CRE hold we conclude that the function is differentiable and use (4.5) to calculate the derivative:

$$\begin{aligned}\exp'(z) \quad &= \quad (\partial_x f_R)(x, y) - \mathrm{i}(\partial_y f_R)(x, y) \\ &= \quad \mathrm{e}^x \cos(y) - \mathrm{i}[-\mathrm{e}^x \sin(y)] \\ &= \quad \mathrm{e}^x (\cos(y) + \mathrm{i}\sin(y)) \\ &= \quad \exp(z) \ .\end{aligned}$$

□

Now that we have the exponential, to introduce the logarithm, let us attempt to solve

$$\mathrm{e}^z \quad = \quad a + \mathrm{i}b$$

for the unknown $z = x + \mathrm{i}y \in \mathbb{C}$:

$$\begin{aligned}\mathrm{e}^x (\cos(y) + \mathrm{i}\sin(y)) \quad &= \quad a + \mathrm{i}b \\ &\updownarrow \\ \mathrm{e}^x \cos(y) \quad &= \quad a \\ \mathrm{e}^x \sin(y) \quad &= \quad b \, .\end{aligned}$$

Divide the two equations to get

$$\tan(y) = \frac{b}{a}$$

and conversely taking the sum of squares of the two equations yields

$$e^{2x} = a^2 + b^2 .$$

We find

$$\begin{aligned} x &= \log\left(\sqrt{a^2+b^2}\right) \\ y &= \arctan\left(\frac{b}{a}\right) . \end{aligned}$$

As remarked already in Definition 1.6, some special care has to be given to the arctan as discussed in Appendix C. We make the

Definition 5.2 (Complex logarithm)**.** We define the "function" $\log : \mathbb{C} \to \mathbb{C}$ via

$$\log(z) := \log(|z|) + \mathrm{i}\arg(z) \qquad (z \in \mathbb{C}) .$$

The reason why the word "function" appears in quotation marks is because to define log we are using arg (from Definition 1.6) which is a *multi-valued function*: for any given $z \in \mathbb{C}$, $\arg(z)$ is actually a *set* of values given by

$$\arg(z) = \operatorname{Arg}(z) + 2\pi\mathbb{Z}$$

with $\operatorname{Arg}(z)$ the *principal* argument, which is defined to be $\operatorname{atan2}(y, x)$ taking values in $(-\pi, \pi]$ (see Definition C.1). Like the principal argument Arg, we also define the principal logarithm:

$$\operatorname{Log}(z) := \log(|z|) + \mathrm{i}\operatorname{Arg}(z) .$$

We note in passing that

$$\begin{aligned} \log(e^z) &= \log(e^x) + \mathrm{i}\arg\left(e^x e^{\mathrm{i}y}\right) \\ &= \log(e^x) + \mathrm{i}\arg\left(e^{\mathrm{i}y}\right) \\ &= x + \mathrm{i}y + 2\pi\mathrm{i}\mathbb{Z} \\ &= z + 2\pi\mathrm{i}\mathbb{Z}\,! \end{aligned}$$

On the other hand

$$\operatorname{Log}(e^z) = x + \mathrm{i}(y \mod 2\pi)$$

in such a way that $(y \mod 2\pi) \in [-\pi, \pi]$, so neither log nor Log are true left inverses of exp, with the special case that

$$\operatorname{Log}(e^z) = z \qquad (z \in \mathbb{C} : \operatorname{\mathbb{Im}}\{z\} \in (-\pi, \pi]) .$$

On the other hand,

$$\begin{aligned} e^{\log(z)} &= e^{\log(|z|)+\mathrm{i}\arg(z)} \\ &= |z|\, e^{\mathrm{i}\arg(z)} \\ &= z \end{aligned}$$

because even though $\arg(z)$ yields many values, within the exponent it simply does not matter and we get back what we started with. So exp is a true left inverse of log and also of Log.

Example 5.3. We note the following special case:

1. $\operatorname{Log}(1) = 0$ but $\log(1) = 2\pi\mathrm{i}\mathbb{Z}$.
2. $\operatorname{Log}(-1) = \mathrm{i}\pi$ but $\log(-1) = \mathrm{i}\arg(-1) = \mathrm{i}(\pi + 2\pi\mathbb{Z}) = \mathrm{i}\pi(2\mathbb{Z}+1)$.

One should be cautious that unlike for real positive values, for complex numbers:

$$\operatorname{Log}(z_1 z_2) \neq \operatorname{Log}(z_1) + \operatorname{Log}(z_2)$$

in general, and the two sides of the equation may differ by a multiple of $2\pi\mathrm{i}$.

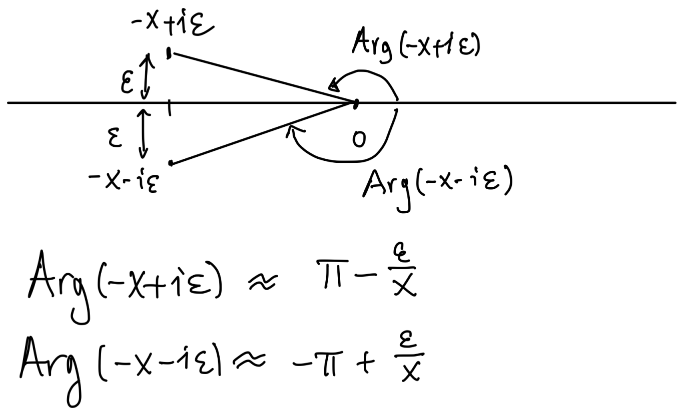

Figure 13: The jump discontinuity of Log along its branch cut.

Example 5.4. Take the following special values:

1. $\operatorname{Log}(\mathrm{i}\cdot(-1)) = \operatorname{Log}(-\mathrm{i}) = -\frac{\pi}{2}\mathrm{i}$. On the other hand,

$$\operatorname{Log}(\mathrm{i}) + \operatorname{Log}(-1) \quad = \quad \mathrm{i}\frac{\pi}{2} + \mathrm{i}\pi = \frac{3\pi}{2}\mathrm{i} \neq -\frac{\pi}{2}\mathrm{i}\,.$$

Definition 5.5 (Choice of branch)**.** Instead of choosing $\operatorname{Arg}(z) \in (-\pi, \pi]$, we could actually choose any $\alpha \in \mathbb{R}$ and define

$$\operatorname{Arg}_\alpha(z) \quad \in \quad (-\alpha, -\alpha + 2\pi]\ .$$

This is called a choice of *branch* for the multivalued argument function and hence for the logarithm as well. The choice

$$\operatorname{Arg} \quad := \quad \operatorname{Arg}_\pi$$

is conventional, appears in [BC13], and we shall follow suit. The line

$$\mathrm{e}^{\mathrm{i}\alpha}[0, \infty)$$

is called a *branch cut* and the point mutual to all branch cuts (the origin) is called a *branch point.*

Since log is a multi-valued function, it does not immediately make sense to ask whether it is continuous. However, what about Log?

Claim 5.6. The principal complex logarithm Log is not continuous on the line

$$\{\, x \in \mathbb{C} \mid x \leq 0 \,\}\ .$$

Proof. For any point $x \geq 0$, let us study the limits

$$\operatorname{Log}(-x \pm \mathrm{i}\varepsilon)$$

as $\varepsilon \to 0^+$. We have

$$\begin{aligned}
\operatorname{Log}(-x \pm \mathrm{i}\varepsilon) \quad &= \quad \log\left(\sqrt{x^2+\varepsilon^2}\right) + \mathrm{i}\operatorname{Arg}(-x \pm \mathrm{i}\varepsilon) \\
&= \quad \log(|x|) + \log\left(\sqrt{1+\left(\frac{\varepsilon}{x}\right)^2}\right) + \mathrm{i}\operatorname{Arg}(-x \pm \mathrm{i}\varepsilon)\ .
\end{aligned}$$

The term $\log\left(\sqrt{1+\left(\frac{\varepsilon}{x}\right)^2}\right)$ is uninteresting in the limit and we ignore it. On the other hand, for the argument, we have

$$\mathrm{Arg}\left(-x\pm \mathrm{i}\varepsilon\right) \quad = \quad \begin{cases}\pi-\frac{\varepsilon}{x} & +\text{ version}\\ -\pi+\frac{\varepsilon}{x} & -\text{ version}\end{cases}$$

clearly from glancing at Figure 13. We find

$$\begin{aligned}\mathrm{Log}\left(-x\pm \mathrm{i}\varepsilon\right) &\approx \log\left(|x|\right)\pm \mathrm{i}\left(\pi-\frac{\varepsilon}{x}\right)+\mathcal{O}\left(\left(\frac{\varepsilon}{x}\right)^2\right)\\ &\overset{\varepsilon\to 0^+}{=} \log\left(|x|\right)\pm \mathrm{i}\pi\,.\end{aligned}$$

Since the putative limit depends on the direction of approach, it clearly does not exist. $\square$

Corollary 5.7. *The function* Log *is* not *holomorphic on* $\mathbb{C}$.

This demonstration along with the actual name branch *cut* suggestion what a solution might be: restrict the domain of Log to make it holomorphic. A conventional maximal choice is

$$\mathbb{C}\setminus\{\,x\in\mathbb{C}\mid x\leq 0\,\}$$

which is an open set since it is the complement of a closed set.

Lemma 5.8. *In fact, with this choice* Log *becomes holomorphic on*

$$\mathbb{C}\setminus\{\,x\in\mathbb{C}\mid x\leq 0\,\}$$

and

$$\mathrm{Log}_\alpha'\left(z\right) \quad = \quad \frac{1}{z}\,.$$

Proof. We demonstrate this by verifying the CRE (4.4). Let us do it for Log_α:

$$\begin{aligned}\partial_x \mathrm{Log}_\alpha\left(x+\mathrm{i}y\right) &= \partial_x \log\left(\sqrt{x^2+y^2}\right)+\mathrm{i}\partial_x \arctan_\alpha\left(\frac{y}{x}\right)\\ &= \frac{1}{\sqrt{x^2+y^2}}\frac{2x}{2\sqrt{x^2+y^2}}+\mathrm{i}\frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right)\\ &= \frac{x}{x^2+y^2}-\mathrm{i}\frac{y}{x^2+y^2}\end{aligned}$$

whereas

$$\begin{aligned}\partial_y \mathrm{Log}_\alpha\left(x+\mathrm{i}y\right) &= \frac{y}{x^2+y^2}+\mathrm{i}\frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{1}{x}\\ &= \frac{y}{x^2+y^2}+\mathrm{i}\frac{x}{x^2+y^2}\,.\end{aligned}$$

Hence we see that the Cauchy-Riemann equations hold for Log_α. From these equations we also see that

$$\begin{aligned}\mathrm{Log}_\alpha'\left(z\right) &= \left(\partial_x f_R\right)\left(z\right)-\mathrm{i}\left(\partial_y f_R\right)\left(z\right)\\ &= \frac{x}{x^2+y^2}-\mathrm{i}\frac{y}{x^2+y^2}\\ &= \frac{\overline{z}}{|z|^2}\\ &= \frac{1}{z}.\end{aligned}$$

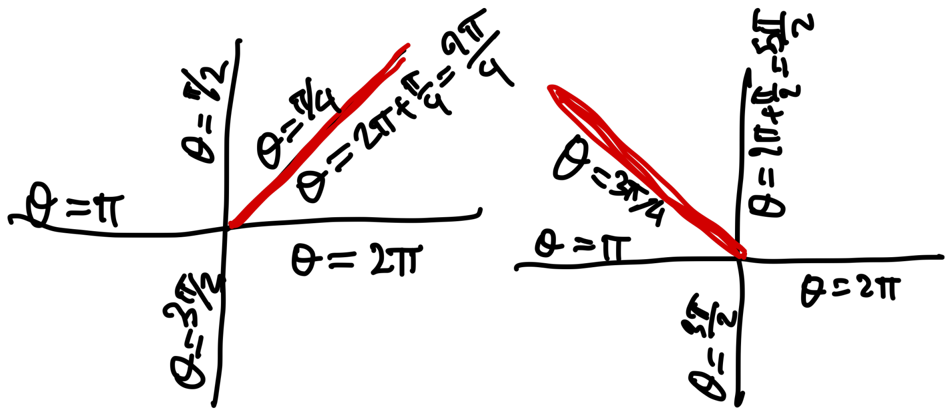

Figure 14: Another two choices of the branch cut.

$\square$

Example 5.9. If instead of $\alpha = \pi$ we pick the branch $\alpha = -\frac{\pi}{4}$ (see Figure 14), we get

$$\operatorname{Arg}_{-\frac{\pi}{4}}(z) \quad \in \quad \left(\frac{\pi}{4}, \frac{9\pi}{4}\right] .$$

Then

$$\begin{aligned}
\operatorname{Log}_{-\frac{\pi}{4}}\left(\mathrm{i}^2\right) &= \operatorname{Log}_{-\frac{\pi}{4}}(-1) \\
&= \log(|-1|) + \mathrm{i}\operatorname{Arg}_{-\frac{\pi}{4}}(-1) \\
&= \mathrm{i}\pi \in \left(\frac{\pi}{4}, \frac{9\pi}{4}\right]
\end{aligned}$$

and

$$2\operatorname{Log}_{-\frac{\pi}{4}}(\mathrm{i}) \quad = \quad 2\operatorname{Arg}_{-\frac{\pi}{4}}(\mathrm{i}) = 2\mathrm{i}\frac{\pi}{2} = \mathrm{i}\pi \in \left(\frac{\pi}{4}, \frac{9\pi}{4}\right] .$$

On the other hand, with $\alpha = -\frac{3\pi}{4}$ (see Figure 14) we have

$$\begin{aligned}
\operatorname{Log}_{-\frac{3\pi}{4}}(-1) &= \mathrm{i}\pi \\
\operatorname{Log}_{-\frac{3\pi}{4}}(\mathrm{i}) &= \mathrm{i}\frac{5\pi}{2} .
\end{aligned}$$

Now, $2\operatorname{Log}_{-\frac{3\pi}{4}}(\mathrm{i}) = 5\pi$ and so

$$2\operatorname{Log}_{-\frac{3\pi}{4}}(\mathrm{i}) - \operatorname{Log}_{-\frac{3\pi}{4}}(-1) \quad = \quad 4\pi .$$

5.2 The power function

We have already encountered the power function if we raise a complex number to integer powers (which is defined via multiplication of complex numbers and the inverse of a complex number. We now extend this to *any* complex power

Definition 5.10 (The complex power)**.** Let $a + \mathrm{i}b \in \mathbb{C}$. Define a function $\mathbb{C} \setminus \{0\} \to \mathbb{C}$ via

$$z \quad \mapsto \quad z^{a+\mathrm{i}b} := \exp\left((a + \mathrm{i}b)\log(z)\right) .$$

Expanding a bit this definition, we get

$$\begin{aligned} z^{a+ib} &= \exp\left(\left(a+\mathrm{i}b\right)\left(\log\left(|z|\right)+\mathrm{i}\arg\left(z\right)\right)\right) \\ &= \exp\left(a\log\left(|z|\right)-b\arg\left(z\right)+\mathrm{i}\left(b\log\left(|z|\right)+a\arg\left(z\right)\right)\right) \\ &= |z|^{a}\,\mathrm{e}^{-b\arg(z)}\mathrm{e}^{\mathrm{i}(b\log(|z|)+a\arg(z))}\,. \end{aligned}$$

We see that since the power function is defined via log (which is defined via arg), it is *also multi-valued,* and so it is natural that there would be a

Definition 5.11 (The principal complex power)**.** Let $a+\mathrm{i}b\in\mathbb{C}$. The principal complex power is a function defined on $\mathbb{C}\setminus\{\,z\in\mathbb{C}\mid x<0,y=0\,\}$ via

$$z \mapsto z^{a+\mathrm{i}b}:=\exp\left(\left(a+\mathrm{i}b\right)\operatorname{Log}\left(z\right)\right)$$

where Log is the principal logarithm defined above.

Example 5.12. Let us consider the special case $a+\mathrm{i}b=n\in\mathbb{N}_{\geq 0}$. In this case, we get

$$\begin{aligned} z^{n} &= \mathrm{e}^{n\log(z)} \\ &= \mathrm{e}^{n(\log(|z|)+\mathrm{i}\arg(z))} \\ &= |z|^{n}\,\mathrm{e}^{n\mathrm{i}\arg(z)} \end{aligned}$$

which agrees with our previous definition via complex multiplication.

A more interesting example is

Example 5.13. Consider $\frac{1}{n}$ for $n\in\mathbb{N}_{>0}$. In this case we get

$$z^{\frac{1}{n}} = |z|^{\frac{1}{n}}\,\mathrm{e}^{\frac{1}{n}\mathrm{i}\arg(z)}\,.$$

Now, since $\arg\left(z\right)$ is many-valued with its valued differing by multiples of 2π, we see that once we take n multiples, we will get again integer multiples all together in the exponent. Hence, even though arg has *infinitely many values,* $z\mapsto z^{\frac{1}{n}}$ *has merely n values! They are*

$$|z|^{\frac{1}{n}}\left\{\mathrm{e}^{\frac{1}{n}\mathrm{i}\operatorname{Arg}(z)},\mathrm{e}^{\frac{1}{n}\mathrm{i}(\operatorname{Arg}(z)+2\pi)},\dots,\mathrm{e}^{\frac{1}{n}\mathrm{i}(\operatorname{Arg}(z)+(n-1)2\pi)}\right\}.$$

A more concrete

Example 5.14. For the power $\frac{2}{3}$ we have three possible values:

$$\begin{aligned} z^{\frac{2}{3}} &= \exp\left(\frac{2}{3}\log\left(z\right)\right) \\ &= |z|^{\frac{2}{3}}\left\{\exp\left(\mathrm{i}\frac{2}{3}\operatorname{Log}\left(z\right)\right),\exp\left(\mathrm{i}\frac{2}{3}\left(\operatorname{Log}\left(z\right)+2\pi\right)\right),\exp\left(\mathrm{i}\frac{2}{3}\left(\operatorname{Log}\left(z\right)+4\pi\right)\right)\right\}. \end{aligned}$$

Again one has to be careful that not always

$$\left(zw\right)^{\alpha} = z^{\alpha}w^{\alpha}$$

if one uses the the *principal* power:

Example 5.15. Consider $z=1-\mathrm{i}$ and $w=-1-\mathrm{i}$ with the power $\alpha=\mathrm{i}$ has, using the principal power

$$\left(zw\right)^{\mathrm{i}} = z^{\mathrm{i}}w^{\mathrm{i}}\mathrm{e}^{-2\pi}\,.$$

Proof. We have

$$\begin{aligned}(zw)^{\mathrm{i}} &= \exp\left(\mathrm{i}\,\mathrm{Log}\,(zw)\right)\\ &= \exp\left(\mathrm{i}\log\left(|zw|\right)-\mathrm{Arg}\,(zw)\right)\end{aligned}$$

and

$$z^{\mathrm{i}} = \exp\left(\mathrm{i}\log\left(|z|\right)-\mathrm{Arg}\,(z)\right)$$

so that

$$\frac{(zw)^{\mathrm{i}}}{z^{\mathrm{i}}w^{\mathrm{i}}} = \exp\left(-\mathrm{Arg}\,(zw)+\mathrm{Arg}\,(z)+\mathrm{Arg}\,(w)\right)\,.$$

However, now,

$$\mathrm{Arg}\,(z)=\mathrm{Arg}\,(1-\mathrm{i})=-\frac{\pi}{4}$$

and

$$\mathrm{Arg}\,(w)=\mathrm{Arg}\,(-1-\mathrm{i})=-\frac{3\pi}{4}$$

whereas

$$\mathrm{Arg}\,(zw) = \mathrm{Arg}\,((1-\mathrm{i})(-1-\mathrm{i}))=\mathrm{Arg}\,(-2)=\pi$$

and so

$$-\mathrm{Arg}\,(zw)+\mathrm{Arg}\,(z)+\mathrm{Arg}\,(w) = -\pi-\pi=-2\pi\,.$$

□

6 Complex integration

Let us begin our discussion of complex integration with the type of integrals we've already studied about in multivariable calculus. Clearly we know how to define integrals of functions

$$g:\mathbb{R} \to \mathbb{C}\,.$$

Indeed, such a function can always be written as $g=u+\mathrm{i}v$ for some $u,v:\mathbb{R}\to\mathbb{R}$ whence

$$\int_{t\in[a,b]} g(t)\,\mathrm{d}t \equiv \int_{t\in[a,b]} u(t)\,\mathrm{d}t+\mathrm{i}\int_{t\in[a,b]} v(t)\,\mathrm{d}t \tag{6.1}$$

assuming that u,v are separately integrable (say, Riemann integrable) on $[a,b]$. Said differently, taking real and imaginary parts and integrating, commute simply because the Riemann integral is linear. We clearly have the fundamental theorem of calculus also in this setting, in the sense that even if g is a complex-valued function of one real variable, it is easy to define its derivative

$$g':\mathbb{R} \to \mathbb{C}$$

as

$$g' \equiv u'+\mathrm{i}v'$$

and the fundamental theorem of calculus (applied individually on $\int u'$ and on $\int v'$) says

$$\int_{t\in[a,b]} g'(t)\,\mathrm{d}t = g(b)-g(a)\in\mathbb{C}\,. \tag{6.2}$$

In summary, if the domain is real, there isn't much new in complex analysis.

Let us protocol here one important property of this type of integral:

Proposition 6.1. *For any absolutely integrable $g : [a, b] \to \mathbb{C}$, we have*

$$\left|\int_a^b g\right| \quad \leq \quad \int_a^b |g| \ .$$

Proof. Since $\int_a^b g$ is just some complex number, write

$$\int_a^b g \quad = \quad \left|\int_a^b g\right| \mathrm{e}^{\mathrm{i}\theta}$$

for some $\theta \in \mathbb{R}$. Then

$$\begin{aligned}
\left|\int_a^b g\right| &= \mathrm{e}^{-\mathrm{i}\theta} \int_a^b g \\
&= \int_a^b \mathrm{e}^{-\mathrm{i}\theta} g(t)\, \mathrm{d}t && (\theta \text{ is a constant}) \\
&= \mathbb{Re}\left\{\int_a^b \mathrm{e}^{-\mathrm{i}\theta} g(t)\, \mathrm{d}t\right\} && (\text{integral is real}) \\
&= \int_a^b \mathbb{Re}\left\{\mathrm{e}^{-\mathrm{i}\theta} g(t)\right\} \mathrm{d}t && (\text{taking real part is linear}) \\
&\leq \int_a^b |g(t)|\, \mathrm{d}t
\end{aligned}$$

where in the last step we have used HW1Q6, which says that

$$|z| \quad = \quad \max_{\alpha \in (-\pi, \pi]} \mathbb{Re}\left\{\mathrm{e}^{\mathrm{i}\alpha} z\right\} \qquad (z \in \mathbb{C}) \ .$$

□

6.1 Contours

To start talking about integrals of functions of a *complex variable* we need the notion of contours.

Definition 6.2 (Contour)**.** A contour is a map $\gamma : [a, b] \to \mathbb{C}$ which is *continuous.*

1. *It is* simple *iff it is injective.*
2. *It is* simple-closed *iff it is injective except for its end points obeying $\gamma(a) = \gamma(b)$. In that case we say that the* interior *of* $\operatorname{im}(\gamma) \equiv \gamma([a, b])$ *is the set of all points encircled by γ, and denote that set of points by* $\operatorname{int}(\gamma)$ *(cf. the interior of a set, Definition 3.15; this now is the* interior of a contour*).*
3. *If it is simple-closed, it is furthermore positively-oriented iff it goes in counter-clockwise (CCW) direction (and negatively-oriented iff it goes in clockwise (CW) direction).*

Example 6.3. $\gamma : [0, 2\pi] \to \mathbb{C}$ defined by

$$t \quad \mapsto \quad \mathrm{e}^{\mathrm{i}t}$$

is a CCW oriented simple-closed contour. Conversely,

$$t \quad \mapsto \quad \mathrm{e}^{-\mathrm{i}t}$$

is CW oriented and

$$t \quad \mapsto \quad \mathrm{e}^{2\mathrm{i}t}$$

is *not* simple, since it winds around itself twice.

Next, we need an important concept of *reparametrization.*

Definition 6.4. Let $\varphi : [\alpha, \beta] \to [a, b]$ be a bijection which is continuously differentiable with $\varphi' > 0$ (which implies φ is monotone increasing). Then given a contour $\gamma : [a, b] \to \mathbb{C}$, the contour $\gamma \circ \varphi : [\alpha, \beta] \to \mathbb{C}$ is now a new *φ-reparametrized* contour. Since $\varphi' > 0$, the orientation of $\gamma \circ \varphi$ matches that of γ (in the sense of CW or CCW).

A familiar notion from multivariable calculus is imported directly into this context as

Definition 6.5 (The arc-length of a contour)**.** If γ is continuously differentiable then

$$L(\gamma) \quad := \quad \int_a^b |\gamma'|$$

is the *arc-length* of γ.

How does the arc-length change under re-parametrization?

$$\begin{aligned} L(\gamma \circ \varphi) \qquad &= \qquad \int_\alpha^\beta \left|(\gamma \circ \varphi)'\right| \\ &= \qquad \int_\alpha^\beta |\gamma' \circ \varphi| \, |\varphi'| \\ &\overset{\varphi' \geq 0}{=} \qquad \int_\alpha^\beta |\gamma' \circ \varphi| \, \varphi' \\ &\overset{\text{Change of var.}}{=} \qquad \int_a^b |\gamma'| \\ &= \qquad L(\gamma) \ . \end{aligned}$$

So, apparently, reparametrization keeps the arc-length of the contour, since it is an intrinsic geometric property of it.
Sometimes it is necessary to talk about the tangent line to a given contour:

Definition 6.6 (Tangent line)**.** The map

$$[a, b] \ni t \mapsto \frac{\gamma'(t)}{|\gamma'(t)|} \in \mathbb{C}$$

is called the tangent, assuming that $\gamma'(t) \neq 0$ for all $t \in [a, b]$.

Certain regularity conditions on contours are customary in complex analysis. The names are slightly different than what they mean in analysis in general:

Definition 6.7 (Smoothness of contours in complex analysis)**.** A contour $\gamma : [a, b] \to \mathbb{C}$ is *smooth* iff γ is continuously differentiable, and if furthermore, $\gamma' \neq 0$ on $[a, b]$ (this differs from the general usage of the word smooth in that usually smooth means *infinitely* differentiable and has no requirement on whether the derivatives happen to take the value zero). The derivatives at the end points should be interpreted as one-sided limits. A contour is called *piecewise smooth* if it fails these conditions only at a finite number of points within $[a, b]$.

6.2 Contour integrals

Remember *line integrals* from multivariable calculus: given a vector field $V : \mathbb{R}^2 \to \mathbb{R}^2$ and a contour $\gamma : [a, b] \to \mathbb{R}^2$ whose image is $\Gamma := \mathrm{im}(\gamma) \equiv \gamma([a, b])$, we define the line integral of V *along* Γ as:

$$I(V, \gamma) := \int_\Gamma V \equiv \int_{t=a}^{t=b} \langle V(\gamma(t)), \dot{\gamma}(t) \rangle \, \mathrm{d}t \in \mathbb{R} \,. \tag{6.3}$$

We have already made reference to this object when we were discussing harmonic conjugates in Section 4.4. The meaning of $I(V, \gamma)$ is two-fold:

1. It is the "average" value of (the component parallel to $\dot{\gamma}$ of) V along γ, up to a normalization.

2. It is the appropriate "inverse operation" to taking the gradient, in the sense that the fundamental theorem of line integrals yields

$$\operatorname{grad}\left(I\left(V,\gamma\right)\right) \quad = \quad V$$

where the gradient is calculated w.r.t. $\gamma\left(b\right)$, so the line integral is the right inverse of grad, and furthermore, if $V = \operatorname{grad}\left(G\right)$ for some scalar field G then

$$I\left(\operatorname{grad}\left(G\right),\gamma\right) = G\left(\gamma\left(b\right)\right) - G\left(\gamma\left(a\right)\right) \,. \tag{6.4}$$

We note that in defining line integrals of vector fields we have not really invoked the interpretation of "area under the curve" which is customary when talking about integrals of functions $\mathbb{R} \to \mathbb{R}$. We also note in passing also the physics interpretation of $I\left(F,\gamma\right)$ as the *work* performed by the force field $F : \mathbb{R}^2 \to \mathbb{R}^2$ along the path γ; in electromagnetism the line integral of the electric field yields the electric potential difference between two points.

Proceeding with complex analysis, we focus our attention now on obtaining the appropriate construction which will serve as the integral which is the appropriate inverse operation to the complex derivative of a holomorphic function, i.e., let us attempt to construct an integral such that, for $f : \mathbb{C} \to \mathbb{C}$ holomorphic, we would get

$$\int_{z_0}^{z} f' \quad \overset{??}{=} \quad f\left(z\right) - f\left(z_0\right) \,.$$

Since the right-hand is a complex number, so should the left hand side be. This rules out a naive application of (6.4). It turns out that the following construction does the job:

Definition 6.8 (Integral of complex function along a contour)**.** Let $\gamma : [a,b] \to \mathbb{C}$ be a piecewise smooth contour and let $\Gamma := \operatorname{im}\left(\gamma\right) \equiv \gamma\left([a,b]\right)$ be the image of the contour. We define the integral of $f : \mathbb{C} \to \mathbb{C}$ *along* Γ as

$$\int_\Gamma f := \int_{t=a}^{b} f\left(\gamma\left(t\right)\right)\gamma'\left(t\right)\mathrm{d}t \tag{6.5}$$

assuming that the RHS is integrable in the real Riemann sense. Sometimes, when the contour γ is implicit we will also use the (common in the literature) notation

$$\int_\Gamma f \quad =: \quad \int_{z_0}^{z_1} f\left(z\right)\mathrm{d}z \,.$$

This will be especially useful for integrals which are independent of the contour except its end points. If the contour is closed, one uses the circle notation

$$\oint_\Gamma f \equiv \oint_\Gamma f\left(z\right)\mathrm{d}z \,.$$

For this to make sense we must demonstrate that $\int_\Gamma f$ indeed depends only on Γ but not on γ (except for the sign which is determined by γ's orientation), which we do below in Claim 6.11. The notation $\int_\Gamma f$ remains somewhat ambiguous since Γ is, strictly speaking, a set and hence has no orientation. We hope this discrepancy in the notation would not cause too much havoc.

We note that since the integrand in (6.5) is a complex function of the real variable $t \in [a,b]$, the integral is clearly defined via our earlier (6.1).

One of the most important contour integrals is depicted in the

Example 6.9. Let $f : \mathbb{C} \setminus \{\, 0 \,\} \to \mathbb{C}$ be given by $z \mapsto \frac{1}{z}$ and define the simple closed contour $\gamma : [0, 2\pi] \to \mathbb{C}$ by $\gamma\left(t\right) = \mathrm{e}^{int}$ for some $n \in \mathbb{Z}$. As we have seen, if $|n| > 1$ this contour is *not* simple (which is OK). Then

$$\oint_\Gamma f \quad = \quad 2\pi\mathrm{i}n \,.$$

Proof. Calculating the derivative of γ we have

$$\gamma'\left(t\right) = \mathrm{i}n\mathrm{e}^{\mathrm{i}nt} \tag{6.6}$$

and so

$$\begin{aligned} f(\gamma(t))\gamma'(t) &= \frac{1}{\mathrm{e}^{\mathrm{i}nt}}\mathrm{i}n\mathrm{e}^{\mathrm{i}nt} \\ &= \mathrm{i}n\,. \end{aligned}$$

Plugging this into the integral we find

$$\begin{aligned} \oint_\Gamma f &= \int_{t=0}^{2\pi} f(\gamma(t))\gamma'(t)\,\mathrm{d}t \\ &= \int_{t=0}^{2\pi} \mathrm{i}n\mathrm{d}t \\ &= 2\pi\mathrm{i}n \end{aligned}$$

as desired. □

Example 6.10. Another instructive example is integrating an entire function, say, $z \mapsto z^m$ for some $m \in \mathbb{N}_{\geq 0}$ along the same circular contour $\gamma(t) = \mathrm{e}^{\mathrm{i}nt}$ with $n \in \mathbb{Z}$. We have

$$\oint_\Gamma f = 0\,.$$

Proof. First note that if $n = 0$ this is actually a constant contour whose derivative is zero and hence the integrand is zero.

Otherwise, the derivative of γ was just calculated above in (6.6) so

$$\begin{aligned} f(\gamma(t))\gamma'(t) &= \left(\mathrm{e}^{\mathrm{i}nt}\right)^m \mathrm{i}n\mathrm{e}^{\mathrm{i}nt} \\ &= \mathrm{i}n\mathrm{e}^{\mathrm{i}(m+1)nt} \end{aligned}$$

and when we integrate it we get

$$\begin{aligned} \int_{t=0}^{2\pi} \mathrm{i}n\mathrm{e}^{\mathrm{i}(m+1)nt}\mathrm{d}t &= \mathrm{i}n\frac{1}{\mathrm{i}(m+1)n}\,\mathrm{e}^{\mathrm{i}(m+1)nt}\Big|_{t=0}^{2\pi} \\ &= 0\,. \end{aligned}$$

□

Next, one may ask whether Definition 6.8 could be related back to the multivariable context. I don't find this interpretation overly insightful, but if one insisted:

$$\begin{aligned} \int_\Gamma f &:= \int_{t=a}^{b} f(\gamma(t))\gamma'(t)\,\mathrm{d}t \\ &= \int_{t=a}^{b} \{[f_R(\gamma(t))\gamma_R'(t) - f_I(\gamma(t))\gamma_I'(t)] + \mathrm{i}\,[f_R(\gamma(t))\gamma_I'(t) + f_I(\gamma(t))\gamma_R'(t)]\}\,\mathrm{d}t \\ &= \int_{t=a}^{b} \left\langle \begin{bmatrix} f_R(\gamma(t)) \\ -f_I(\gamma(t)) \end{bmatrix}, \begin{bmatrix} \gamma_R'(t) \\ \gamma_I'(t) \end{bmatrix} \right\rangle \mathrm{d}t + \\ &\quad +\mathrm{i}\int_{t=a}^{b} \left\langle \begin{bmatrix} f_I(\gamma(t)) \\ f_R(\gamma(t)) \end{bmatrix}, \begin{bmatrix} \gamma_R'(t) \\ \gamma_I'(t) \end{bmatrix} \right\rangle \mathrm{d}t\,. \end{aligned}$$

We find that if we define two vector fields

$$U := \begin{bmatrix} f_R \\ -f_I \end{bmatrix}, V := \begin{bmatrix} f_I \\ f_R \end{bmatrix} \tag{6.7}$$

then

$$\int_\Gamma f = \int_\Gamma U + \mathrm{i}\int_\Gamma V \tag{6.8}$$

where the right-hand side is to be interpreted in the sense of line integrals of vector fields from multivariable calculus, as in (6.3). We also note that V is the 90-degree rotation of U, i.e., one may think of the integral of V it as the "circulation" of U.

Claim 6.11. $\int_{t=a}^{b} f(\gamma(t))\gamma'(t)\,\mathrm{d}t$ does not depend on the parametrization of γ and hence depends only on Γ except on the orientation of γ.

Proof. Let $\varphi : [\alpha, \beta] \to [a, b]$ be a reparametrization as in Definition 6.4. Then

$$\begin{aligned}\int_{t=\alpha}^{\beta} f(\gamma(\varphi(t)))(\gamma\circ\varphi)'(t)\,\mathrm{d}t &\overset{\text{chain}}{=} \int_{t=\alpha}^{\beta} f(\gamma(\varphi(t)))\gamma'(\varphi(t))\varphi'(t)\,\mathrm{d}t \\ &= \int_{t=a}^{b} f(\gamma(t))\gamma'(t)\,\mathrm{d}t\end{aligned}$$

where the last line follows by the change of variable formula for real integrals. ☐

We emphasize the contour's orientation does matter, in the sense that:

$$\int_{t=a}^{b} f(\tilde{\gamma}(t))\tilde{\gamma}'(t)\,\mathrm{d}t = -\int_{\Gamma} f$$

where $\tilde{\gamma} : [a, b] \to \mathbb{C}$ is a contour that goes in the reverse direction to a path γ.

Lemma 6.12. *Contour integrals are linear, in the sense that*

$$\int_{\Gamma} \alpha f + g = \alpha\int_{\Gamma} f + \int_{\Gamma} g$$

where $\alpha \in \mathbb{C}$ and $f, g : \mathbb{C} \to \mathbb{C}$ are continuous.

Proof. This is an immediate consequence of the linearity of the real Riemann integral. ☐

Lemma 6.13. *Contour integrals are additive, in the sense that*

$$\int_{\Gamma_1\cup\Gamma_2} f = \int_{\Gamma_1} f + \int_{\Gamma_2} f\,.$$

Proof. This again is an immediate consequence of the same statement for real Riemann integrals. ☐

Let us see that this contour integral is indeed the inverse operation to the derivative:

Proposition 6.14. *For any holomorphic $f : \mathbb{C} \to \mathbb{C}$,*

$$\int_{\Gamma} f' = f(z) - f(z_0)$$

where Γ corresponds to any contour starting at z_0 and ending at z. In particular, $\int_{\Gamma} f'$ depends only on the end points of Γ and $\oint f' = 0$ for any closed contour.

Proof. We take some contour $\gamma : [a, b] \to \mathbb{C}$ which has Γ as its range. Then

$$\begin{aligned}\int_{\Gamma} f' &\equiv \int_{t=a}^{b} f'(\gamma(t))\gamma'(t)\,\mathrm{d}t \\ &= \int_{t=a}^{b} (f\circ\gamma)'(t)\,\mathrm{d}t\end{aligned}$$

where the last line follows by the chain rule. Now, however, the integral is interpreted via (6.1) since the integrand is a complex function of one real variable, and so (6.2) applies and we get

$$\int_{\Gamma} f' \quad = \quad f\left(\gamma\left(b\right)\right) - f\left(\gamma\left(a\right)\right) .$$

Now, since $\gamma\left(b\right) = z$ and $\gamma\left(a\right) = z_0$ we are finished. ☐

This yields an extremely simple proof of

Lemma 6.15. *If $f : \mathbb{C} \to \mathbb{C}$ is holomorphic with $f' = 0$ then f is the constant function.*

Proof. We write

$$f\left(z\right) - f\left(w\right) = \int_{w}^{z} f' = 0$$

for any $z, w \in \mathbb{C}$ and any path in between them. ☐

Conversely to Proposition 6.14, we also want to verify that the contour integral is complex differentiable with respect to its end point.

Lemma 6.16. *If $f : \mathbb{C} \to \mathbb{C}$ is holomorphic and has the property that*

$$\mathbb{C} \ni z \quad \mapsto \quad \int_{z_0}^{z} f\left(\tilde{z}\right) \mathrm{d}\tilde{z} =: F\left(z\right)$$

is a well-defined function for some fixed reference point $z_0 \in \mathbb{C}$ (in the sense that F is independent of the path $z_0 \to z$ chosen) then the function $F : \mathbb{C} \to \mathbb{C}$ is holomorphic and $F' = f$.

We note that later on we will see that for f holomorphic in an open set, F is *always* a well-defined function. But for now we prove this statement assuming it.

Proof. To proceed, we have

$$F\left(z\right) \quad := \quad \int_{z_0}^{z} f\left(\tilde{z}\right) \mathrm{d}\tilde{z}$$

for some contour from some fixed $z_0 \in \mathbb{C}$ to $z \in \mathbb{C}$. Indeed, we know F is a well-defined function which does not depend on the choice of the contour by hypothesis. Now,

$$\begin{aligned} F\left(z+w\right) - F\left(z\right) \quad &= \quad \int_{z_0}^{z+w} f\left(\tilde{z}\right) \mathrm{d}\tilde{z} - \int_{z_0}^{z} f\left(\tilde{z}\right) \mathrm{d}\tilde{z} \\ &= \quad \int_{z}^{z+w} f\left(\tilde{z}\right) \mathrm{d}\tilde{z} \end{aligned}$$

where in passing to the second line, we have assumed that the paths agree so we can subtract them. Also note that if $g\left(z\right) = 1$ for all z, then

$$\int_{z}^{z+w} g\left(\tilde{z}\right) \mathrm{d}\tilde{z} = \int_{z}^{z+w} \mathrm{d}\tilde{z} = \int_{z}^{z+w} \left(\tilde{z}\right)' \mathrm{d}\tilde{z} = w$$

using Proposition 6.14. Hence

$$\begin{aligned} \frac{F\left(z+w\right) - F\left(z\right)}{w} - f\left(z\right) \quad &= \quad \frac{\int_{z}^{z+w} f\left(\tilde{z}\right) \mathrm{d}\tilde{z}}{w} - f\left(z\right) \\ &= \quad \frac{\int_{z}^{z+w} f\left(\tilde{z}\right) \mathrm{d}\tilde{z} - \left(\int_{z}^{z+w} \mathrm{d}\tilde{z}\right) f\left(z\right)}{w} \\ &= \quad \frac{\int_{z}^{z+w} \left(f\left(\tilde{z}\right) - f\left(z\right)\right) \mathrm{d}\tilde{z}}{w} . \end{aligned}$$

Since f is continuous, for any $\varepsilon > 0$ there's some $\delta > 0$ such that if $\tilde{z} \in B_\delta(z)$ then $f(z) \in B_\varepsilon(f(\tilde{z}))$. Since w is arbitrarily small, let us pick a path $z \to z + w$ which lies entirely within $B_\delta(z)$, which implies that $|f(\tilde{z}) - f(z)| < \varepsilon$ on this path. Then

$$\begin{aligned} \left| \frac{F(z+w) - F(z)}{w} - f(z) \right| &= \left| \frac{\int_z^{z+w} (f(\tilde{z}) - f(z))\, \mathrm{d}\tilde{z}}{w} \right| \\ &\leq \frac{1}{|w|} \varepsilon L(z \to z+w) \end{aligned}$$

where $L(z \to z+w)$ is the length of the path chosen. Since we are free to choose this path, let us choose the path so that

$$L(z \to z+w) = C|w|$$

for some constant $C > 0$ (this can always be arranged). Since ε was arbitrary, this implies that $F' = f$. □

We proceed with an extremely useful tool in evaluating contour integrals, the so-called

Lemma 6.17 (ML lemma)**.** *For a continuous function $f : \mathbb{C} \to \mathbb{C}$ and contour $\gamma : [a,b] \to \mathbb{C}$ with image Γ, let*

$$M := \sup_{z \in \Gamma} |f(z)| .$$

Then

$$\left| \int_\Gamma f \right| \leq ML(\gamma) .$$

Proof. We have

$$\begin{aligned} \left| \int_\Gamma f \right| &\equiv \left| \int_{t=a}^b f(\gamma(t))\gamma'(t)\, \mathrm{d}t \right| \\ &\leq \int_{t=a}^b |f(\gamma(t))\gamma'(t)|\, \mathrm{d}t \\ &\leq M \int_{t=a}^b |\gamma'(t)|\, \mathrm{d}t \\ &\equiv ML(\gamma) \end{aligned}$$

where in going from the first to the second line we have used Proposition 6.1. □

6.3 Anti-derivatives

Definition 6.18 (Anti-derivative)**.** $F : \mathbb{C} \to \mathbb{C}$ is the *anti-derivative* (also called *primitive*) of $f : \mathbb{C} \to \mathbb{C}$ in the connected open set $\Omega \subseteq \mathbb{C}$ iff $F'(z) = f(z)$ for all $z \in \Omega$. F is unique up to additive constant, if it exists (of course F must be holomorphic on Ω for this to make sense).

Example 6.19. Define $f : \mathbb{C} \to \mathbb{C}$ via $f(z) = \mathrm{e}^{\pi z}$. Then $f'(z) = \pi \mathrm{e}^{\pi z}$. So $F(z) = \frac{1}{\pi}\mathrm{e}^{\pi z}$ is the anti-derivative of f.

Example 6.20. $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ defined via $z \mapsto \frac{1}{z^2}$ has anti-derivative $z \mapsto -\frac{1}{z}$.

Collecting together the results from the previous section regarding anti-derivatives and contour integrals, we summarize:

Theorem 6.21. *For a continuous $f : \Omega \to \mathbb{C}$ in some open subset $\Omega \subseteq \mathbb{C}$, the following are equivalent:*

1. *f has anti-derivative F on Ω.*
2. *$\int_\Gamma f = F(\gamma(b)) - F(\gamma(a))$ for any piecewise differentiable contour $\gamma : [a, b] \to \Omega$.*
3. *$\oint_\Gamma f = 0$ if $\Gamma \subseteq \Omega$ is a closed contour.*

Proof. We have already shown (1)$\Longrightarrow$(2) in Proposition 6.14. If γ is merely piecewise differentiable rather than differentiable this will remain true thanks to a telescoping sum.

For (2)$\Longrightarrow$(3), let γ be any closed contour with $\operatorname{im}(\gamma) = \Gamma$. Let $z, w \in \Gamma$ and divide γ into two paths γ_1, γ_2 which both end and start on z, w respectively and whose concatenation yields γ: $\gamma_1 : z \to w$ and $\gamma_2 : w \to z$. Since the integral is independent of paths, we have

$$\begin{aligned}
\int_{\gamma_1} f &= \int_{-\gamma_2} f \\
&\Updownarrow \\
\int_{\gamma_1} f - \int_{-\gamma_2} f &= 0 \\
&\Updownarrow \\
\int_{\gamma_1} f + \int_{\gamma_2} f &= 0 \\
&\Updownarrow \\
\oint_{\gamma} f &= 0\,.
\end{aligned}$$

Finally, (3)$\Longrightarrow$(1) was proven in Lemma 6.16. $\square$

6.4 The Cauchy-Goursat theorem

We have already seen that if $f : \mathbb{C} \to \mathbb{C}$ happens to have an anti-derivative F, then the contour integral around any closed contour is zero

$$\oint_\Gamma f = 0\,. \tag{6.9}$$

In what follows, we want to give *more* conditions for when (6.9) holds (which, in turn, by Theorem 6.21 implies more conditions on when it has an anti-derivative). It will turn out that it is enough to assume that f is holomorphic within the interior $\operatorname{int}(\gamma)$ of Γ.

The main result of interest is

Theorem 6.22 (Cauchy-Goursat's integral theorem)**.** *If $f : \mathbb{C} \to \mathbb{C}$ is holomorphic at all points in the interior to and on a closed simple contour γ then*

$$\oint_\Gamma f = 0\,.$$

Before tending to this theorem, let us present the simpler predecessor due to Cauchy:

Theorem 6.23 (Cauchy's integral theorem)**.** *If $f : \mathbb{C} \to \mathbb{C}$ is holomorphic at all points in the interior* $\operatorname{int}(\gamma)$ *of some closed simple contour γ and furthermore f' is continuous there, then*

$$\oint_\Gamma f = 0\,.$$

Proof. Let $\gamma : [a,b] \to \mathbb{C}$ be some simple closed contour. Then

$$\begin{aligned}\oint_\Gamma f &\equiv \int_a^b f(\gamma(t))\,\gamma'(t)\,\mathrm{d}t \\ &= \int_a^b \left[f_R(\gamma(t))\,\gamma_R'(t) - f_I(\gamma(t))\,\gamma_I'(t)\right]\mathrm{d}t + \\ &\quad +\mathrm{i}\int_a^b \left[f_R(\gamma(t))\,\gamma_I'(t) - f_I(\gamma(t))\,\gamma_R'(t)\right]\mathrm{d}t\,.\end{aligned}$$

Next, recall Green's theorem, which states that if $V : \mathbb{R}^2 \to \mathbb{R}^2$ is a *continuously-differentiable* vector field and $\Omega \subseteq \mathbb{R}^2$ is a simply-connected region, then

$$\oint_{\partial\Omega} V = \int_\Omega \operatorname{curl}(V)\,.$$

Let us define the vector field

$$V := \begin{bmatrix} f_R \\ -f_I \end{bmatrix}$$

and re-interpret $\gamma : [a,b] \to \mathbb{R}^2$. Then

$$f_R(\gamma(t))\,\gamma_R'(t) - f_I(\gamma(t))\,\gamma_I'(t) = \langle V(\gamma(t)),\gamma'(t)\rangle_{\mathbb{R}^2}$$

so that

$$\begin{aligned}\int_a^b \left[f_R(\gamma(t))\,\gamma_R'(t) - f_I(\gamma(t))\,\gamma_I'(t)\right]\mathrm{d}t &= \oint_\Gamma V \\ &\overset{\text{Greens}}{=} \int_\Omega \operatorname{curl}(V)\end{aligned}$$

where $\Omega := \operatorname{int}(\gamma)$ is the interior of γ. But

$$\operatorname{curl}(V) = -\partial_x f_I - \partial_y f_R = 0$$

where the last equality is the second Cauchy-Riemann equation (4.4) applied on f. In a similar manner, we may apply the other CRE to get that also the imaginary part of the integral is zero.

Note that it was crucial to assume that f' was continuous because Greens theorem requires the *continuous differentiability of* V. ☐

It may seem like the distinction between Cauchy's integral theorem and the Cauchy-Goursat integral theorem is merely a technicality in regularity assumptions on f'. It is not. Goursat's extension is of crucial importance to the field of complex analysis in the following way:

If we apply the Cauchy-Goursat theorem on f assuming only that it is holomorphic (but not that its derivative is continuous) we can get as a consequence (using Theorem 6.26) that f' is also analytic (i.e. that f is smooth and has a power series expansion) which is what was preempted in Definition 4.2. Hence, assuming only one derivative implies the full analyticity, which singles out an extremely natural, single, regularity condition within the field of complex analysis: $\mathbb{C}$-differentiability.

We begin the proof of Goursat's theorem with a special case about triangles

Theorem 6.24 (Goursat's integral theorem for triangles)**.** *If* $\Omega \in \operatorname{Open}(\mathbb{C})$ *and* $T \subseteq \Omega$ *is a triangular set, then*

$$\oint_T f = 0$$

whenever f *is holomorphic on* Ω*.*

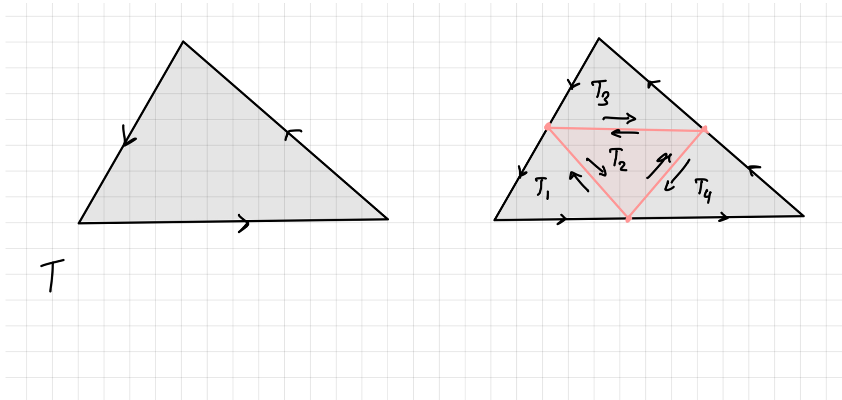

Figure 15: The triangular subdivision in Goursat's theorem.

Proof. The idea of the proof is to keep subdividing T into smaller triangles (as presented in [SS03]).

Let us study the first such subdivision. Take the initial triangle T, as indicated on the left of Figure 15. Bisect each side of that triangle to create four smaller triangles, $T_1, \dots, T_4$ within T. Importantly, the orientation along T_2 is precisely reversed those overlapping sides of the other triangles. This choice ensures that

$$\oint_T f = \oint_{T_1} f + \cdots + \oint_{T_4} f$$

since the contour integrals along the overlapping sides between T_2 and the other triangles cancel out. On one of these four triangles the maximum is attained, so that we have

$$\left|\oint_T f\right| \leq 4 \left|\oint_{T_j} f\right| \qquad (\exists j \in \{ 1, \dots, 4 \}) \; . \tag{6.10}$$

Let us name that particular T_j for which this maximum is attained as $T^{(1)}$. In this process we obtain a *sequence* of nested triangles

$$T, T^{(1)}, T^{(2)}, \dots$$

and for each such triangle we denote its diameter by $d^{(n)}$ and its perimeter by $p^{(n)}$. Clearly we have

$$\begin{aligned} p^{(n)} &= \frac{1}{2} p^{(n-1)} \\ d^{(n)} &= \frac{1}{2} d^{(n-1)} \end{aligned}$$

and also, recursively applying (6.10) n times we have

$$\left|\oint_T f\right| \leq 4^n \left|\oint_{T^{(n)}} f\right| .$$

Since we have a nested sequence of *compact* sets whose diameter converges to zero, we use Proposition 3.43 to conclude

$$\exists ! p \in \bigcap_{n=1}^{\infty} \Delta^{(n)}$$

where $\Delta^{(n)}$ denotes the (closed) set of points enclosed within $T^{(n)}$. Since f is assumed to be holomorphic, we have its $\mathbb{C}$-linear approximation as

$$f(z) = f(p) + f'(p)(z-p) + g(z)$$

for some function $g:\mathbb{C}\to\mathbb{C}$ which has the property that $g(z)\to 0$ as $z\to p$ necessarily strictly faster than $|z-p|$, i.e.,

$$\frac{g(z)}{|z-p|}\overset{z\to p}{\longrightarrow}0\,. \tag{6.11}$$

Integrating the above equation over $T^{(n)}$ we get

$$\begin{aligned}\oint_{T^{(n)}} f(z)\,\mathrm{d}z &= \oint_{T^{(n)}}[f(p)+f'(p)(z-p)+g(z)]\,\mathrm{d}z\\ &= f(p)\oint_{T^{(n)}}\mathrm{d}z+f'(p)\oint_{T^{(n)}}(z-p)\,\mathrm{d}z+\oint_{T^{(n)}}g(z)\,\mathrm{d}z\,.\end{aligned}$$

Now, since $z\mapsto 1$ has an anti-derivative, its closed contour integral vanishes. The same is true for $z\mapsto z-p$ (whose anti-derivative is, e.g. $z\mapsto\frac{1}{2}(z-p)^2$). So we are left with

$$\oint_{T^{(n)}} f(z)\,\mathrm{d}z = \oint_{T^{(n)}} g(z)\,\mathrm{d}z\,.$$

Since p is in the interior of $\Delta^{(n)}$ and z is on the boundary $T^{(n)}$, we must have

$$|z-p| \leq d^{(n)}$$

and hence we find using Lemma 6.17

$$\begin{aligned}\left|\oint_{T^{(n)}} f(z)\,\mathrm{d}z\right| &= \left|\oint_{T^{(n)}} g(z)\,\mathrm{d}z\right|\\ &\leq \left(\sup_{z\in T^{(n)}}\frac{|g(z)|}{|z-p|}\right)d^{(n)}p^{(n)}\\ &= \left(\sup_{z\in T^{(n)}}\frac{|g(z)|}{|z-p|}\right)4^{-(n-1)}d^{(1)}p^{(1)}\,.\end{aligned}$$

Combining this together with our previous estimate on $\oint_T f$, we find

$$\left|\oint_T f\right| \leq 4\left(\sup_{z\in T^{(n)}}\frac{|g(z)|}{|z-p|}\right)d^{(1)}p^{(1)}$$

which converges to zero as $n\to\infty$ by (6.11) and the fact $d^{(n)}\to 0$ and so as $n\to\infty$, $z\to p$. □

From triangles we can build rectangles and most simple contours. There is more to be said about the issues of just what kind of contours this works for but we delay this for later and content ourselves with this triangular proof as demonstrating the basic mechanism with which Goursat's theorem is true.

Example 6.25. It is important for us that holomorphicity holds for *every* point in the interior of the contour to apply Theorem 6.22. An important counter-example is $z\overset{f}{\mapsto}\frac{1}{z}$ which is only defined in $\mathbb{C}\setminus\{0\}$ (which is *not* simply-connected). Then, for instance, taking the contour

$$[0,2\pi]\ni t \overset{\gamma}{\mapsto} \mathrm{e}^{\mathrm{i}t}\in\mathbb{C}$$

clearly f *fails* to be holomorphic at the origin (it's not even defined there) so that it is *not* true that f is holomorphic everywhere in the interior of γ. Indeed, performing the integral directly as in Example 6.9 yields a result which is

$$2\pi\mathrm{i} \neq 0\,!$$

6.5 The Cauchy integral formula

The following remarkable consequence of the Cauchy-Goursat integral theorem is the Cauchy integral formula, which is at the heart of complex analysis.

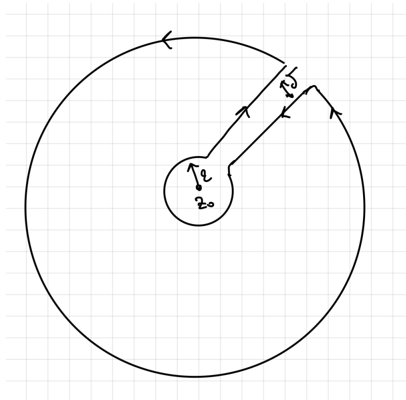

Figure 16: The so-called key-hole contour.

Theorem 6.26. *Let $\Omega \subseteq \mathbb{C}$ be a simply connected set and $f : \Omega \to \mathbb{C}$ holomorphic on it. Then for any $\gamma : [a, b] \to \Omega$ simply closed contour taken in CCW, if $z_0 \in \operatorname{int}(\gamma)$,*

$$f^{(n)}(z_0) = \frac{n!}{2\pi \mathrm{i}} \oint \frac{f(z)}{(z - z_0)^{n+1}} \mathrm{d}z \qquad (n \in \mathbb{N}_{\geq 0}) \,. \tag{6.12}$$

In particular, for $n = 0$ we get

$$f(z_0) = \frac{1}{2\pi \mathrm{i}} \oint \frac{f(z)}{z - z_0} \mathrm{d}z \,. \tag{6.13}$$

Note that in contrast to the holomorphic function f, the function

$$z \quad \mapsto \quad \frac{f(z)}{z - z_0}$$

is not defined at $z = z_0$ and so its closed contour integral is *not* zero and there is no contradiction to Theorem 6.22.

Proof. Let us start with the case $n = 0$. Consider the keyhole contour $\gamma_{\varepsilon,\delta}$ depicted in Figure 16, which is a contour that goes around a circle of radius $\varepsilon > 0$ around $z_0 \in \mathbb{C}$ and does *not* encircle it, and keeps a corridor of width $\delta > 0$ from infinity to z_0. The key-hole contour only exists on a simply-connected domain. As a result,

$$z \quad \mapsto \quad \frac{f(z)}{z - z_0}$$

is holomorphic in $\operatorname{int}(\gamma)$ so that

$$\oint_{\Gamma_{\varepsilon,\delta}} \frac{f(z)}{z - z_0} \mathrm{d}z \quad = \quad 0 \,.$$

By taking the limit $\delta \to 0$ we see that since f is continuous, $\int_C \frac{f(z)}{z - z_0} \mathrm{d}z = 0$ where C are the contributions from the two sides of the corridor, since in the limit $\delta \to 0$ they will precisely go on opposite directions of the same straight line. The remaining contour contribution is $\partial B_\varepsilon(z_0)$ (CW) and the big exterior circle D (CCW):

$$\oint_{\Gamma_{\varepsilon,\delta}} \frac{f(z)}{z - z_0} \mathrm{d}z \overset{\delta \to 0}{=} \oint_D \frac{f(z)}{z - z_0} \mathrm{d}z - \oint_{\partial B_\varepsilon(z_0)} \frac{f(z)}{z - z_0} \mathrm{d}z = 0$$

where the orientation on $\partial B_{\varepsilon}(z_0)$ was CW, so we put in a minus sign and now integrate CCW. We find

$$\begin{aligned}\oint_D \frac{f(z)}{z-z_0}\mathrm{d}z &= \oint_{\partial B_{\varepsilon}(z_0)} \frac{f(z)}{z-z_0}\mathrm{d}z \\ &= \oint_{\partial B_{\varepsilon}(z_0)} \frac{f(z)}{z-z_0}\mathrm{d}z - 2\pi\mathrm{i} f(z_0) + 2\pi\mathrm{i} f(z_0)\,.\end{aligned}$$

However, using Example 6.9, we have

$$\begin{aligned}\left|\frac{1}{2\pi\mathrm{i}}\oint_{\partial B_{\varepsilon}(z_0)} \frac{f(z)}{z-z_0}\mathrm{d}z - f(z_0)\right| &= \left|\frac{1}{2\pi\mathrm{i}}\oint_{\partial B_{\varepsilon}(z_0)} \frac{f(z)-f(z_0)}{z-z_0}\mathrm{d}z\right| \\ &= \left|\frac{1}{2\pi\mathrm{i}}\int_{t=0}^{2\pi} \frac{f\left(z_0+\varepsilon\mathrm{e}^{\mathrm{i}t}\right)-f(z_0)}{\varepsilon\mathrm{e}^{\mathrm{i}t}}\varepsilon\mathrm{e}^{\mathrm{i}t}\mathrm{i}\mathrm{d}t\right| \\ &\leq \frac{1}{2\pi}\int_0^{2\pi}\left|f\left(z_0+\varepsilon\mathrm{e}^{\mathrm{i}t}\right)-f(z_0)\right|\mathrm{d}t \\ &\leq \max_{t\in[0,2\pi]}\left|f\left(z_0+\varepsilon\mathrm{e}^{\mathrm{i}t}\right)-f(z_0)\right| \\ &\overset{\varepsilon\to 0}{\longrightarrow} 0\,.\end{aligned}$$

where the last limit follows due to the continuity of f at z_0, and this completes the $n=0$ proof.

To show the general case, argue by induction on n. Assume then the statement for n and verify it for $n+1$:

$$f^{(n+1)}(z_0) \equiv \lim_{z\to 0}\frac{f^{(n)}(z_0+z)-f^{(n)}(z_0)}{z}$$

and by the induction hypothesis, the prelimit has an integral representation as

$$\begin{aligned}\frac{f^{(n)}(z_0+z)-f^{(n)}(z_0)}{z} &= \frac{1}{z}\left[\frac{n!}{2\pi\mathrm{i}}\oint\frac{f(w)}{(w-z_0-z)^{n+1}}\mathrm{d}w - \frac{n!}{2\pi\mathrm{i}}\oint\frac{f(w)}{(w-z_0)^{n+1}}\mathrm{d}w\right] \\ &= \frac{1}{z}\frac{n!}{2\pi\mathrm{i}}\oint f(w)\left[\frac{1}{(w-z_0-z)^{n+1}}-\frac{1}{(w-z_0)^{n+1}}\right]\mathrm{d}w\,.\end{aligned}$$

Now use the formula

$$a^n-b^n = (a-b)\sum_{k=0}^{n-1}a^{n-k-1}b^k \tag{6.14}$$

with $a=\frac{1}{w-z_0-z}$, $b=\frac{1}{w-z_0}$ to get

$$\begin{aligned}\frac{1}{z}\left[\frac{1}{(w-z_0-z)^{n+1}}-\frac{1}{(w-z_0)^{n+1}}\right] &= \frac{1}{z}\left(\frac{1}{w-z_0-z}-\frac{1}{w-z_0}\right)\sum_{k=0}^{n}a^{n-k}b^k \\ &= \frac{1}{z}\frac{1}{w-z_0-z}z\frac{1}{w-z_0}\sum_{k=0}^{n}a^{n-k}b^k \\ &= \frac{1}{w-z_0-z}\frac{1}{w-z_0}\sum_{k=0}^{n}a^{n-k}b^k \\ &= ab\sum_{k=0}^{n}a^{n-k}b^k\,.\end{aligned}$$

Now, as $z \to 0$, $a \to b$ and we get

$$\frac{1}{z}\left[\frac{1}{(w-z_0-z)^{n+1}}-\frac{1}{(w-z_0)^{n+1}}\right] \overset{z\to 0}{\to} b^2 \sum_{k=0}^{n} b^n = (n+1)\,b^{n+2}$$

so that

$$\begin{aligned}\lim_{z\to 0}\frac{f^{(n)}(z_0+z)-f^{(n)}(z_0)}{z} &= \lim_{z\to 0}\frac{n!}{2\pi \mathrm{i}}\oint f(w)\frac{1}{z}\left[\frac{1}{(w-z_0-z)^{n+1}}-\frac{1}{(w-z_0)^{n+1}}\right]\mathrm{d}w \\ &\overset{\star}{=} \frac{n!}{2\pi \mathrm{i}}\oint f(w)(n+1)\frac{1}{(w-z_0)^{n+2}}\mathrm{d}w \\ &= \frac{(n+1)!}{2\pi \mathrm{i}}\oint \frac{f(w)}{(w-z_0)^{n+2}}\mathrm{d}w\,.\end{aligned}$$

To complete the proof we must argue why it is justified to exchange the limit with the integral in the line marked by $\star$. This can be done using the bounded convergence theorem [Rud86, Theorem 1.34], which holds since the integrand

$$f(w)\frac{1}{z}\left[\frac{1}{(w-z_0-z)^{n+1}}-\frac{1}{(w-z_0)^{n+1}}\right]$$

is a bounded function for all z, as long as z is sufficiently small, since $w-z$ never gets close to z_0 then. ☐

Corollary 6.27. *Any holomorphic function $f:\mathbb{C}\to\mathbb{C}$ is actually smooth (has continuous derivatives of all orders).*

Proof. The existence of the formula (6.12). ☐

Corollary 6.28. *Any holomorphic function $f:\mathbb{C}\to\mathbb{C}$ is equal to its average on a circle, in the sense that*

$$f(z_0) = \frac{1}{2\pi}\int_{\theta=0}^{2\pi} f\left(z_0+R\mathrm{e}^{\mathrm{i}\theta}\right)\mathrm{d}\theta \qquad (z_0\in\mathbb{C}, R>0)\,. \tag{6.15}$$

Proof. If we use (6.13) explicitly with the contour

$$\gamma(\theta) = z_0+R\mathrm{e}^{\mathrm{i}\theta} \qquad (\theta\in[0,2\pi])$$

then we find

$$\gamma'(\theta) = R\mathrm{e}^{\mathrm{i}\theta}\mathrm{i}$$

so that

$$\frac{f(\gamma(\theta))}{(\gamma(\theta)-z_0)}\gamma'(\theta) = \frac{f\left(z_0+R\mathrm{e}^{\mathrm{i}\theta}\right)}{R\mathrm{e}^{\mathrm{i}\theta}}R\mathrm{e}^{\mathrm{i}\theta}\mathrm{i}\,.$$

☐

Corollary 6.29. *Any harmonic function $F:\mathbb{R}^2\to\mathbb{R}$ is equal to its average on a circle, in the sense that*

$$F(z_0) = \frac{1}{2\pi}\int_{\theta=0}^{2\pi} F\left(z_0+R\mathrm{e}^{\mathrm{i}\theta}\right)\mathrm{d}\theta \qquad (z_0\in\mathbb{C}, R>0)\,.$$

Proof. Let $F:\mathbb{R}^2\to\mathbb{R}$ harmonic be given. Then we know via Proposition 4.15 that F is the real part of some holomorphic

$$f:\mathbb{C} \quad\to\quad \mathbb{C}\,.$$

Taking now the real part of the equation (6.15) we arrive at our desired result. ☐

6.6 Cauchy's inequality

Theorem 6.30 (Cauchy's inequality). *If $f:\mathbb{C}\to\mathbb{C}$ is holomorphic on $\operatorname{clo}(B_R(z_0))$ then*

$$\left|f^{(n)}(z_0)\right| \quad\leq\quad \frac{n!}{R^n}\sup_{z\in\operatorname{clo}(B_R(z_0))}|f(z)|\,.$$

Proof. This is a consequence of the Cauchy integral formula for $f^{(n)}(z_0)$. Indeed, parametrizing a contour which is a circle of radius R as in Example 6.9, we have

$$\begin{aligned}\left|f^{(n)}(z_0)\right| &= \left|\frac{n!}{2\pi\mathrm{i}}\oint\frac{f(z)}{(z-z_0)^{n+1}}\mathrm{d}z\right| \\ &= \frac{n!}{2\pi}\left|\int_{\theta=0}^{2\pi}\frac{f\left(z_0+R\mathrm{e}^{\mathrm{i}\theta}\right)}{\left(R\mathrm{e}^{\mathrm{i}\theta}\right)^{n+1}}R\mathrm{i}\mathrm{e}^{\mathrm{i}\theta}\mathrm{d}\theta\right| \\ &\leq \frac{n!}{2\pi}\frac{\|f\|_{L^\infty(B_R(z_0))}}{R^n}2\pi\end{aligned}$$

where

$$\|f\|_{L^\infty(B_R(z_0))} \quad\equiv\quad \sup_{z\in\operatorname{clo}(B_R(z_0))}|f(z)|\,.$$

☐

6.7 Morera's theorem

A converse of Cauchy's integral theorem could be stated as

Theorem 6.31 (Morera). *Let $f:B_R(z_0)\to\mathbb{C}$ be a continuous function such that for any triangle T within $B_R(z_0)$,*

$$\oint_T f \quad=\quad 0\,.$$

Then f is holomorphic.

Proof. By Theorem 6.21 we know that f has some anti-derivative F such that $F'=f$. In particular F is holomorphic and hence by Corollary 6.27 smooth. In particular it has two derivatives and so f' exists, i.e., f is holomorphic. ☐

6.8 Liouville's theorem and other miracles

Theorem 6.32 (Liouville). *If a function $f:\mathbb{C}\to\mathbb{C}$ is entire (i.e. holomorphic on $\mathbb{C}$) and bounded then f is actually a constant function.*

Proof. We have using Cauchy's inequality Theorem 6.30 that

$$|f'(z_0)| \quad \leq \quad \frac{1}{R}\|f\|_{L^\infty(B_R(z_0))}\,.$$

Since f is bounded, there is some $M>0$ such that $\|f\|_{L^\infty(B_R(z_0))} < M$ for *all* z_0. It follows that

$$|f'(z_0)| \quad \leq \quad \frac{M}{R}\,.$$

Since R may be taken arbitrarily large, it follows that $f' = 0$ which means that f is the constant function via Lemma 6.15. ☐

Theorem 6.33 (The fundamental theorem of algebra)**.** *Any non-constant polynomial has at least one root in* $\mathbb{C}$*.*

Proof. Let

$$p(z) := a_n z^n + \cdots + a_0$$

be some non-constant polynomial. If p has no roots, then $\frac{1}{p}$ is a bounded holomorphic function (see e.g. HW2). Now applying Liouville's theorem on $\frac{1}{p}$ we learn that it is a constant, which is a contradiction! ☐

Corollary 6.34. *Any polynomial of degree $n \geq 1$ has precisely n roots in $\mathbb{C}$. Denoting these roots by $\lambda_1, \dots, \lambda_n$, one may factorize p as*

$$p(z) \quad = \quad a_n \prod_{j=1}^{n} (z - \lambda_j)$$

where $a_n \in \mathbb{C}$ is the coefficient of the highest degree term in p.

Proof. Applying the previous theorem on p we find at least *one* root, λ_1. Then, writing $z = z - \lambda_1 + \lambda_1$, we have

$$\begin{aligned} p(z) &= a_n z^n + \cdots + a_0 \\ &= a_n (z - \lambda_1 + \lambda_1)^n + \cdots + a_0 \\ &= a_n \sum_{k=0}^{n} \binom{n}{k} (z-\lambda_1)^k \lambda_1^{n-k} + \cdots + a_0\,. && \text{(binomial theorem)} \end{aligned}$$

This new expression may now be re-arranged by powers of $z - \lambda_1$ to get a new polynomial in the variable $z - \lambda_1$

$$\begin{aligned} p(z) \quad &= \quad q(z-\lambda_1) \\ &=: \quad b_n (z-\lambda_1)^n + \cdots + b_1 (z - \lambda_1) + b_0 \end{aligned}$$

and we know that: (1) $b_n = a_n$ as the highest degree coefficient must match, and, since $p(\lambda_1) = 0$, $b_0 = 0$. We may thus factor $z - \lambda_1$ out of q. We iterate this procedure until we exhaust all roots of the polynomial. ☐

6.9 Bounds on holomorphic functions [extra]

Theorem 6.35. *If $f : \Omega \to \mathbb{C}$ is holomorphic with Ω being the infinite horizontal strip of width 2 centered on the horizontal real axis, then $\exists \nu > 0$ such that for some $C < \infty$,*

$$|f(z)| \quad \leq \quad C(1+|z|)^\nu$$

and furthermore, for any $n \in \mathbb{N}$ there exists some $C_n < \infty$ such that

$$\left|f^{(n)}(x)\right| \quad \leq \quad C_n (1+|x|)^\nu \qquad (x \in \mathbb{R})\,.$$

[TODO: contrast with quasi analytic extensions]

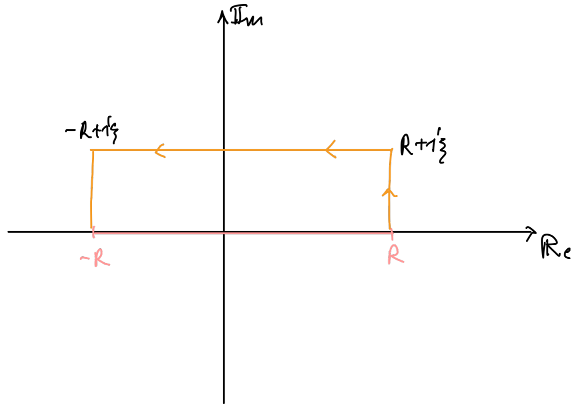

Figure 17: Rectangular contour used in calculating $\int_{x\in\mathbb{R}}\mathrm{e}^{-\pi x^2}\mathrm{e}^{-2\pi\mathrm{i}x\xi}\mathrm{d}x$.

6.10 Examples of some contour integrals

Example 6.36 (Fourier transform of a Gaussian)**.** Let $\xi\in\mathbb{R}$. Then

$$\int_{x\in\mathbb{R}}\mathrm{e}^{-\pi x^2}\mathrm{e}^{-2\pi\mathrm{i}x\xi}\mathrm{d}x \quad = \mathrm{e}^{-\pi\xi^2} \quad .$$

The same result may be extended to $\xi\in\mathbb{C}$.

Proof. First, if $\xi=0$ we get the formula

$$1 \quad = \quad \int_{x\in\mathbb{R}}\mathrm{e}^{-\pi x^2}\mathrm{d}x$$

which is usually solved by polar coordinates:

$$\begin{aligned}
\int_{x\in\mathbb{R}}\mathrm{e}^{-\pi x^2}\mathrm{d}x &= \sqrt{\left(\int_{x\in\mathbb{R}}\mathrm{e}^{-\pi x^2}\mathrm{d}x\right)^2} \\
&= \sqrt{\int_{(x,y)\in\mathbb{R}^2}\mathrm{e}^{-\pi\|(x,y)\|^2}\mathrm{d}x\mathrm{d}y} \\
&= \sqrt{\int_{\theta=0}^{2\pi}\int_{r=0}^{\infty}r\mathrm{e}^{-\pi r^2}\mathrm{d}\theta\mathrm{d}r} \\
&= \sqrt{2\pi\int_{r=0}^{\infty}\left(-\frac{1}{2\pi}\partial\mathrm{e}^{-\pi r^2}\right)\mathrm{d}r} \\
&= \sqrt{-\left(\mathrm{e}^{-\pi\infty}-1\right)} \\
&= 1\,.
\end{aligned}$$

Next, if $\xi>0$, define $f:\mathbb{C}\to\mathbb{C}$ via $f(z)=\mathrm{e}^{-\pi z^2}$. What is the relevance of this function to our problem?

$$\begin{aligned}
f(x+\mathrm{i}\xi) &= \mathrm{e}^{-\pi(x+\mathrm{i}\xi)^2} \\
&= \mathrm{e}^{-\pi\left(x^2-\xi^2+2\mathrm{i}\xi x\right)} \\
&= \mathrm{e}^{\pi\xi^2}\mathrm{e}^{-\pi x^2}\mathrm{e}^{-2\pi\mathrm{i}\xi x}
\end{aligned}$$

which is precisely our integrand up to a constant factor $\mathrm{e}^{\pi\xi^2}$; so our goal is to prove that

$$I \quad := \quad \int_{-\infty}^{\infty}f(x+\mathrm{i}\xi)\,\mathrm{d}x = 1\,.$$

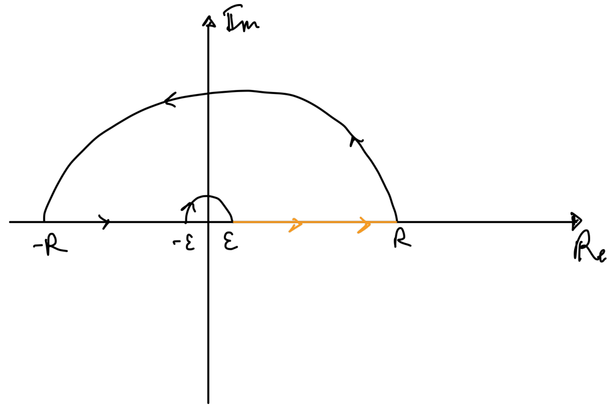

Figure 18: The indented semi-circle.

Now, f is entire. Let us define the closed simple rectangular contour γ as depicted in Figure 17 which depends on a parameter R, not via formula by via words as follows:

1. Lower horizontal leg: go from $-R$ to R along the real axis.
2. Right vertical leg: go from R to $R+\mathrm{i}\xi$.
3. Upper horizontal leg: go from $R+\mathrm{i}\xi$ to $-R+\mathrm{i}\xi$.
4. Left vertical leg closing the loop: go from $-R+\mathrm{i}\xi$ to $-R$.

By Cauchy's theorem, we have

$$\oint_\Gamma f \quad = \quad 0\,.$$

But, also, thanks to Lemma 6.13,

$$\oint_\Gamma f \quad = \quad \int_{-R}^{R} f(x)\,\mathrm{d}x + \int_0^\xi f(R+\mathrm{i}y)\,\mathrm{i}\mathrm{d}y + \int_R^{-R} f(x+\mathrm{i}\xi)\,\mathrm{d}x + \int_\xi^0 f(R+\mathrm{i}y)\,\mathrm{d}y\,.$$

On the vertical legs, we have using Proposition 6.1

$$\begin{aligned}\left|\int_0^\xi f(R+\mathrm{i}y)\,\mathrm{i}\mathrm{d}y\right| \quad &= \quad \left|\int_0^\xi \mathrm{e}^{-\pi(R+\mathrm{i}y)^2}\mathrm{i}\mathrm{d}y\right| \\ &\leq \quad \mathrm{e}^{-\pi R^2}\int_0^\xi \mathrm{e}^{\pi y^2}\mathrm{d}y \\ &\leq \quad \mathrm{e}^{-\pi R^2}\xi\mathrm{e}^{\pi\xi^2}\end{aligned}$$

whereas the upper horizontal leg equals $-I$ since its orientation is reversed. Combining everything together we get:

$$0 \quad = \quad 1 - I + 2\mathcal{O}\left(\mathrm{e}^{-\pi R^2}\xi\mathrm{e}^{\pi\xi^2}\right)$$

so that in the limit $R\to\infty$ (at fixed ξ) we find $I=1$ as desired. We remark that if $\xi<0$ one must take the rectangle *below* the real axis. □

Example 6.37. We have

$$\int_0^\infty \frac{1-\cos(x)}{x^2}\mathrm{d}x \quad = \quad \frac{\pi}{2}\,.$$

Proof. Consider the function $f:\mathbb{C}\to\mathbb{C}$ given by

$$f(z) = \frac{1-\mathrm{e}^{\mathrm{i}z}}{z^2}\,.$$

If $z=x$ is real and taking the real part of f we get precisely our desired integrand. On the other hand, consider the following contour γ as depicted in Figure 18 which depends on parameters R and ε:

1. Go along the real axis from ε to R.
2. Go along the semicircle of radius R CCW from R to $-R$.
3. Go along the real axis from $-R$ to $-\varepsilon$.
4. Go along the semicircle of radius ε CW from $-\varepsilon$ to ε.

Within the interior of our contour, f is holomorphic.

We note that on the large semicircle, since

$$\left|\frac{1-\mathrm{e}^{\mathrm{i}z}}{z^2}\right| \leq \frac{2}{|z|^2}\,,$$

that integral is bounded (via Lemma 6.17) by

$$\begin{aligned}\left|\int_{R\text{ semicirc.}} f\right| &\leq \frac{2}{R^2}\pi R\\ &= \frac{2\pi}{R}\,.\end{aligned}$$

On the other hand, on the small semicircle we have

$$\begin{aligned}\int_{\varepsilon\text{ semi}} f &= \int_{\theta=\pi}^{0}\frac{1-\mathrm{e}^{\mathrm{i}\varepsilon\mathrm{e}^{\mathrm{i}\theta}}}{\varepsilon^2\mathrm{e}^{2\mathrm{i}\theta}}\mathrm{i}\varepsilon\mathrm{e}^{\mathrm{i}\theta}\mathrm{d}\theta\\ &= \int_{\theta=\pi}^{0}\frac{1-\mathrm{e}^{\mathrm{i}\varepsilon\mathrm{e}^{\mathrm{i}\theta}}}{\varepsilon\mathrm{e}^{\mathrm{i}\theta}}\mathrm{i}\mathrm{d}\theta\\ &\overset{\varepsilon\to0^+}{=} \int_{\theta=\pi}^{0}\mathrm{d}\theta\\ &= -\pi\,.\end{aligned}$$

Note that to make the line marked with $\varepsilon\to0^+$ rigorous we have to invoke Theorem D.4. This is justified because the sequence of functions

$$\left\{\theta\mapsto\frac{1-\mathrm{e}^{\mathrm{i}\varepsilon\mathrm{e}^{\mathrm{i}\theta}}}{\varepsilon\mathrm{e}^{\mathrm{i}\theta}}\right\}_{\varepsilon>0}$$

converges (pointwise in θ) to $-\mathrm{i}$ as $\varepsilon\to0^+$ (indeed, this is the limit definition of the derivative), each element is clearly Riemann integrable in θ on $[0,\pi]$ (just bound it by $\frac{2}{\varepsilon}$). Hence the theorem applies.

Finally, the integral over the real axis is *twice* what we need (the integrand is an even function on the real axis) so that using Cauchy's theorem we find

$$\lim_{R\to\infty,\varepsilon\to0}\int_{-R}^{-\varepsilon}\frac{1-\mathrm{e}^{\mathrm{i}x}}{x^2}\mathrm{d}x+\int_{\varepsilon}^{R}\frac{1-\mathrm{e}^{\mathrm{i}x}}{x^2}\mathrm{d}x = \pi$$

Now taking the real part of this equation yields our result since our integrand is even, whereas taking the imaginary part yields the obvious

$$\lim_{R\to\infty,\varepsilon\to0}\int_{-R}^{-\varepsilon}\frac{\sin(x)}{x^2}\mathrm{d}x+\int_{\varepsilon}^{R}\frac{\sin(x)}{x^2}\mathrm{d}x = 0$$

which is true even before taking the limit since sin is odd. □

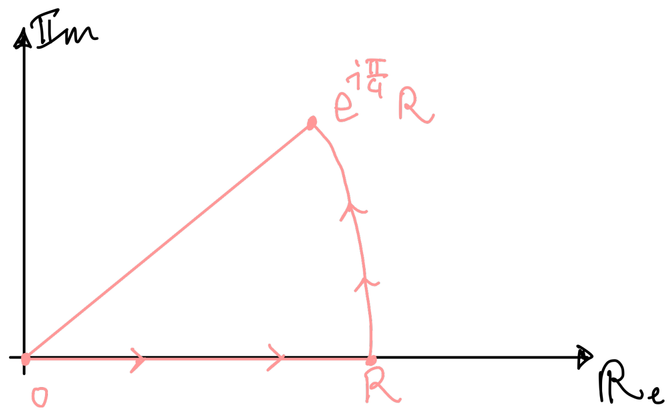

Figure 19: The sector contour.

We stress that $x \mapsto \frac{1}{x^2}$ is *not* integrable at the origin, so this example ultimately works only due to precise cancellations at the origin:

$$\frac{1-\cos(x)}{x^2} \approx \frac{1}{2} + \mathcal{O}\left(x^2\right)$$

whereas we do have integrability at infinity

$$\int_1^\infty \frac{1}{x^2}\mathrm{d}x = 1\,.$$

Example 6.38 (Fresnel integral)**.** We have

$$\int_0^\infty \sin\left(x^2\right)\mathrm{d}x = \int_0^\infty \cos\left(x^2\right)\mathrm{d}x = \frac{\sqrt{2\pi}}{4}\,.$$

Proof. Let us write

$$\sin\left(x^2\right) = -\mathbb{Im}\left\{\mathrm{e}^{-\mathrm{i}x^2}\right\}$$

and define the function Use e^{-z^2} on the contour depicted in Figure 19. On its horizontal leg we have

$$\begin{aligned}\int_0^R \mathrm{e}^{-x^2}\mathrm{d}x &\to \int_0^\infty \mathrm{e}^{-x^2}\mathrm{d}x \\ &= \frac{1}{2}\int_{-\infty}^\infty \mathrm{e}^{-x^2}\mathrm{d}x \\ &= \frac{1}{2}\sqrt{\pi}\,.\end{aligned}$$

On its arc-like leg we have

$$\int_0^{\frac{\pi}{4}} \mathrm{e}^{-\left(R\mathrm{e}^{\mathrm{i}\theta}\right)^2} R\mathrm{e}^{\mathrm{i}\theta}\mathrm{i}\mathrm{d}\theta = \mathrm{i}R\int_0^{\frac{\pi}{4}} \mathrm{e}^{-R^2\cos(2\theta)}\mathrm{e}^{\mathrm{i}\left[\theta-R^2\sin(2\theta)\right]}\mathrm{d}\theta$$

whose absolute value is bounded by

$$\leq R\int_0^{\frac{\pi}{4}} \mathrm{e}^{-R^2\cos(2\theta)}\mathrm{d}\theta\,.$$

Now, to proceed we may follow either one of two routes. The first one is to cite Theorem D.4: Since the sequence of functions

$$\left\{ \left[0, \frac{\pi}{4}\right] \ni \theta \mapsto R\mathrm{e}^{-R^2\cos(2\theta)} \right\}_{R>0}$$

converges (pointwise in θ) to zero as $R \to \infty$, and since they are all integrable on $\left[0, \frac{\pi}{4}\right]$, and all bounded by the integrable function

$$\theta \quad \mapsto \quad 1$$

the theorem applies to yield

$$\lim_{R\to\infty} R \int_0^{\frac{\pi}{4}} \mathrm{e}^{-R^2\cos(2\theta)}\mathrm{d}\theta \quad = \quad 0\,.$$

If one wanted to avoid this theorem, there is the following way: The function

$$\left[0, \frac{\pi}{4}\right] \ni \theta \quad \mapsto \quad \cos(2\theta)$$

is *concave* [Rud86, Definition 3.1]: for any $\theta_1, \theta_2 \in \left[0, \frac{\pi}{4}\right]$, the graph of $\theta \mapsto \cos(2\theta)$ is always *above* the straight line between the two points

$$(\theta_1, \cos(\theta_1)) \to (\theta_2, \cos(\theta_2))\ .$$

This can be verified, e.g., by the second derivative test:

$$\cos''(2\theta) \quad = \quad -4\cos(2\theta) \leq 0\,.$$

As a result, we get the lower bound

$$\cos(2\theta) \quad \geq \quad 1 - \frac{\pi}{4}\theta \qquad \left(\theta \in \left[0, \frac{\pi}{4}\right]\right) \tag{6.16}$$

which implies the upper bound on the integrand

$$\mathrm{e}^{-R^2\cos(2\theta)} \quad \leq \quad \mathrm{e}^{-R^2\left(1-\frac{\pi}{4}\theta\right)} \qquad \left(\theta \in \left[0, \frac{\pi}{4}\right]\right)$$

and hence on the integral

$$\begin{aligned}
R\int_0^{\frac{\pi}{4}} \mathrm{e}^{-R^2\cos(2\theta)}\mathrm{d}\theta \quad &\leq \quad R\int_0^{\frac{\pi}{4}} \mathrm{e}^{-R^2\left(1-\frac{\pi}{4}\theta\right)}\mathrm{d}\theta \\
&= \quad R\mathrm{e}^{-R^2}\int_0^{\frac{\pi}{4}} \mathrm{e}^{\frac{\pi}{4}R^2\theta}\mathrm{d}\theta \\
&= \quad R\mathrm{e}^{-R^2}\frac{4}{\pi R^2}\left(\mathrm{e}^{\frac{\pi}{4}R^2\frac{\pi}{4}} - 1\right) \\
&= \quad \frac{4}{\pi R}\mathrm{e}^{-\left(1-\frac{\pi^2}{16}\right)R^2}\left(1 - \mathrm{e}^{-\frac{\pi^2}{16}R^2}\right)
\end{aligned}$$

Now, crucially, since $1 - \frac{\pi^2}{16} \approx 0.38 > 0$, we get this expression converging to zero very quickly.

Proceeding to the the radial leg we have

$$\begin{aligned}
\int_R^0 \mathrm{e}^{-\left(r\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}\right)^2}\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}\mathrm{d}r \quad &= \quad \mathrm{e}^{\mathrm{i}\frac{\pi}{4}}\int_R^0 \mathrm{e}^{-\mathrm{i}r^2}\mathrm{d}r \\
&= \quad -\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}\int_0^R \mathrm{e}^{-\mathrm{i}r^2}\mathrm{d}r\,.
\end{aligned}$$

Now, since $z \mapsto \mathrm{e}^{-z^2}$ is entire we find

$$0 \quad \overset{R\to\infty}{=} \quad \frac{1}{2}\sqrt{\pi} - \mathrm{e}^{\mathrm{i}\frac{\pi}{4}}\int_0^\infty \mathrm{e}^{-\mathrm{i}r^2}\mathrm{d}r\,.$$

Taking the imaginary part of this equation yields

$$0 = \frac{1}{2}\sqrt{\pi}\sin\left(-\frac{\pi}{4}\right) - \mathbb{Im}\left\{\int_0^\infty \mathrm{e}^{-ir^2}\mathrm{d}r\right\}$$

and so

$$\begin{aligned}\int_0^\infty \sin\left(x^2\right)\mathrm{d}x &= \frac{1}{2}\sqrt{\pi}\sin\left(\frac{\pi}{4}\right)\\ &= \frac{1}{2}\sqrt{\pi}\frac{1}{\sqrt{2}}\\ &= \frac{1}{2}\sqrt{\frac{\pi}{2}}\\ &= \frac{\sqrt{2\pi}}{4}\,.\end{aligned}$$

Similarly taking the real part of the equation yields the cosine integral. □

Example 6.39 (The sinc integral)**.** We have

$$\int_0^\infty \operatorname{sinc}(x)\,\mathrm{d}x = \frac{\pi}{2}\,.$$

Proof. Consider the function $f(z) = \frac{\mathrm{e}^{\mathrm{i}z}}{z}$ which is holomorphic in the interior of the indented semi-circle Figure 18. On the leg from ε to R we have

$$\begin{aligned}\int_\varepsilon^R f(x)\,\mathrm{d}x &= \int_\varepsilon^R \frac{\mathrm{e}^{\mathrm{i}x}}{x}\mathrm{d}x\\ &= \int_\varepsilon^R \frac{\cos(x)+\mathrm{i}\sin(x)}{x}\mathrm{d}x\end{aligned}$$

so that $\mathbb{Im}\left\{\int_\varepsilon^R f(x)\,\mathrm{d}x\right\}$ is precisely the integral we want to calculate. On the other hand, on the large circle we have

$$\begin{aligned}\int_{\theta=0}^\pi f\left(R\mathrm{e}^{\mathrm{i}\theta}\right)R\mathrm{e}^{\mathrm{i}\theta}\mathrm{i}\mathrm{d}\theta &= \int_{\theta=0}^\pi \frac{\mathrm{e}^{\mathrm{i}R\mathrm{e}^{\mathrm{i}\theta}}}{R\mathrm{e}^{\mathrm{i}\theta}}R\mathrm{e}^{\mathrm{i}\theta}\mathrm{i}\mathrm{d}\theta\\ &= \mathrm{i}\int_{\theta=0}^\pi \mathrm{e}^{\mathrm{i}R\cos(\theta)}\mathrm{e}^{-R\sin(\theta)}\mathrm{d}\theta\end{aligned}$$

Its absolute value is bounded by

$$\leq \int_{\theta=0}^\pi \mathrm{e}^{-R\sin(\theta)}\mathrm{d}\theta\,.$$

Again one could use Theorem D.4 (justify this as above) or, more elementary: we have $\sin(\theta) \geq \frac{2}{\pi}\theta$ if $\theta \in \left[0, \frac{\pi}{2}\right]$ and conversely $\sin(\theta) \geq 2 - \frac{2}{\pi}\theta$ if $\theta \in \left[\frac{\pi}{2}, \pi\right]$ (see Lemma A.1), so that

$$\begin{aligned}\int_{\theta=0}^\pi \mathrm{e}^{-R\sin(\theta)}\mathrm{d}\theta &\geq \int_{\theta=0}^{\frac{\pi}{2}} \mathrm{e}^{-R\frac{2}{\pi}\theta}\mathrm{d}\theta + \int_{\theta=\frac{\pi}{2}}^\pi \mathrm{e}^{-R\left(2-\frac{2}{\pi}\theta\right)}\mathrm{d}\theta\\ &= \left(-\frac{\pi}{2R}\right)\left(\mathrm{e}^{-R}-1\right) + \mathrm{e}^{-2R}\left(\frac{\pi}{2R}\left(\mathrm{e}^{2R}-\mathrm{e}^{R}\right)\right)\\ &\to 0\,.\end{aligned}$$

On the small ε semicircle, we have

$$\begin{aligned}\int_{\theta=\pi}^{0} f\left(\varepsilon e^{i\theta}\right)\varepsilon e^{i\theta} i d\theta &= i\int_{\theta=\pi}^{0} e^{i\varepsilon e^{i\theta}} d\theta \\ &\overset{\varepsilon\to 0}{=} i\int_{\pi}^{0} d\theta \\ &= -i\pi\end{aligned}$$

where in the penultimate line we have used again Theorem D.4 on the integrable sequence of functions

$$\left\{\theta\mapsto e^{i\varepsilon e^{i\theta}}\right\}_{\varepsilon>0}$$

which converges pointwise to $\theta\mapsto 1$ as $\varepsilon\to 0$. We conclude the result. ☐

Note that sinc is not absolutely integrable! It is sign cancellations near the origin which make it integrable near the origin (said differently, sinc $\to 1$ at the origin, which is perfectly regular).

Example 6.40. We have for any $a>0$, $b\in\mathbb{R}$,

$$\int_0^\infty e^{-ax}\cos(bx)\,dx = \frac{a}{a^2+b^2}\,.$$

Proof. Let us write

$$\int_0^\infty e^{-ax}\cos(bx)\,dx = \mathbb{Re}\left\{\int_0^\infty e^{-ax+ibx}dx\right\}$$

and consider the function $z\mapsto e^{-Az}$ with $A>0$ to be determined, which is entire. Consider a sector contour as in Figure 19 (though now with an angle ω (also to be determined) instead of $\frac{\pi}{4}$). Then on the real leg we have

$$\int_0^R e^{-Ax}dx = \frac{1}{-A}\left(e^{-AR}-1\right)\overset{R\to\infty}{\longrightarrow}\frac{1}{A}\,.$$

On the arc we have

$$\int_0^\omega e^{-ARe^{i\theta}}Re^{i\theta}id\theta$$

whose absolute value goes to zero again as in Example 6.38, assuming $\omega<\frac{\pi}{2}$.

We are thus left with the radial integral

$$\int_{r=R}^0 e^{-Are^{i\omega}}e^{i\omega}dr = e^{i\omega}\int_{r=R}^0 e^{-Are^{i\omega}}dr\,.$$

We now pick A,ω so that

$$-Are^{i\omega}\overset{!}{=}-ar+ibr$$

which can be solved for A,ω. With this choice, find the result for the integral as

$$\begin{aligned}\frac{1}{A} &= -e^{i\omega}\int_{r=R}^0 e^{-Are^{i\omega}}dr \\ &= -e^{i\omega}\int_{r=R}^0 e^{-ar+ibr}dr\end{aligned}$$

and hence

$$\int_{r=0}^R e^{-ar+ibr}dr = \frac{e^{i\omega}}{A}$$

so that the real part is

$$\int_0^\infty \mathrm{e}^{-ax}\cos(bx)\,\mathrm{d}x \quad = \quad \frac{\cos(\omega)}{A}\,.$$

Going back to $-Are^{\mathrm{i}\omega} \overset{!}{=} -ar + ibr$ we find $\frac{\cos(\omega)}{A}$: Taking the sum of squares of the real and imaginary parts yields $A^2 = a^2 + b^2$ and taking the real part yields $\cos(\omega) = \frac{a}{A}$. Hence

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-ax}\cos(bx)\,\mathrm{d}x \quad &= \quad \frac{a}{A^2} \\ &= \quad \frac{a}{a^2+b^2}\end{aligned}$$

as desired. Finally we must verify that $\omega > \frac{\pi}{2}$ which means $\cos(\omega) > 0$. This holds as long as $a > 0$, which was one of our assumptions, so the derivation was indeed valid. □

6.11 Sequences of holomorphic functions [extra]

Recall that if $\{ f_n \}_n$ is a sequence of functions $\Omega \to \mathbb{C}$ then it is said to converge pointwise iff for every $z \in \Omega$, the sequence of complex numbers $\{ f_n(z) \}_n$ converges (which defines the limit function $f(z)$ at any point $z \in \Omega$).

Conversely, the sequence is said to converge *uniformly* iff the convergence of the sequence $\{ f_n(z) \}_n$ does not depend on z: For every $\varepsilon > 0$ there is some $N_\varepsilon \in \mathbb{N}$ (independent of z) such that if $n \geq N_\varepsilon$ then

$$|f_n(z) - f(z)| \quad < \quad \varepsilon\,.$$

See also the discussion in Appendix D.

Theorem 6.41. *If $\{ f_n \}_{n\in\mathbb{N}}$ is a sequence of holomorphic functions $\Omega \to \mathbb{C}$ which converges* pointwise *to some $f : \Omega \to \mathbb{C}$, such that for every $K \subseteq \Omega$, the convergence $f_n|_K \to f|_K$ is uniform, then f is holomorphic.*

Proof. The set $D := \operatorname{clo}(B_R(z)) \subseteq \Omega$ is compact. By Goursat, for any triangle within D,

$$\oint_T f_n \quad = \quad 0 \qquad (n \in \mathbb{N})\,.$$

Since $f_n \to f$ uniformly in D, so by [TODO: cite Rudin], f is continuous on D and furthermore

$$\oint_T f_n \quad \to \quad \oint_T f$$

which necessarily implies $\oint_T f = 0$ for any triangle $T \subseteq D$. Hence Morera's theorem Theorem 6.31 implies f is holomorphic in D, and since D was general, we draw the same conclusion on the entirety of Ω. □

This should be contrasted with a uniform limit of differentiable $\Omega \to \mathbb{R}$ functions, which is certainly not necessarily differentiable!

7 Series and poles

A *series* is a sequence $\{ z_n \}_{n\in\mathbb{N}} \subseteq \mathbb{C}$ of the form

$$z_n \quad = \quad \sum_{j=1}^{n} w_j$$

built out of some other sequence $\{ w_j \}_{j\in\mathbb{N}}$. We say that the series $\sum_{j=1}^\infty w_j$ converges iff the sequence of partial sums $\{ z_n \}_n$ converges. Clearly if a series converges, then by linearity of limits,

$$\sum_{j=1}^\infty w_j \quad = \quad \left(\sum_{j=1}^\infty \mathbb{R}\mathrm{e}\{w_j\}\right) + \mathrm{i}\left(\sum_{j=1}^\infty \mathbb{I}\mathrm{m}\{w_j\}\right)$$

since $\mathbb{Re}\{a\} \equiv \frac{1}{2}(a+\overline{a})$ and $\mathbb{Im}\{a\} \equiv \frac{1}{2\mathrm{i}}(a-\overline{a})$; the series $\{w_j\}_{j\in\mathbb{N}}$ converges iff the two real series on the RHS converge.

Definition 7.1 (absolute convergence of series)**.** A series $\sum_{n=1}^{\infty} z_n$ converges *absolutely* iff the series $\sum_{n=1}^{\infty} |z_n|$ converges.

Claim 7.2. If a series $\sum_{n=1}^{\infty} z_n$ converges absolutely then it converges.

Proof. Since $|x_n| \leq |z_n|$ and $|y_n| \leq |z_n|$, we know that the two real series $\sum_{n=1}^{\infty} x_n$, $\sum_{n=1}^{\infty} y_n$ converge absolutely. But that implies that they converge by standard theorems, e.g. [Rud76, Theorem 3.45]. □

Definition 7.3 (Power series)**.** A power series is a series depending on a parameter (that is, it's a function) of the form

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n$$

where $\{a_n\}_n \subseteq \mathbb{C}$ is some fixed sequence, $z_0 \in \mathbb{C}$ is a fixed *base point* and $z \in \mathbb{C}$ is the argument of the power series. The function

$$z \quad \mapsto \quad \sum_{n=0}^{\infty} a_n (z-z_0)^n \in \mathbb{C}$$

is defined for those points z for which the series on the RHS converges absolutely. The *radius of convergence* of a series is the (largest) number $R > 0$ such that for all $z \in B_R(z_0)$, the series *absolutely* converges.

We note that a consequence of this definition is that within the radius of convergence, the series converges absolutely!

Lemma 7.4. *A power series converges* uniformly *within compact subsets of its disc of convergence.*

Proof. Let

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n$$

be a power series with $|z-z_0| < r$, where r is strictly smaller than the radius R of the disc of convergence of the series. Define $M_n := |a_n| r^n$. Then

$$|a_n (z-z_0)^n| \quad \leq \quad M_n$$

by definition. Furthermore, with $z = z_0 + r$ we get that

$$\sum_{n=0}^{\infty} M_n \quad = \quad \sum_{n=0}^{\infty} |a_n| r^n$$

which converges since a power series converges absolutely within its disc of convergence (by definition). Hence by the so-called Weierstraß M-test ([Rud76, Theorem 7.10]) the sequence

$$\left\{ \sum_{n=0}^{N} a_n (z-z_0)^n \right\}_N$$

convergences uniformly in z. □

In particular, it should be emphasized that uniformity may fail on the entire domain. The exponential function $\exp : \mathbb{C} \to \mathbb{C}$ is such an example!

Example 7.5 (Geometric series)**.** The power series $\sum_{n=0}^{\infty} z^n$ converges whenever $|z| < 1$, and then, it equals $\frac{1}{1-z}$.

Proof. Recall from HW1 that we have proven

$$\sum_{n=0}^{N} z^n = \frac{1-z^{N+1}}{1-z} \qquad (z \neq 1)\,. \tag{7.1}$$

Now if $|z| < 1$ we may take the limit $N \to \infty$ of both sides to find that if $|z| < 1$, the term z^{N+1} converges to zero and hence the result. □

Example 7.6. The power series $\sum_{n=0}^{\infty} n^n z^n$ has zero radius of convergence, i.e., it never converges.

Proof. By the root test, we have a series converging iff the following limit exists and is smaller than 1:

$$\lim_{n\to\infty} \left(|n^n z^n|\right)^{\frac{1}{n}} \quad = \quad \lim_{n\to\infty} n\,|z| = \infty\,.$$

□

Example 7.7. The power series $\sum_{n=0}^{\infty} \frac{1}{n!} z^n$ has infinite radius of convergence, i.e., it always converges.

Proof. By the ratio test, we should examine

$$\begin{aligned} \limsup_{n\to\infty} \left| \frac{\frac{1}{(n+1)!} z^{n+1}}{\frac{1}{n!} z^n} \right| &= |z| \limsup_{n\to\infty} \frac{1}{n+1} \\ &= 0\,. \end{aligned}$$

□

Example 7.8. The power series $\sum_{n=0}^{\infty} z^n$ has radius of convergence equal to 1.

Proof. Take the root test:

$$\lim_{n\to\infty} |z^n|^{\frac{1}{n}} \quad = \quad |z|\,.$$

On the boundary circle itself, the series diverges since if it did, $\{\, z^n \,\}_n$ would have had to converge to zero, which it does not. □

Example 7.9. The power series $\sum_{n=1}^{\infty} \frac{1}{n} z^n$ has radius of convergence equal to 1.

Proof. Considering the ratio test we find

$$\begin{aligned} \left| \frac{\frac{1}{n+1} z^{n+1}}{\frac{1}{n} z^n} \right| &= |z| \frac{n}{n+1} \\ &\overset{n\to\infty}{\longrightarrow} |z|\,. \end{aligned}$$

On the boundary circle we have convergence unless $z = 1$. Indeed, in the latter case, we know that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. However, in the former case, we get

$$\sum_{n=1}^{\infty} \frac{1}{n} \mathrm{e}^{\mathrm{i}n\theta}$$

and we will see later that this is the power series for $\mathrm{Log}\,(1-\cdot)$ [TODO: add citation]. $\square$

Example 7.10. Think about $\sum_{n=1}^{\infty}\frac{z^n}{n^2}$ and its radius of convergence. What happens on the boundary circle?

7.1 Taylor series expansions

We are familiar with Taylor's theorem in multivariable calculus which allows us to make an expansion of a function $f:\mathbb{R}^n\to\mathbb{R}$ as

$$f(x_0+x) \approx f(x_0)+\langle(\nabla f)(x_0),x\rangle_{\mathbb{R}^n}+\frac{1}{2}\langle x,(\mathbb{H}f)(x_0)\,x\rangle_{\mathbb{R}^n}+\dots \qquad (x_0,x\in\mathbb{R}^n)\;.$$

where $\mathbb{H}f$ is the Hessian matrix (-valued function) of f,

$$(\mathbb{H}f)_{ij} \equiv \partial_i\partial_j f \qquad (i,j=1,\dots,n)\;.$$

To make an approximation to Nth order, a function must have at least N continuous derivatives.

We begin with a notion of analyticity:

Definition 7.11 (Analytic function)**.** A function $f:\mathbb{C}\to\mathbb{C}$ is *analytic* at a point z_0 iff there is a convergent series

$$\sum_{n=0}^{\infty}a_n z_0^n$$

such that

$$f(z_0) = \sum_{n=0}^{\infty}a_n z_0^n\;.$$

But this is somewhat of a vacuous condition, since one could always take $a_0=f(z_0)$ and $a_n=0$ for all $n\geq 1$. More interesting, this property could hold in *on a set.* I.e., a function f is analytic on a set $S\subseteq\mathbb{C}$ iff there is some convergent power series

$$S\ni z\mapsto\sum_{n=0}^{\infty}a_n z^n$$

and such that

$$f(z) = \sum_{n=0}^{\infty}a_n z^n \qquad (z\in S)\,.$$

Down below we will see that the power series expansion of an analytic function about a given point, is unique.

We have seen that a function $f:\mathbb{C}\to\mathbb{C}$ being holomorphic implies it is smooth in Corollary 6.27. But in fact, *more* is true: holomorphicity implies analyticity, in the sense the function's Taylor series approximation converges.

Theorem 7.12 (Taylor)**.** *Let $f:\mathbb{C}\to\mathbb{C}$ be holomorphic within $B_R(z_0)$ and on its boundary. Then f is* analytic *on $B_R(z_0)$. I.e., the Taylor series for f converges and equals f at any $z\in B_R(z_0)$:*

$$\begin{aligned} f(z_0+z) &= \sum_{n=0}^{\infty}\frac{1}{n!}f^{(n)}(z_0)\,z^n \qquad (z_0+z\in B_R(z_0)) \\ &= f(z_0)+f'(z_0)\,z+\frac{1}{2}f''(z_0)\,z^2+\dots \end{aligned}$$

Another way to write this expansion is as

$$f(z)=\sum_{n=0}^{\infty}\frac{1}{n!}f^{(n)}(z_0)\,(z-z_0)^n \qquad (z\in B_R(z_0))\;. \tag{7.2}$$

When $z_0=0$ this series is called a Maclaurin *series.*

This theorem finally justifies the conflation of:

1. Holomorphic.

2. Analytic.

It should be stressed that the radius of convergence of the series (7.2) about z_0 is the distance of z_0 to the nearest singularity of f.

Proof. Assume $z_0 = 0$ and consider the Cauchy integral formula (6.13)

$$f(z) \quad = \quad \frac{1}{2\pi \mathrm{i}} \oint \frac{f(w)}{w-z} \mathrm{d}w$$

where we choose an integration contour $B_r(0)$ of radius $r \in (|z|, R)$, so it along some circle entirely contained within $B_R(0)$ but which contains $z \in B_R(0)$. Hence, for all points w on this circle, $|w| > |z|$. Now, using (7.1) rewrite

$$\begin{aligned}\frac{1}{w-z} &= \frac{1}{w}\frac{1}{1-\frac{z}{w}} \\ &= \frac{1}{w}\left[\sum_{n=0}^{N}\left(\frac{z}{w}\right)^n + \frac{\left(\frac{z}{w}\right)^{N+1}}{1-\left(\frac{z}{w}\right)}\right]\end{aligned}$$

Plug this into our integral formula for f to get

$$\begin{aligned} f(z) &= \frac{1}{2\pi\mathrm{i}}\oint \frac{f(w)}{w-z}\mathrm{d}w \\ &= \frac{1}{2\pi\mathrm{i}}\oint \frac{f(w)}{w}\left[\sum_{n=0}^{N}\left(\frac{z}{w}\right)^n + \frac{\left(\frac{z}{w}\right)^{N+1}}{1-\left(\frac{z}{w}\right)}\right]\mathrm{d}w \\ &= \sum_{n=0}^{N}\left[\frac{1}{2\pi\mathrm{i}}\oint\frac{f(w)}{w^{n+1}}\mathrm{d}w\right]z^n + \left[\frac{1}{2\pi\mathrm{i}}\oint\frac{f(w)}{(w-z)\,w^{N+1}}\mathrm{d}w\right]z^{N+1}\,. \end{aligned}$$

Now, we recognize from (6.12) that

$$\frac{1}{2\pi\mathrm{i}}\oint\frac{f(w)}{w^{n+1}}\mathrm{d}w \quad = \quad \frac{f^{(n)}(0)}{n!} \qquad (n\in\mathbb{N}_{\geq 0})\,.$$

To control the error term, we estimate similarly to Theorem 6.30

$$\left|z^{N+1}\oint\frac{f(w)}{(w-z)\,w^{N+1}}\mathrm{d}w\right| \quad \leq \quad 2\pi r\frac{\|f\|_{\partial B_r(0)}}{\inf_{w\in\partial B_r(0)}|w-z|}\left(\sup_{w\in\partial B_r(0)}\left|\frac{z}{w}\right|\right)^{N+1}$$

and we have $\inf_{w\in\partial B_r(0)}|w-z| = r-|z| > 0$ by choice of r as well as

$$\left|\frac{z}{w}\right| = \frac{|z|}{r} < 1\,.$$

Hence the error term is bounded by

$$\begin{aligned} &\leq \quad 2\pi r\frac{\|f\|_{\partial B_r(0)}}{r-|z|}\left(\frac{|z|}{r}\right)^{N+1} \\ &\to \quad 0 \qquad (N\to\infty)\,. \end{aligned}$$

Since the error term converges to zero, the series converges.

The case $z_0 \neq 0$ is dealt with as follows: define the function

$$g(z) \quad := \quad f(z_0+z) \qquad (z\in B_R(0))\,.$$

Then $g:\mathbb{C}\to\mathbb{C}$ is holomorphic on $B_R(0)$ since f is holomorphic on $B_R(z_0)$. Apply the above analysis to g to obtain

$$g(z) \quad = \quad \sum_{n=0}^{\infty}\frac{1}{n!}g^{(n)}(0)\,z^n \qquad (z\in B_R(0))$$

and note $g^{(n)}(0)=f^{(n)}(z_0)$. Indeed, For $n=0,1$ this is true by definition of g, and by induction:

$$\begin{aligned} g^{(n+1)}(0) &\equiv \lim_{z\to 0}\left[\frac{g^{(n)}(z)-g^{(n)}(0)}{z}\right] \\ &= \lim_{z\to 0}\left[\frac{f^{(n)}(z_0+z)-f^{(n)}(z_0)}{z}\right] \\ &\equiv f^{(n+1)}(z_0)\ . \end{aligned}$$

Hence we find

$$f(z_0+z) \quad = \quad \sum_{n=0}^{\infty}\frac{1}{n!}f^{(n)}(z_0)\,z^n \qquad (z\in B_R(0))$$

which is equivalent (with the change of variable $w=z_0+z$) to

$$f(w) \quad = \quad \sum_{n=0}^{\infty}f^{(n)}(z_0)\,(w-z_0)^n \qquad (w\in B_R(z_0))\ .$$

□

Corollary 7.13. *The power series expansion of an analytic function about a point is unique.*

Proof. Let $f:\mathbb{C}\to\mathbb{C}$ be analytic and assume that

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n=\sum_{n=0}^{\infty}b_n(z-z_0)^n \qquad (z\in B_R(z_0))$$

for some $R>0$. We want to show that $a_n=b_n$ for all $n\geq 0$.

By linearity of limits, the above implies

$$0 \quad = \quad \sum_{n=0}^{\infty}(a_n-b_n)\,z^n \qquad (z\in B_R(0))\ .$$

However, if

$$0 \quad = \quad \sum_{n=0}^{\infty}c_n z^n \qquad (z\in B_R(0))$$

with $R>0$ then $c_n=0$ for all $n\geq 0$. The case $n=0$ is obtained by plugging in $z=0$ into this equation. By taking the kth derivative of the above formula with respect to z (we are allowed to exchange limits because the convergence of the series is uniform in its disc of convergence, by Lemma 7.4; this is where $R>0$ is used) we find

$$0 \quad = \quad \sum_{n=k}^{\infty}c_n n(n-1)\dots(n-k+1)\,z^{n-k}$$

and now plugging $z=0$ into the above we find that $c_k=0$. □

Example 7.14 (Complex sine function)**.** The complex sine function $\sin : \mathbb{C} \to \mathbb{C}$ is the entire function defined via

$$\sin(z) \quad := \quad \frac{e^{iz} - e^{-iz}}{2i} \qquad (z \in \mathbb{C}) \ .$$

Since exp is entire, so is sin. This means it should have a Taylor series with an infinite radius of convergence. Let us find it about the origin:

$$\begin{aligned} \sin^{(1)}(z) &= \cos(z) \\ \sin^{(2)}(z) &= -\sin(z) \\ \sin^{(3)}(z) &= -\cos(z) \\ \sin^{(4)}(z) &= \sin(z) \\ &\dots \end{aligned}$$

so

$$\sin^{(n)}(0) \quad = \quad \begin{cases} 1 & n \equiv 1 \mod 4 \\ 0 & n \equiv 2 \mod 4 \\ -1 & n \equiv 3 \mod 4 \\ 0 & n \equiv 4 \mod 4 \end{cases}$$

so

$$\begin{aligned} \sin(z) &= \sum_{n=0}^{\infty} \frac{1}{n!} \sin^{(n)}(0)\, z^n \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} \end{aligned}$$

and this series converges for all $z \in \mathbb{C}$ (but uniformly only within compact subsets).

Example 7.15. Consider the function $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = \mathrm{Log}(1-z)$, which we know is holomorphic on $\mathbb{C} \setminus \{\, x \mid x \leq 0 \,\}$ so when $\mathbb{Re}\{z\} < 1$. So using Taylor's theorem we should be able to write down a Taylor series expansion about the point $z_0 = 0$:

$$f(z) \quad = \quad \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n \qquad (|z| < 1) \ .$$

All we have to do now is figure out what $f^{(n)}(0)$ is:

$$\begin{aligned} f^{(1)}(z) &= \frac{1}{1-z}(-1) \\ f^{(2)}(z) &= \frac{1}{(1-z)^2} \\ f^{(3)}(z) &= \frac{2}{(1-z)^3} \\ f^{(4)}(z) &= \frac{2 \cdot 3}{(1-z)^4} \\ &\dots \\ f^{(n)}(z) &= \frac{(n-1)!}{(1-z)^n} \ . \end{aligned}$$

So we find, with $f^{(0)}(0) \equiv \mathrm{Log}(1) = 0$,

$$f^{(n)}(z) = -\sum_{n=1}^{\infty} \frac{1}{n} z^n \qquad (|z| < 1) \, .$$

7.2 Miracles of analyticity

In what follows we see some dramatic consequences of the fact an analytic function behaves much like a polynomial: it cannot be non-trivial yet have non-isolated zeros. Indeed, in Corollary 6.34 we saw that a polynomial of degree n has precisely (counting multiplicity) n roots (=zeros). In particular it has a finite number of zeros. Analytic functions, which are convergent power series, are limits of polynomials, and hence should have similar properties.

Theorem 7.16. *Let $f : \Omega \to \mathbb{C}$ be holomorphic where Ω is open and connected, and assume further that f vanishes on a sequence of distinct points with a limit point* in Ω. *Then it is actually the zero function, i.e.,*

$$f = 0 \, .$$

Proof. We follow the presentation of [SS03]. Let w be *the* limit point of the sequence $\{ w_m \}_m \subseteq \Omega$ on which f is zero:

$$\lim_{n\to\infty} w_n = w$$

and

$$f(w_m) = 0 \qquad (m \in \mathbb{N}) \, .$$

Let $\varepsilon > 0$ be sufficiently small so that $B_\varepsilon(w) \subseteq \Omega$ (possible as Ω is open, and recall, by hypothesis $w \in \Omega$). We first want to show that

$$f|_{B_\varepsilon(w)} = 0 \, .$$

To that end, let $\{ a_n \}_n \subseteq \mathbb{C}$ be the power series expansion of f in $B_\varepsilon(w)$:

$$f(z) = \sum_{n=0}^{\infty} a_n (z-w)^n \qquad (z \in B_\varepsilon(w)) \, .$$

Now if f is not identically zero within $B_\varepsilon(w)$, not all $\{ a_n \}_n$ are zero, so there must exist some smallest $N \in \mathbb{N}$ such that $a_N \neq 0$. Then it must be that

$$f(z) = a_N (z-w)^N (1 + g(z-w))$$

for some g which obeys $g(z-w) \to 0$ as $z \to w$, since all higher order terms converge *faster* than $(z-w)^N$ to zero as $z \to w$. Now for any m,

$$f(w_m) = 0$$

by assumption, but also, since all $\{ w_m \}_m$ are distinct and converge to w, then for m sufficiently large we have

$$\begin{aligned} \left| a_N (w_m - w)^N (1 + g(w_m - w)) \right| &\geq |a_N| \, |w_m - w|^N \left| 1 - \underbrace{|g(w_m - w)|}_{\text{small for } m \text{ large}} \right| \\ &> \frac{1}{2} |a_N| \, |w_m - w|^N \qquad \text{(say, there's some m so that this holds)} \\ &> 0 \end{aligned}$$

which is a contradiction with $f(w_m) = 0$. Hence $f|_{B_\varepsilon(w)} = 0$ necessarily.

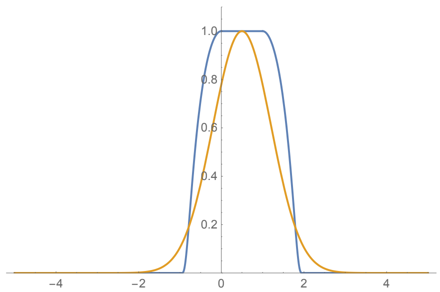

Figure 20: Analytic Gaussian (orange) vs. smooth bump function (blue).

Next, let $U \subseteq \Omega$ be the interior of the set

$$\{ z \in \Omega \mid f(z) = 0 \} .$$

It is by definition open (interiors are always open) and non-empty because it contains $B_\varepsilon(w)$. Actually it is also closed, since if $\{ z_n \}_n \subseteq U$ converges to some $z \in \mathbb{C}$, then $f(z_n) = 0$, and then $f(z) = 0$ by continuity of f. So $z \in U$ and so U contains all its limit points. But then $U \subseteq \Omega$ is a non-empty *clopen* set. But the only such sets within a connected Ω are Ω itself, via Lemma 4.18. □

This theorem is quite strong, and in particular it says that if an analytic function varnishes on some open subset then it is identically zero!

As a result, one cannot hope for an *analytic* bump function of compact support:

Definition 7.17 (bump function)**.** A bump function is a smooth approximation of a characteristic function.

For example, to approximate

$$\chi_{[0,1]}(x) \quad \equiv \quad \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & 0 > x \vee x > 1 \end{cases} \qquad (x \in \mathbb{R})$$

we may try the analytic Gaussian

$$\mathbb{R} \ni x \mapsto \exp\left(-\left(x - \frac{1}{2} \right)^2 \right)$$

but that has no compact support and also no flat top. If we insist we may come up with the smooth version of $\chi_{[0,1]}$ as

$$\mathbb{R} \ni x \quad \mapsto \quad \begin{cases} 0 & x < -1 \\ \exp\left(1 - \frac{1}{1-x^2}\right) & -1 \leq x \leq 0 \\ 1 & 0 \leq x \leq 1 \\ \exp\left(1 - \frac{1}{1-(x-1)^2}\right) & 1 \leq x \leq 2 \\ 0 & x > 2 \end{cases}$$

but the theorem above says that we'll never be able to find a version of this bump function which is both analytic and with compact support: smooth is the best we can manage, and hence, there will not be a power series expansion! See Figure 20.

We proceed with more consequences of the above theorem:

Corollary 7.18. *Suppose $f, g : \Omega \to \mathbb{C}$ are holomorphic in some connected open Ω and that there is a sequence of distinct points $\{ z_n \}_n \subseteq \Omega$ which converges to some $z \in \Omega$, and such that*

$$f(z_n) = g(z_n) \qquad (n \in \mathbb{N}) .$$

Then $f = g$. In particular if $S \subseteq \Omega$ is open and non-empty and if further

$$f|_S \quad = \quad g|_S$$

then $f = g$.

Definition 7.19 (Analytic continuation)**.** Let $f : \Omega \to \mathbb{C}$ be holomorphic on some open connected $\Omega \subseteq \mathbb{C}$. Assume further that $\tilde{f} : \tilde{\Omega} \to \mathbb{C}$ is holomorphic, $\tilde{\Omega} \supseteq \Omega$ is open and connected, and that $\tilde{f}\Big|_{\Omega} = f$. Then $\tilde{f}$ is called *the* analytic continuation of f onto $\tilde{\Omega}$. It is indeed unique.

Proof. The proof of uniqueness is a corollary of the above. □

7.3 Poles, residues and the Laurent series

We have just seen that unless an analytic function f is identically zero, if $f(z_0) = 0$ then z_0 is an *isolated* zero, meaning

$$f(z_0 + z) \quad \neq \quad 0$$

for all $z \in B_\varepsilon(z_0)$ for some $\varepsilon > 0$. In fact, more is true

Lemma 7.20. *If $f : \Omega \to \mathbb{C}$ is analytic on some open connected set $\Omega \subseteq \mathbb{C}$, $f \neq 0$ identically, and $f(z_0) = 0$ for some $z_0 \in \Omega$, then there is some $\varepsilon > 0$ and some analytic $g : B_\varepsilon(z_0) \to \mathbb{C}$ which never vanishes, and finally, some unique $n \geq 1$, such that*

$$f(z) \quad = \quad (z - z_0)^n g(z) \qquad (z \in B_\varepsilon(z_0))$$

in which case we say f has a zero of order n at z_0.

Proof. Write a power series expansion of f at z_0

$$f(z) \quad = \quad \sum_{n=0}^{\infty} a_n (z - z_0)^n \, .$$

Since $f \neq 0$, $\exists n \geq 1$ such that $a_n \neq 0$. For that n, we have

$$f(z) \quad = \quad (z - z_0)^n \underbrace{[a_n + a_{n+1}(z - z_0) + \dots]}_{=:g(z)} \, .$$

Since $a_n \neq 0$, g is a holomorphic function which does not vanish for all $z - z_0$ sufficiently small.

To prove uniqueness, assume

$$(z - z_0)^n g(z) \quad = \quad (z - z_0)^m h(z)$$

for all $|z - z_0|$ sufficiently small, for some g, h which do not vanish. If $m > n$, divide by $(z - z_0)^n$ to get

$$g(z) \quad = \quad (z - z_0)^{m-n} h(z)$$

and take the limit $z \to z_0$ to get a contradiction with $g(z_0) \neq 0$. □

Definition 7.21 (Poles)**.** The function $f : \mathbb{C} \to \mathbb{C}$ is said to have a pole at z_0 of order $n \geq 1$ iff $\frac{1}{f}$ is a holomorphic function which has a zero at z_0 of order n.

When putting these pieces together we learn that

Lemma 7.22. *If $f : \mathbb{C} \to \mathbb{C}$ has a pole of order $n \geq 1$ at $z_0 \in \mathbb{C}$ then there exists some $\varepsilon > 0$ and some holomorphic non-vanishing $g : \mathbb{C} \to \mathbb{C}$ such that*

$$f(z) \quad = \quad (z - z_0)^{-n} g(z) \qquad (z \in B_\varepsilon(z_0)) \, .$$

Proof. Apply the above theorem on $\frac{1}{f}$. $\square$

Definition 7.23 (Meromorphic function)**.** A function $f:\mathbb{C}\setminus S\to\mathbb{C}$ is said to be *meromorphic* iff it is holomorphic and if S is a set (possibly infinite) of isolated points which do not accumulate and on which f has finite order poles.

Theorem 7.24 (Laurent series expansion and residue)**.** *Let $f:\mathbb{C}\to\mathbb{C}$ be holomorphic except a pole of order $1\leq n<\infty$ at $z_0\in\mathbb{C}$. Then f has an expansion of the form*

$$f(z)=\underbrace{\frac{a_{-n}}{(z-z_0)^n}+\frac{a_{-(n-1)}}{(z-z_0)^{n-1}}+\cdots+\frac{a_{-1}}{z-z_0}}_{\text{principal part of } f \text{ at } z_0}+g(z) \tag{7.3}$$

for some holomorphic g (which has its own Taylor expansion). The coefficient a_{-1} is called the residue of f at z_0, and is denoted by

$$\text{residue}_{z_0}(f)\quad\equiv\quad a_{-1}\,.$$

Proof. Use the above lemma to write

$$f(z)\quad=\quad(z-z_0)^{-n}g(z)$$

and plug in a power series expansion for g about z_0:

$$\begin{aligned}f(z)\quad&=\quad(z-z_0)^{-n}\sum_{l=0}^{\infty}b_l(z-z_0)^l\\&=\quad\sum_{l=0}^{\infty}b_l(z-z_0)^{l-n}\,.\end{aligned}$$

$\square$

Corollary 7.25. *If f has one isolated pole of order $1\leq n<\infty$ at z_0 then*

$$\oint_\Gamma f\mathrm{d}z\quad=\quad2\pi\mathrm{i}\,\text{residue}_{z_0}(f)$$

for any Γ which encircles z_0, where $\text{residue}_{z_0}(f)=a_{-1}$ *is the residue of f at z_0 as above, i.e., the coefficient of the term $\frac{1}{z-z_0}$ in the expansion about z_0.*

Proof. In HW4Q6 we have proven using Cauchy's integral formula

$$\oint_\Gamma(z-z_0)^k\mathrm{d}z\quad=\quad2\pi\mathrm{i}\delta_{k,-1}$$

where Γ is any contour which encircles z_0. Now, if we write the expansion for f as

$$f(z)\quad=\quad\sum_{k=-n}^{\infty}a_k(z-z_0)^{-k}$$

plug it into the integral and exchange integration and summation (allowed thanks to Lemma 7.4) we find

$$\oint f(z)\mathrm{d}z\quad=\quad2\pi\mathrm{i}a_{-1}\equiv2\pi\mathrm{i}\,\text{residue}_{z_0}(f)\,.$$

$\square$

Corollary 7.26. *A more general formula is*

$$\oint_{\Gamma} f(z)\,\mathrm{d}z = 2\pi \mathrm{i} \sum_{j=1}^{N} \mathrm{residue}_{z_j}(f) \tag{7.4}$$

where $\{ z_j \}_{j=1}^{N}$ *are the set of* N *poles of* f *within* $\mathrm{int}(\gamma)$*. This is a direct generalization of Theorem 6.22 to the case that* f *is not holomorphic* but *has a finite number of finite-order poles.*

Proof. Say we have two (isolated) poles and the contour Γ encircles both of them. Bisect the contour with an additional leg L which defines two new contours Γ_1 and Γ_2, each encircling one of the poles, and having L as one of their legs (but in opposite direction). Since L appears in opposite orientation, it cancels out and we have

$$\oint_{\Gamma} f(z)\,\mathrm{d}z = \oint_{\Gamma_1} f(z)\,\mathrm{d}z + \oint_{\Gamma_2} f(z)\,\mathrm{d}z\,.$$

$\square$

We also have the general formula

Lemma 7.27. *If* f *has a pole of order* $1 < n < \infty$ *at* z_0 *then*

$$\boxed{\mathrm{residue}_{z_0}(f) = \lim_{z\to z_0} \frac{1}{(n-1)!}\partial_z^{n-1}(z-z_0)^n f(z)}\,. \tag{7.5}$$

In particular, if f *has a simple pole (of order* $n=1$*) at* z_0 *then*

$$\mathrm{residue}_{z_0}(f) = \lim_{z\to z_0}(z-z_0)f(z)\,. \tag{7.6}$$

Proof. Use (7.3) to write

$$(z-z_0)^n f(z) = a_{-n} + a_{-n+1}(z-z_0) + \cdots + a_{-1}(z-z_0)^{n-1} + g(z)(z-z_0)^n$$

$\square$

For later we also record another way to phrase these formulas, whose proof already follows from what was stated above

Theorem 7.28 (Laurent series expansion)**.** *Let* $f : \mathbb{C} \to \mathbb{C}$ *be given such that it is holomorphic in the annual region*

$$\Omega := B_R(z_0) \setminus B_r(z_0)$$

for some $0 < r < R$ *and some* $z_0 \in \mathbb{C}$*. Then* f *has the following* Laurent series expansion *in positive and negative powers:*

$$f(z_0+z) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(z_0)\, z^n + \sum_{n=1}^{\infty} b_n z^{-n} \qquad (z_0 + z \in \Omega)$$

with

$$b_n = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma} f(w)(w-z_0)^{n-1}\,\mathrm{d}w \qquad (n \in \mathbb{N}_{\geq 1})$$

and Γ *is a closed contour within* Ω *which encircles* z_0*.*

In the simplest example for a simple pole, residue calculation would amount to the coefficient of the term proportional to

$$\frac{1}{z-z_0}\,.$$

Example 7.29. Let $g : \mathbb{C} \to \mathbb{C}$ be entire. The function $f : \mathbb{C} \setminus \{ z_0 \} \to \mathbb{C}$ given by

$$f(z) = \frac{g(z)}{z - z_0}$$

is meromorphic and

$$\text{residue}_{z_0}(f) \quad = \quad g(z_0) \, .$$

Proof. This could be done in one of two ways. The easy way is using Lemma 7.27 which yields

$$\text{residue}_{z_0}(f) \quad = \quad \lim_{z \to z_0} g(z) \, .$$

Now since g is analytic, it is continuous and hence we find

$$\text{residue}_{z_0}(f) = \lim_{z \to z_0} g(z) = g\left(\lim_{z \to z_0} z\right) = g(z_0) \, .$$

Alternatively, we can use the residue formula, which says

$$\text{residue}_{z_0}(f) \quad = \quad \frac{1}{2\pi \mathrm{i}} \oint_\Gamma f(z) \, \mathrm{d}z$$

where Γ is any simple closed CCW contour which encircles z_0. Then we have, using Cauchy's integral formula Theorem 6.26,

$$\begin{aligned} \frac{1}{2\pi \mathrm{i}} \oint_\Gamma f(z) \, \mathrm{d}z \quad &= \quad \frac{1}{2\pi \mathrm{i}} \oint_\Gamma \frac{g(z)}{z - z_0} \mathrm{d}z \\ &= \quad g(z_0) \, . \end{aligned}$$

□

Example 7.30. Sometimes the limit is really necessary, for example, consider the function

$$f(z) \quad = \quad \frac{5}{z - z_0 + (z - z_0)^{42}} \, .$$

We claim that

$$\text{residue}_{z_0}(f) \quad = \quad 5 \, .$$

Proof. Consider the limit

$$\begin{aligned} \lim_{z \to z_0} (z - z_0) f(z) &= \lim_{z \to z_0} \frac{5(z - z_0)}{z - z_0 + (z - z_0)^{42}} \\ &= 5 \lim_{z \to z_0} \frac{1}{1 + (z - z_0)^{41}} \\ &= 5 \frac{1}{1 + \lim_{z \to z_0} (z - z_0)^{41}} && \text{(Continuity)} \\ &= 5 \, . \end{aligned}$$

□

7.4 Integration using the residue formula

Example 7.31. Consider $\int_{x=-\infty}^{\infty}\frac{1}{1+x^2}\mathrm{d}x=\pi$.

Proof. Define $f(z):=\frac{1}{1+z^2}$ for all z which make sense. f has two simple (=of order 1) poles at $\pm\mathrm{i}$. Consider the semicircular contour Γ_R composed of

$$[0,\pi]\ni t\quad\mapsto\quad R\mathrm{e}^{\mathrm{i}t}$$

and

$$[-R,R]\ni x\quad\mapsto\quad x\,.$$

Using (7.4) we have

$$\oint_{\Gamma_R} f(z)\,\mathrm{d}z \quad=\quad 2\pi\mathrm{i}\sum_i \mathrm{residue}_{z_i}(f)$$

where z_i is a pole of f within $\mathrm{int}(\Gamma_R)$. Since the poles of f are $\pm\mathrm{i}$, and Γ_R is the upper semi-circle of radius R centered at the origin, it is only the pole at i which will contribute to the residue formula. Hence we have

$$\oint_{\Gamma_R} f(z)\,\mathrm{d}z \quad=\quad 2\pi\mathrm{i}\,\mathrm{residue}_{\mathrm{i}}(f)\,.$$

Next, we must calculate the residue. To that end, let us write

$$f(z)=\frac{1}{1+z^2}=\frac{1}{(z-\mathrm{i})(z+\mathrm{i})}$$

and use the formula in Lemma 7.27 to get

$$\mathrm{residue}_{\mathrm{i}}(f)=\lim_{z\to\mathrm{i}}(z-\mathrm{i})f(z)=\lim_{z\to\mathrm{i}}\frac{1}{z+\mathrm{i}}=\frac{1}{2\mathrm{i}}\,.$$

Hence we find

$$\oint_{\Gamma_R} f(z)\,\mathrm{d}z \quad=\quad \frac{2\pi\mathrm{i}}{2\mathrm{i}}=\pi\,.$$

On the other hand, we now take the limit $R\to\infty$ of this equation to get

$$\pi \quad=\quad \lim_{R\to\infty}\int_{-R}^{R} f(x)\,\mathrm{d}x+\lim_{R\to\infty}\int_{t=0}^{\pi} f\left(R\mathrm{e}^{\mathrm{i}t}\right)R\mathrm{e}^{\mathrm{i}t}\mathrm{i}\mathrm{d}t\,.$$

The first term is precisely what we wanted to calculate, so if we can show the second term converges to zero we'd be done. Let us study it:

$$\begin{aligned}\left|\int_{t=0}^{\pi} f\left(R\mathrm{e}^{\mathrm{i}t}\right)R\mathrm{e}^{\mathrm{i}t}\mathrm{i}\mathrm{d}t\right| &= \left|\int_{t=0}^{\pi}\frac{1}{1+R^2\mathrm{e}^{2\mathrm{i}t}}R\mathrm{e}^{\mathrm{i}t}\mathrm{i}\mathrm{d}t\right| \\ &\leq R\int_{t=0}^{\pi}\frac{1}{\left|1+\mathrm{e}^{2\mathrm{i}t}R^2\right|}\mathrm{d}t\,.\end{aligned}$$

Now,

$$\begin{aligned}\left|1+\mathrm{e}^{2\mathrm{i}t}R^2\right|^2 &= 1+R^4-2R^2\cos(2t) \\ &\geq 1+R^4-2R^2 \\ &= \left(R^2-1\right)^2\end{aligned}$$

so we get

$$
\begin{aligned}
\star &\leq \frac{R}{R^2-1}\pi \\
&\overset{R\to\infty}{\sim} \frac{\pi}{R} \\
&\to 0\,.
\end{aligned}
$$

□

Note that in the above, one could have used the lower semicircle with the other pole just as well.

Example 7.32. For any $a \in (0,1)$,

$$\int_{x=-\infty}^{\infty} \frac{\mathrm{e}^{ax}}{1+\mathrm{e}^{x}}\mathrm{d}x = \frac{\pi}{\sin(\pi a)}\,.$$

Proof. Define $f(z) := \frac{\mathrm{e}^{az}}{1+\mathrm{e}^{z}}$ and consider the CCW closed rectangular contour Γ_R with vertices at

$$-R, R, R+2\pi\mathrm{i}, -R+2\pi\mathrm{i}\,.$$

We have the following (simple) poles of f:

$$
\begin{aligned}
1+\mathrm{e}^{z} &\overset{!}{=} 0 \\
\mathrm{e}^{z} &= -1 \\
z &= \log(-1) \\
&= \mathrm{i}\arg(-1) \\
&= \mathrm{i}(\pi+2\pi n) \qquad (n\in\mathbb{Z})\,.
\end{aligned}
$$

Within the contour Γ_R only the pole $\mathrm{i}\pi$ is contained. Let us calculate the reside at $\mathrm{i}\pi$ using Lemma 7.27 again:

$$
\begin{aligned}
\mathrm{residue}_{\mathrm{i}\pi}(f) &= \lim_{z\to\mathrm{i}\pi}(z-\mathrm{i}\pi)f(z) \\
&= \lim_{z\to\mathrm{i}\pi}(z-\mathrm{i}\pi)\frac{\mathrm{e}^{az}}{1+\mathrm{e}^{z}} \\
&= \lim_{z\to\mathrm{i}\pi}\mathrm{e}^{az}\frac{(z-\mathrm{i}\pi)}{\mathrm{e}^{z}-\mathrm{e}^{\mathrm{i}\pi}}\,. && (\text{Use } \mathrm{e}^{\mathrm{i}\pi}=\text{-1})
\end{aligned}
$$

Now one may recognize the second term in the pre-limit as $\frac{1}{\exp'(\mathrm{i}\pi)}$. Hence we have

$$
\begin{aligned}
\mathrm{residue}_{\mathrm{i}\pi}(f) &= \mathrm{e}^{a\mathrm{i}\pi}\frac{1}{\mathrm{e}^{\mathrm{i}\pi}} \\
&= -\mathrm{e}^{a\mathrm{i}\pi}\,.
\end{aligned}
$$

This step might have been able to be done via HW2Q9 too ([TODO: check this]). Thus we find

$$\oint_{\Gamma_R} f(z)\,\mathrm{d}z = -2\pi\mathrm{i}\mathrm{e}^{a\mathrm{i}\pi}\,.$$

On the other hand,

$$\oint_{\Gamma_R} f(z)\,\mathrm{d}z = \int_{-R}^{R} f(x)\,\mathrm{d}x + \int_{R}^{-R} f(x+2\pi\mathrm{i})\,\mathrm{d}x + \int_{0}^{2\pi} f(R+\mathrm{i}t)\,\mathrm{i}\mathrm{d}t + \int_{2\pi}^{0} f(-R+\mathrm{i}t)\,\mathrm{i}\mathrm{d}t\,.$$

Now,

$$\begin{aligned}\int_{R}^{-R} f\left(x+2\pi\mathrm{i}\right)\mathrm{d}x &= \int_{R}^{-R}\frac{\mathrm{e}^{a(x+2\pi\mathrm{i})}}{1+\mathrm{e}^{(x+2\pi\mathrm{i})}}\mathrm{d}x\\ &= \mathrm{e}^{2\pi\mathrm{i}a}\int_{R}^{-R}\frac{\mathrm{e}^{ax}}{1+\mathrm{e}^{x}}\mathrm{d}x\\ &= -\mathrm{e}^{2\pi\mathrm{i}a}\int_{-R}^{R}\frac{\mathrm{e}^{ax}}{1+\mathrm{e}^{x}}\mathrm{d}x\end{aligned}$$

whereas

$$\begin{aligned}\left|\int_{0}^{2\pi} f\left(R+\mathrm{i}t\right)\mathrm{i}\mathrm{d}t\right| &= \left|\int_{0}^{2\pi}\frac{\mathrm{e}^{a(R+\mathrm{i}t)}}{1+\mathrm{e}^{(R+\mathrm{i}t)}}\mathrm{d}t\right|\\ &\leq \mathrm{e}^{aR}\int_{0}^{2\pi}\frac{1}{\left|1+\mathrm{e}^{R+it}\right|}\mathrm{d}t\\ &\leq 2\pi\mathrm{e}^{-(1-a)R}\,.\end{aligned}$$

Since we have assumed $a<1$, this converges exponentially fast to zero. The same happens with the other vertical leg. We thus find

$$\begin{aligned}\lim_{R\to\infty}\oint_{\Gamma_R} f\left(z\right)\mathrm{d}z &= \int_{x=-\infty}^{\infty}\frac{\mathrm{e}^{ax}}{1+\mathrm{e}^{x}}\mathrm{d}x-\mathrm{e}^{2\pi\mathrm{i}a}\int_{x=-\infty}^{\infty}\frac{\mathrm{e}^{ax}}{1+\mathrm{e}^{x}}\mathrm{d}x\\ &= \left(1-\mathrm{e}^{2\pi\mathrm{i}a}\right)\int_{x=-\infty}^{\infty}\frac{\mathrm{e}^{ax}}{1+\mathrm{e}^{x}}\mathrm{d}x\,.\end{aligned}$$

So that

$$\begin{aligned}\int_{x=-\infty}^{\infty}\frac{\mathrm{e}^{ax}}{1+\mathrm{e}^{x}}\mathrm{d}x &= 2\pi\mathrm{i}\frac{-\mathrm{e}^{a\mathrm{i}\pi}}{1-\mathrm{e}^{2\pi\mathrm{i}a}}\\ &= \frac{\pi}{\sin\left(\pi a\right)}\end{aligned}$$

as desired. □

Example 7.33. For any $\xi\in\mathbb{R}$,

$$\int_{x=-\infty}^{\infty}\frac{\mathrm{e}^{-2\pi\mathrm{i}x\xi}}{\cosh\left(\pi x\right)}\mathrm{d}x = \frac{1}{\cosh\left(\pi\xi\right)}\,.$$

7.5 Removable and essential singularities

Definition 7.34. For some open $\Omega\subseteq\mathbb{C}$, a holomorphic function $f:\Omega\setminus\{z_0\}\to\mathbb{C}$ is said to have a removable singularity at z_0 iff there exists a unique analytic extension $\tilde{f}:\Omega\to\mathbb{C}$ of f (see Definition 7.19).

Example 7.35. The function $f:\mathbb{C}\setminus\{0\}\to\mathbb{C}$ given by $z\mapsto\frac{\sin(z)}{z}$ may be extended to $\tilde{f}:\mathbb{C}\to\mathbb{C}$ via

$$\tilde{f}\left(z\right):=\begin{cases}f\left(z\right) & z\neq 0\\ 1 & z=0\end{cases}\,.$$

Then clearly $\tilde{f}$ is an extension of f, and moreover, $\tilde{f}$ is holomorphic at the origin. Indeed,

$$\begin{aligned}\lim_{z\to 0}\frac{\tilde{f}(z)-\tilde{f}(0)}{z} &= \lim_{z\to 0}\frac{\frac{\sin(z)}{z}-1}{z}\\ &= \lim_{z\to 0}\frac{\sin(z)-z}{z^2}\\ &= \lim_{z\to 0}\frac{\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n+1}-z}{z^2}\\ &= \lim_{z\to 0}\frac{\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n+1}}{z^2}\\ &= \lim_{z\to 0}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n-1}\,.\end{aligned}$$

We may now exchange the limit and the sum thanks to Lemma 7.4 so that each term converges to zero and hence the whole series does.

In conclusion, $z\mapsto\frac{\sin(z)}{z}$ has a removable singularity at the origin.

Theorem 7.36 (Riemann's theorem on removable singularities)**.** *Let $\Omega\subseteq\mathbb{C}$ be open and $f:\Omega\setminus\{z_0\}\to\mathbb{C}$ holomorphic. If f is bounded then z_0 is a removable singularity.*

Proof. Our goal is to define $\tilde{f}:\Omega\to\mathbb{C}$, the unique analytic extension of f. Consider the function

$$h(z) := \begin{cases}(z-z_0)^2 f(z) & z\neq z_0\\ 0 & z=z_0\end{cases}.$$

Then, since f is bounded near z_0, $\lim_{z\to z_0} f(z)$ remains bounded and hence $\lim_{z\to z_0}(z-z_0)^2 f(z)=0$ so that h is actually continuous at z_0. We claim that h is in fact holomorphic around z_0. Clearly it is holomorphic on $B_\varepsilon(z_0)\setminus\{z_0\}$. At z_0, we verify by hand that

$$\begin{aligned}\lim_{z\to z_0}\frac{h(z)-0}{z-z_0} &= \lim_{z\to z_0}\frac{(z-z_0)^2 f(z)}{z-z_0}\\ &= \lim_{z\to z_0}(z-z_0)f(z)\\ &= 0\,.\end{aligned}$$

The last line follows due to the boundedness near z_0 of f. Thus, by Theorem 7.12 we find that on $B_\varepsilon(z_0)$,

$$h(z) = \sum_{n=0}^{\infty}a_n(z-z_0)^n$$

for some $\{a_n\}_n\subseteq\mathbb{C}$. Now, we know that $h(z_0)\equiv 0$ so $a_0=0$. We actually have calculated $h'(z_0)\equiv\lim_{z\to z_0}\frac{h(z)-0}{z-z_0}=0$. But $a_1=h'(z_0)$ so $a_1=0$ too. Hence

$$h(z) = \sum_{n=2}^{\infty}a_n(z-z_0)^n\,.$$

Let us define $\tilde{f}:\Omega\to\mathbb{C}$ by

$$\tilde{f}(z) := \sum_{n=2}^{\infty}a_n(z-z_0)^{n-2}\,.$$

Clearly by its power series definition, $\tilde{f}$ is analytic. Moreover, on $\Omega \setminus \{ z_0 \}$,

$$f(z) = \frac{h(z)}{(z-z_0)^2} = \frac{\sum_{n=2}^{\infty} a_n (z-z_0)^n}{(z-z_0)^2} = \sum_{n=2}^{\infty} a_n (z-z_0)^{n-2} \equiv \tilde{f}(z)$$

so that $\tilde{f}$ is indeed *the* analytic continuation of f. ☐

In general, we divide the behavior of singularities at a point z_0 to three different categories:

1. Removable singularities, when $f(z)$ is bounded as $z \to z_0$.
2. Pole singularities, when $|f(z)| \to \infty$ as $z \to z_0$ because $\frac{1}{f}$ is analytic and has a zero at z_0.
3. Essential singularities: any divergence which is not a finite-order pole or removable (=zero order pole).

Example 7.37. The function $\mathbb{C} \setminus \{ 0 \} \ni z \mapsto \mathrm{e}^{\frac{1}{z}}$ has an *essential* singularity at the origin.

Proof. Clearly as $z \to 0$, $\left|\frac{1}{z}\right| \to \infty$ so that $\mathrm{e}^{\frac{1}{z}}$ is *not* bounded. So the origin is not a removable singularity. Moreover, considering the reciprocal function $z \mapsto \mathrm{e}^{-\frac{1}{z}}$, it does not have a zero of finite order. Indeed, a putative Taylor expansion shows

$$\partial_z^n \mathrm{e}^{-\frac{1}{z}} \quad = \quad \mathrm{e}^{-\frac{1}{z}} g(z)$$

where g is some meromorphic function. Hence the derivatives at zero are all zero in all orders, so that $z \mapsto \mathrm{e}^{-\frac{1}{z}}$ has a zero of infinite order. Another way to say this is to say that there exists no finite $n \in \mathbb{N}$ so that

$$\lim_{z\to 0} z^n \mathrm{e}^{\frac{1}{z}}$$

exists and is bounded. ☐

7.6 Calculation of residues

Example 7.38. The residue of $z \mapsto \operatorname{sinc}(z)$ at the origin is zero (which makes sense as it has a *removable* singularity there).

Proof. We have already seen that the function may be extended to an analytic one, so that the residue formula

$$\begin{aligned} \operatorname{residue}_0(\operatorname{sinc}) &= \frac{1}{2\pi \mathrm{i}} \oint \operatorname{sinc}(z)\,\mathrm{d}z \\ &= \frac{1}{2\pi \mathrm{i}} \oint \widetilde{\operatorname{sinc}}(z)\,\mathrm{d}z \\ &= \widetilde{\operatorname{sinc}}(0) && \text{(By Theorem 6.26)} \\ &\equiv 0\,. \end{aligned}$$

☐

Example 7.39. The function $f(z) := \pi \cot(\pi z)$ defined on $\mathbb{C} \setminus \mathbb{Z}$ has simple poles on the integers with residues equal to $+1$ there.

Proof. We have $\cot(\pi z) = \frac{\cos(\pi z)}{\sin(\pi z)}$ so that a zero of the denominator corresponds to a pole of cot. Indeed, let us verify via Lemma 7.27: let $n \in \mathbb{Z}$; then

$$\begin{aligned}\lim_{z\to n}(z-n)\,\pi\cot(\pi z) &= \pi(-1)^n\lim_{z\to n}\frac{z-n}{\sin(\pi z)}\\ &= \pi(-1)^n\lim_{z\to 0}\frac{z}{\sin(\pi(z+n))}\\ &= \pi(-1)^n\lim_{z\to 0}\frac{z}{\sin(\pi z)\cos(\pi n)+\cos(\pi z)\sin(\pi n)}\\ &= \lim_{z\to 0}\frac{\pi z}{\sin(\pi z)}\\ &= 1\,.\end{aligned}$$

So this is indeed a pole of order 1, and the residue, by this calculation, equals 1. ☐

The function $z\mapsto\pi\cot(\pi z)$ is related to the so-called *Dirac comb:*

$$x\mapsto\sum_{n\in\mathbb{Z}}\delta(x-n)$$

as follows: We have, for any analytic function $f:\mathbb{C}\to\mathbb{C}$

$$\begin{aligned}\sum_{n\in\mathbb{Z}}f(n) &= \int_{x\in\mathbb{R}}f(x)\sum_{n\in\mathbb{Z}}\delta(x-n)\,\mathrm{d}x\\ &= \lim_{R\to\infty}\frac{1}{2\pi\mathrm{i}}\oint_{\Gamma_R}f(z)\,\pi\cot(\pi z)\,\mathrm{d}z\,. && \text{(By (7.4))}\end{aligned}$$

where Γ_R is any simple CCW closed contour which contains the interval $[-R,R]$.

Example 7.40. The function $z\mapsto\frac{\mathrm{e}^z}{z^3}$ has a pole of order 3 and a residue equal to $\frac{1}{2}$ at the origin.

Proof. We use the formula (7.5) to get

$$\begin{aligned}\mathrm{residue}_0\left(z\mapsto\frac{\mathrm{e}^z}{z^3}\right) &= \lim_{z\to 0}\frac{1}{2}\partial_z^3 z^3\frac{\mathrm{e}^z}{z^3}\\ &= \lim_{z\to 0}\frac{1}{2}\mathrm{e}^z\\ &= \frac{1}{2}\,.\end{aligned}$$

☐

Example 7.41. The function $z\mapsto\frac{\mathrm{e}^z}{z^2-1}$ has a pole of order 1 at $z_0=1$ with residue $\frac{\mathrm{e}}{2}$.

Proof. We use (7.5) to get

$$\begin{aligned}\mathrm{residue}_1\left(\frac{\mathrm{e}^z}{z^2-1}\right) &= \lim_{z\to 1}(z-1)\frac{\mathrm{e}^z}{z^2-1}\\ &= \lim_{z\to 1}(z-1)\frac{\mathrm{e}^z}{(z-1)(z+1)}\\ &= \frac{\mathrm{e}}{2}\,.\end{aligned}$$

☐

Example 7.42. The function $z \mapsto \frac{1}{z^7 - z_0^7}$ has 7 simple poles at $z_0 \left\{ 1, \exp\left(2\pi \mathrm{i} \frac{1}{7}\right), \dots, \exp\left(2\pi \mathrm{i} \frac{6}{7}\right) \right\}$. The nth pole has residue

$$\frac{1}{z_0^6} \frac{1}{\prod_{j=0, j\neq n}^{6} \left(\exp\left(2\pi \mathrm{i} \frac{n}{7}\right) - \exp\left(2\pi \mathrm{i} \frac{j}{7}\right)\right)}\,.$$

Proof. Since the polynomial $z \mapsto z^7 - z_0^7$ is a degree 7 polynomial, we know it has precisely seven roots by Corollary 6.34, so it could be written as $(z - w_1) \cdots (z - w_7)$ where w_i are the seven roots. These roots are easily found as the solution $z \in \mathbb{C}$ to the equation

$$\begin{aligned} z^7 - z_0^7 &= 0 \\ z^7 &= z_0^7\,. \end{aligned}$$

Proceeding as in HW1Q5, we find

$$\begin{aligned} z &= z_0 1^{\frac{1}{7}} \\ &= z_0 \exp\left(2\pi \mathrm{i} n\right)^{\frac{1}{7}} \qquad (n \in \mathbb{Z}) \\ &= z_0 \exp\left(2\pi \mathrm{i} \frac{n}{7}\right) \end{aligned}$$

so we have seven roots of unity,

$$z \in z_0 \left\{ 1, \exp\left(2\pi \mathrm{i} \frac{1}{7}\right), \dots, \exp\left(2\pi \mathrm{i} \frac{6}{7}\right) \right\}.$$

Hence each of these points is a simple pole:

$$\begin{aligned} \lim_{z \to z_0 \exp\left(2\pi \mathrm{i} \frac{n}{7}\right)} \left(z - z_0 \exp\left(2\pi \mathrm{i} \frac{n}{7}\right)\right) \frac{1}{z^7 - z_0^7} &= \lim_{z \to z_0 \exp\left(2\pi \mathrm{i} \frac{n}{7}\right)} \frac{1}{\prod_{j=0, j\neq n}^{6} \left(z - z_0 \exp\left(2\pi \mathrm{i} \frac{j}{7}\right)\right)} \\ &= \frac{1}{\prod_{j=0, j\neq n}^{6} \left(z_0 \exp\left(2\pi \mathrm{i} \frac{n}{7}\right) - z_0 \exp\left(2\pi \mathrm{i} \frac{j}{7}\right)\right)}\,. \end{aligned}$$

□

7.7 The argument principle

The following principle, which relates to the notion of a *winding number* and more generally of a Fredholm index [BB89] (via the Krein-Widom-Devinatz theorem, [Dou98, Theorem 7.36]), gives us a geometric way to count the zeros and poles of a meromorphic function.

Theorem 7.43 (Argument principle). *Let $\Omega \subseteq \mathbb{C}$ be open and $f : \Omega \setminus \{ z_j \}_j \to \mathbb{C}$ be meromorphic. Assume that there is some disc $D \subseteq \Omega$ such that on ∂D, f has no zeros and no poles. Then if $\gamma : [0,1] \to \Omega$ is the simple closed CCW contour which goes about ∂D,*

$$\frac{1}{2\pi \mathrm{i}} \oint_\Gamma \frac{f'(z)}{f(z)} \mathrm{d}z = \operatorname{index}_D(f) = \operatorname{wind}(f \circ \gamma) \tag{7.7}$$

where $\operatorname{index}_D(f)$ is the number of zeros of f within D minus the number of poles of f within D. Here number is counted with the order of the zero or pole. $\operatorname{wind}(f \circ \gamma)$ is the (signed) number of times $f \circ \gamma : [0,1] \to \mathbb{C}$ winds about the origin of $\mathbb{C}$, known as the winding number.

Before tending to the proof of the argument principle, it is instructive to consider some examples. Consider, then $z \mapsto z$ and the unit disc. We know that $\operatorname{index}_D(z \mapsto z) = 1$ since it has one zero at the origin and no poles. Moreover, we have the integrand on the LHS of Theorem 7.43 equal to

$$\frac{1}{z}$$

which is an integral covered by Theorem 6.26 which indeed yields $+1$. Finally, we know that the contour $\gamma : [0,1] \to \mathbb{C}$ given by $t \mapsto \mathrm{e}^{2\pi\mathrm{i}t}$ winds about the unit circle in a CCW fashion once, and that composed with $z \mapsto z$, which is the identity map, this winding (which is also a winding about the origin) remains.

Conversely, we may consider the function $z \mapsto \frac{1}{z}$ which has one pole and no zeros on $\mathbb{C} \setminus \{ 0 \}$. Hence again with the unit disc, it has $\mathrm{index}_D \left(z \mapsto \frac{1}{z} \right) = -1$. But now,

$$\begin{aligned} \frac{f'(z)}{f(z)} &= \frac{-\frac{1}{z^2}}{\frac{1}{z}} \\ &= -\frac{1}{z} \end{aligned}$$

whose integral is, again by Theorem 6.26, -1. Also, we know that $f \circ \gamma$ is the contour $t \mapsto \mathrm{e}^{-2\pi\mathrm{i}t}$ which winds about the origin once clockwise, and hence has winding number -1.

What happens if we combine the two? For instance, consider $f(z) = \frac{1}{z}(z-a)$ with $|a| < 1$. Then we expect $\mathrm{index}_D(f) = 1 - 1 = 0$ and we indeed get

$$\begin{aligned} \frac{f'(z)}{f(z)} &= \frac{-\frac{1}{z^2}(z-a) + \frac{1}{z}}{\frac{1}{z}(z-a)} \\ &= -\frac{1}{z} + \frac{1}{z-a} \end{aligned}$$

so that when we run the contour integral we get two contributions.

To see the winding number, consider that

$$\frac{f'}{f} = (\log \circ f)'$$

at least if we do not cross into the negative real axis with f. But we also know then that

$$\begin{aligned} \frac{1}{2\pi\mathrm{i}} \oint_\Gamma \frac{f'(z)}{f(z)} \mathrm{d}z &= \frac{1}{2\pi\mathrm{i}} \int_{t=0}^1 (\log \circ f \circ \gamma)'(t)\, \mathrm{d}t \\ &= \frac{1}{2\pi\mathrm{i}} \left[\log(f(\gamma(1))) - \log(f(\gamma(0))) \right] . \end{aligned}$$

Now recall

$$\log(z) = \log(|z|) + \mathrm{i}\arg(z)$$

so

$$\log(f(\gamma(1))) - \log(f(\gamma(0))) = \log(|f(\gamma(0))|) - \log(|f(\gamma(1))|) + \mathrm{i}\left[\arg(f(\gamma(1))) - \arg(f(\gamma(0)))\right] .$$

Since $\gamma(0) = \gamma(1)$, we have certainly $|f(\gamma(1))| = |f(\gamma(0))|$ so the first term vanishes. Not so with the argument. We are using the multi-valued argument and we are trying to use it inside of an integral, so it better be continuous. In order to do that, we must refer to the multi-valued version of the argument, whence it could very well happen that

$$\arg(f(\gamma(1))) \neq \arg(f(\gamma(0))) .$$

Indeed, the difference is *precisely the angle subtended by* $t \mapsto (f \circ \gamma)(t)$ *as* t *varies from* 0 *to* 1. This angle is what we call the winding number. It is a topological quantity, in the sense that "wiggling" f does not change it so long as we do not move zeros or poles of f out of the disc, or, said differently, do not hit zeros or poles as we deform γ.

Proof of Theorem 7.43. Assume first that f has one isolated zero of order $n \geq 1$ at z_0. Then we may write

$$f(z) = (z - z_0)^n g(z)$$

where g is some analytic function which does not vanish near z_0. Then

$$\begin{aligned} \frac{f'(z)}{f(z)} &= \frac{n(z-z_0)^{n-1} g(z) + (z-z_0)^n g'(z)}{(z-z_0)^n g(z)} \\ &= n\frac{1}{z-z_0} + \frac{g'(z)}{g(z)} \end{aligned}$$

Since g never vanishes near z_0, $z \mapsto \frac{g'(z)}{g(z)}$ is analytic near z_0 and we find that on some ε circle about z_0,

$$\frac{1}{2\pi \mathrm{i}} \oint_{\partial B_\varepsilon(z_0)} \frac{f'}{f} \quad = \quad n\,.$$

Conversely, if f has one isolated pole of order $n \geq 1$ at z_0 then we may write

$$f(z) \quad = \quad (z - z_0)^{-n} g(z)$$

for some analytic g which does not vanish near z_0 and

$$\begin{aligned} \frac{f'(z)}{f(z)} \quad &= \quad \frac{-n(z-z_0)^{-n-1} g(z) + (z-z_0)^{-n} g'(z)}{(z-z_0)^{-n} g(z)} \\ &= \quad -n\frac{1}{z-z_0} + \frac{g'(z)}{g(z)} \end{aligned}$$

and a similar argument shows

$$\frac{1}{2\pi \mathrm{i}} \oint_{\partial B_\varepsilon(z_0)} \frac{f'}{f} \quad = \quad -n\,.$$

Through the use of keyhole contours, we may reduce the general $\frac{1}{2\pi \mathrm{i}} \oint_{\partial D} \frac{f'}{f}$ to a sum over sufficiently small epsilon balls around each of the isolated poles and zeros, and hence

$$\frac{1}{2\pi \mathrm{i}} \oint_{\partial B_\varepsilon(z_0)} \frac{f'}{f} \quad = \quad \mathrm{index}_D(f)\,.$$

□

An extremely important consequence of the argument follows. It finds many applications in mathematics, physics and computer science.

Corollary 7.44 (Rouché's theorem)**.** *Let $f, g : \Omega \to \mathbb{C}$ be two holomorphic functions in an open set Ω containing a disc D. If g is sufficiently close to f on ∂D, in the sense that*

$$|g(z) - f(z)| \quad < \quad |f(z)| \qquad (z \in \partial D)$$

then g has the same number of zeros within $\mathrm{int}(D)$ *as f.*

This theorem allows us to control the number of zeros of a perturbed function g if we can control the perturbation, i.e., $\|g - f\|_{L_\infty(\partial D)}$.

Proof. Define the interpolation

$$g_t(z) \quad := \quad (1-t) f(z) + t g(z) \qquad (t \in [0,1], z \in \Omega)\,.$$

Let $n_t \in \mathbb{N}$ be the number of zeros of g_t within ∂D. Note that g_t has no zeros ∂D. Indeed,

$$\begin{aligned} |g_t(z)| &= |(1-t) f(z) + t g(z)| \\ &= |f(z) + t(g(z) - f(z))| \\ &\geq |f(z)| - t|g(z) - f(z)| \\ &\geq |f(z)| - |g(z) - f(z)| && \text{(t≤1)} \\ &> 0\,. && \text{(By hypothesis)} \end{aligned}$$

Hence we may apply Theorem 7.43 on g_t with ∂D to get

$$n_t \quad = \quad \frac{1}{2\pi \mathrm{i}} \oint_{\partial D} \frac{g_t'}{g_t}\,.$$

Now we claim that $[0,1] \ni t \mapsto n_t \in \mathbb{N}$ is a continuous function. Indeed, since g_t never vanishes on ∂D, $(t,z) \mapsto \frac{g_t'(z)}{g_t(z)}$ is a continuous function in $t \in [0,1]$, $z \in \partial D$. The integral of a continuous function is however continuous.

But a continuous function which takes integer values must be constant. Indeed, otherwise, the intermediate value theorem would mean that there must be some $t_0 \in (0,1)$ for which n_{t_0} is *not* an integer. Hence $n_0 = n_1$ which is what we wanted to show. $\square$

As an example application of Rouché's powerful theorem, we show that eigenvalues of matrices (or operators) cannot move too much if we can control how much the operators themselves move. To do this, let us briefly discuss analysis of matrices, i.e., the notion of distance between matrices:

Definition 7.45 (Matrix norms)**.** For a matrix $A \in \mathrm{Mat}_{n\times n}(\mathbb{C})$, *the Hilbert-Schmidt norm*, also called *the Frobenius norm*, is

$$\|A\|_{\mathrm{HS}} := \sqrt{\sum_{i,j=1}^{n} |A_{ij}|^2}\,.$$

Compare this to the ℓ^2 norm of a vector $v \in \mathbb{C}^n$ given by

$$\|v\|_{\ell^2} \equiv \sqrt{\sum_{i=1}^{n} |v_i|^2}$$

and to the operator norm of A,

$$\|A\| \equiv \sup\left(\{\, \|Av\|_{\ell^2} \mid v \in \mathbb{C}^n : \|v\|_{\ell^2} = 1 \,\}\right)\,.$$

The operator norm is an upper bound on the largest *singular value*. The singular values $\sigma_1(A), \dots, \sigma_n(A)$ of A are the square roots of the eigenvalues $\lambda_1(A^*A), \dots, \lambda_n(A^*A)$ of A^*A. Since A^*A is a positive matrix (i.e., $\langle v, A^*Av\rangle = \langle Av, Av\rangle = \|Av\|^2 \geq 0$ for all $v \in \mathbb{C}^n$) all these eigenvalues are positive and taking their square roots makes sense). Observe that

$$\begin{aligned}
\|A\|_{\mathrm{HS}}^2 &\equiv \sum_{i,j=1}^{n} |A_{ij}|^2 \\
&= \sum_{i,j=1}^{n} \overline{A_{ij}} A_{ij} \\
&= \sum_{i,j=1}^{n} (A^*)_{ji} A_{ij} \\
&= \sum_{j=1}^{n} (A^*A)_{jj} \\
&= \operatorname{tr}(A^*A) \\
&= \sum_{i=1}^{n} \lambda_i(A^*A) \\
&\equiv \sum_{i=1}^{n} \sigma_i(A)^2
\end{aligned}$$

and

$$\begin{aligned}\|A\|^2 &\equiv \sup_{v:\|v\|=1} \|Av\|^2 \\ &= \sup_{v:\|v\|=1} \langle Av, Av\rangle \\ &= \sup_{v:\|v\|=1} \langle v, A^*Av\rangle \\ &\stackrel{\star}{=} \sup_{i=1,\dots,n} \lambda_i\left(A^*A\right) \\ &= \sup_{i=1,\dots,n} \sigma_i\left(A\right)^2 \\ &= \left(\sup_{i=1,\dots,n} \sigma_i\left(A\right)\right)^2\end{aligned}$$

where in $\star$ we have used the variational characteristic of the largest eigenvalue (the Rayleigh form) and in the last step we've used the monotone increasing property of $(\cdot)^2$. These considerations lead us to the estimates

$$\|A\| \leq \|A\|_{\text{HS}} \leq \sqrt{n}\|A\|\,. \tag{7.8}$$

Another important property of matrix norm is the fact they are submultiplicative, i.e.,

$$\|AB\| \quad\leq\quad \|A\|\|B\|\,.$$

This may be proven as follows. First observe that

$$\begin{aligned}\|Av\| &= \left\|A\frac{v}{\|v\|}\right\|\|v\| \\ &\leq \|A\|\|v\|\,.\end{aligned}$$

Then,

$$\begin{aligned}\|AB\| &\equiv \sup_{\|v\|=1}\|ABv\| \\ &\leq \sup_{\|v\|=1}\|A\|\|Bv\| \\ &= \|A\|\|B\|\,.\end{aligned}$$

Lemma 7.46. *If $A \in \text{Mat}_{n\times n}\left(\mathbb{C}\right)$ is such that $\|A\|$ is small then*

$$\det\left(\mathbb{1}-A\right) \quad\leq\quad 1-\text{tr}\left(A\right)+n\cdot n!\|A\|^2\,.$$

Consequently,

$$\left|1-\det\left(\mathbb{1}-A\right)\right| \leq 2n\cdot n!\|A\|\,. \tag{7.9}$$

This is most likely a terrible bound as n gets large, but it shall suffice for our purposes.

Proof. Assume first that A is diagonalizable, with eigenvalues $\lambda_1,\dots,\lambda_n$; assume they are listed in descending order (of absolute value). Then

$$\begin{aligned}\det\left(\mathbb{1}-A\right) &= \prod_{i=1}^{n}\left(1-\lambda_i\right) \\ &= 1-\sum_{i=1}^{n}\lambda_i+\sum_{i=1}^{n}\sum_{j\neq i}\lambda_i\lambda_j+\dots+\sum_{i_1=1}^{n}\sum_{i_2\neq i_1}\dots\sum_{i_n\neq i_{n-1}}\lambda_{i_1}\dots\lambda_{i_n} \\ &\leq 1-\text{tr}\left(A\right)+n\left(n-1\right)\left|\lambda_1\lambda_2\right|+n\left(n-1\right)\left(n-2\right)\left|\lambda_1\lambda_2\lambda_3\right|+\dots+n!\left|\lambda_1\dots\lambda_n\right| \\ &\leq 1-\text{tr}\left(A\right)+n\left(n-1\right)\left|\lambda_1\right|^2+n\left(n-1\right)\left(n-2\right)\left|\lambda_1\right|^3+\dots+n!\left|\lambda_1\right|^n \\ &\leq 1-\text{tr}\left(A\right)+n\cdot n!\left|\lambda_1\right|^2\end{aligned}$$

But $|\lambda_1| \leq \|A\|$. Indeed, if λ_1 is an eigenvalue of A then there is some $u \in \mathbb{C}^n$ such that $Au = \lambda_1 u$ and furthermore we may choose $\|u\| = 1$. Then

$$\begin{aligned} |\lambda_1| &= |\lambda_1| \|u\| \\ &= \|\lambda_1 u\| \\ &= \|Au\| \\ &\leq \sup_{v \in \mathbb{C}^n : \|v\|=1} \|Av\| \\ &= \|A\| . \end{aligned}$$

Next, if A is not diagonalizable, use the density of diagonalizable matrices in $\mathrm{Mat}_{n\times n}(\mathbb{C})$: For any $A \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ there is a sequence of diagonalizable matrices $\{ A_k \}_{k\in\mathbb{N}} \subseteq \mathrm{Mat}_{n\times n}(\mathbb{C})$ with $A_k \to A$ in operator norm, i.e.

$$\lim_{k\to\infty} \|A_k - A\| = 0 .$$

For each element in the sequence we have

$$\det(\mathbb{1} - A_k) \leq 1 - \mathrm{tr}(A_k) + n \cdot n! \|A_k\|^2 .$$

Take now the limit $k \to \infty$ on both sides. Since both sides are operator norm continuous functions of the matrix (they are continuous functions $\mathbb{C}^{n\times n} \to \mathbb{C}$ when distance in $\mathbb{C}^{n\times n}$ is measured via the operator norm) we find the result. Note this is not circular: we are using continuity (via the limit) in order to find the modulus of continuity. Continuity itself may be proven first without appeal to the modulus of continuity.

To get the second estimate of the lemma, we use

$$\begin{aligned} |1 - \det(\mathbb{1} - A)| &\leq |\mathrm{tr}(A)| + n \cdot n! \|A\|^2 \\ &\leq n\|A\| + n \cdot n! \|A\|^2 \\ &\leq 2n \cdot n! \|A\| . \end{aligned}$$

□

Lemma 7.47 (Eigenvalues of a matrix are stable)**.** *Consider a matrix $A \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ and let D be some disc. The matrix A has eigenvalues $\{ \lambda_1, \dots, \lambda_n \} \subseteq \mathbb{C}$ and assume that they all lie within* $\mathrm{int}\, D$*. Let*

$$\mathrm{gap}_D(A) := \inf_{j=1,\dots,n, z\in\partial D} |\lambda_j - z| > 0 .$$

If $B \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ obeys

$$\|B - A\| < \frac{1}{2n \cdot n!} \mathrm{gap}_D(A)$$

then B has the same number of eigenvalues within D as A.

Proof. We apply Corollary 7.44 on the characteristic polynomial, $z \mapsto p_A(z) = \det(A - z\mathbb{1})$ which is analytic away from the eigenvalues of A. By definition, an eigenvalue is in the disc D iff p_A has a root within it. To apply Rouche, we want to verify that

$$|p_A(z) - p_B(z)| < |p_A(z)| \qquad (z \in \partial D) .$$

Let us estimate

$$\begin{aligned}|p_A(z)-p_B(z)| &= |\det(A-z\mathbb{1})-\det(B-z\mathbb{1})| \\ &= |p_A(z)|\left|1-\frac{\det(B-z\mathbb{1})}{\det(A-z\mathbb{1})}\right| \\ &= |p_A(z)|\left|1-\det\left((B-z\mathbb{1})(A-z\mathbb{1})^{-1}\right)\right| \\ &= |p_A(z)|\left|1-\det\left((A-z\mathbb{1}+B-A)(A-z\mathbb{1})^{-1}\right)\right| \\ &= |p_A(z)|\left|1-\det\left(\mathbb{1}+(B-A)(A-z\mathbb{1})^{-1}\right)\right|\end{aligned}$$

at this point, let us study the matrix

$$(B-A)(A-z\mathbb{1})^{-1}\,.$$

We posit it should be "small". Indeed, if we take $z\in\partial D$, then all eigenvalues of A are a safe finite distance from z (by assumption) and hence

$$A-z\mathbb{1}$$

cannot be too small, or, in other words,

$$\left\|(A-z\mathbb{1})^{-1}\right\| \leq \frac{1}{\mathrm{gap}_D(A)}<\infty\,.$$

Hence by Lemma 7.46 we find

$$\begin{aligned}\left|1-\det\left(\mathbb{1}-(B-A)(A-z\mathbb{1})^{-1}\right)\right| &< 2n\cdot n!\left\|(B-A)(A-z\mathbb{1})^{-1}\right\| \\ &\leq 2n\cdot n!\|B-A\|\left\|(A-z\mathbb{1})^{-1}\right\| \\ &\leq 2n\cdot n!\|B-A\|\frac{1}{\mathrm{gap}_D(A)} \\ &< 1\end{aligned}$$

by hypothesis so that

$$|p_A(z)-p_B(z)|<|p_A(z)|$$

and Corollary 7.44 applies. □

7.8 The open mapping theorem and the maximum modulus principle

We begin with a

Definition 7.48 (Open function)**.** A function $f:\mathbb{C}\to\mathbb{C}$ is termed *open* iff it maps open subsets into open subsets, i.e., if

$$f(U) \in \mathrm{Open}(\mathbb{C}) \qquad (U\in\mathrm{Open}(\mathbb{C}))\,.$$

This should be contrasted with continuity, which is the property of *inverse images*

$$f^{-1}(U) \in \mathrm{Open}(\mathbb{C}) \qquad (U\in\mathrm{Open}(\mathbb{C}))\,.$$

Theorem 7.49 (Open mapping theorem)**.** *If $f:\Omega\to\mathbb{C}$ is holomorphic and non-constant, then f is an open map.*

Proof. Let $U\in\mathrm{Open}(\mathbb{C})$. Our goal is to show that $f(U)\in\mathrm{Open}(\mathbb{C})$, so let $w_0\in f(U)$, and we seek some $\varepsilon>0$ such that $B_\varepsilon(w_0)\subseteq f(U)$. Since $w_0\in f(U)$, there is some $z_0\in U: f(z_0)=w_0$.

Pick now any $w\in B_\varepsilon(w_0)\setminus\{w_0\}$. We want to show that $w\in f(U)$ too, i.e., that there is some $z\in U$ such that $f(z)=w$. To that end, define

$$g(z) := f(z)-w\,.$$

If we can show that g has a zero in U then we'd be finished, right?

Rewrite

$$g(z) = \underbrace{f(z) - w_0}_{=:F(z)} + \underbrace{w_0 - w}_{=:G(z)}$$

(no matter that G is actually a constant...). Now, F has a zero in U since $f(z_0) = w_0$. Let $\tilde{\delta} > 0$ such that $B_{\tilde{\delta}}(z_0) \subseteq U$ (possible since U is open). Now pick $\delta < \tilde{\delta}$ so that $\partial B_\delta(z_0)$ contains no zeros of F (always possible since f is holomorphic, so it does not have accumulation of zeros). In fact, since $\partial B_\delta(z_0)$ is a circle (and is hence closed) there is some $\varepsilon > 0$ such that

$$\inf_{z\in\partial B_\delta(z_0)} |F(z)| \geq \varepsilon .$$

Next, if $w \in B_\varepsilon(w_0)$, we find

$$\begin{aligned} |G(z)| &\equiv |w - w_0| \\ &< \varepsilon \\ &\leq |F(z)| \end{aligned}$$

for all $z \in \partial B_\delta(z_0)$. Hence we apply Rouche's theorem Corollary 7.44 to conclude that $g = F + G$ has the same amount of zeros in $B_\delta(z_0)$ as F does, namely, one zero. ☐

Even though we have already seen the maximum modulus principle in HW4Q11, here is a shorter and nicer proof of it using our new hammer.

Lemma 7.50. *The absolute value $|\cdot| : \mathbb{C} \to [0,\infty)$ is an open mapping away from the origin.*

Proof. Let $U \in \mathrm{Open}(\mathbb{C})$ which does not contain the origin and pick some $\alpha \in |U|$. That means there is some $z \in U$ such that $|z| = \alpha$. Since U is open, there's some $\delta > 0$ such that $B_\delta(z) \subseteq U$. Observe that if $\arg(z) = \arg(\tilde{z})$ then

$$|z - \tilde{z}| = ||z| - |\tilde{z}|| .$$

Indeed,

$$\begin{aligned} |z - \tilde{z}|^2 &= |z|^2 + |\tilde{z}|^2 - 2\,\mathbb{Re}\left\{\overline{\tilde{z}}z\right\} \\ &= (|z| - |\tilde{z}|)^2 + 2|z||\tilde{z}|(1 - \cos(\text{angle between them})) . \end{aligned}$$

So define $\varepsilon := \frac{1}{2}\delta$, and now, given any $\beta \in B_\varepsilon(\alpha)$, take $\tilde{z} := \frac{\beta}{|z|}z$ (possible since $0 \notin U$). Then by the above,

$$|z - \tilde{z}| = \frac{1}{2}\delta$$

and hence $\tilde{z} \in U$, and, by construction, $|\tilde{z}| = \beta$. ☐

Lemma 7.51. *If $f : X \to Y$ and $g : Y \to Z$ are two open maps, then $g \circ f : X \to Z$ is open.*

Proof. Let $U \in \mathrm{Open}(X)$. We want to show that $(g \circ f)(U) \in \mathrm{Open}(Z)$. But $(g \circ f)(U) \equiv g(f(U))$. Since f is open, $f(U)$ is open, and since g is open, $g(f(U))$ is open. ☐

Corollary 7.52 (Maximum modulus principle)**.** *If $f : \Omega \to \mathbb{C}$ is a non-constant holomorphic function then $|f|$ cannot attain a maximum in the interior of Ω.*

Proof. Since f is holomorphic, Theorem 7.49 says it is open. Since $|\cdot|$ is open, the above two lemmas imply that $|\cdot| \circ f \equiv |f| : \Omega \to [0, \infty)$ is open. Suppose now, by way of contradiction, that $|f|$ does have a maximum on some $z_0 \in \operatorname{int}(\Omega)$. Since $\operatorname{int}(\Omega)$ is open, there exists some $\varepsilon > 0$ such that $B_\varepsilon(z_0) \subseteq \Omega$. But then, the image of that disc under $|f|$ is

$$|f|(B_\varepsilon(z_0))$$

is an open subset of $[0, \infty)$. This, however, cannot be true if it also has a maximal value, $|f(z_0)|$, since it will necessarily contain its maximum. □

7.9 The Cauchy Principal Value Integral

We have learnt in Corollary 7.26 that if $f : \Omega \to \mathbb{C}$ has a simple pole at $z_0 \in \operatorname{int}(\Gamma)$ then

$$\oint_\Gamma f = 2\pi \mathrm{i}\, \mathrm{residue}_{z_0}(f) .$$

What happens if z_0 sits *on* the contour? We will see that in case f has a *simple* pole which sits *on* Γ,

$$\oint_\Gamma f = \pi \mathrm{i}\, \mathrm{residue}_{z_0}(f) .$$

Before tending to that question, this integral raises a more basic question: is there a way to understand such integrals at all, given that $x \mapsto \frac{1}{x}$ is not integrable at the origin?

Let us start with an

Example 7.53. The integral of $x \mapsto \frac{1}{x^3}$ on $[-1, 1]$ does not make sense, because the function is not integrable at the origin. Despite this, it should also be zero, because $x \mapsto \frac{1}{x^3}$ is an odd function, and hence there should be a cancellation of the bad things happening around the origin.

Definition 7.54 (Cauchy's principal value)**.** The Cauchy principal value of a real-valued function on an interval is everything in the integral of that function except the very singularity. More precisely, let a function $f : (a, b) \to \mathbb{R}$ be given (a, b possibility at $\pm\infty$) which has a non-integrable singularity at some $x_0 \in (a, b)$. Then

$$⨍_a^b f \equiv \mathscr{P} \int_a^b f \equiv \text{p.v.} \int_a^b f := \lim_{\varepsilon \to 0^+} \left[\int_a^{x_0 - \varepsilon} f + \int_{x_0 + \varepsilon}^b f \right] .$$

Importantly, the two integrals are computed (with the *same* regularization $\varepsilon > 0$) and only afterwards the limit is taken. This definition may be extended in an obvious way if f has more than one singularity on (a, b).

Since contour integrals of functions $f : \mathbb{C} \to \mathbb{C}$ are interpreted as functions of a real argument, i.e., with $\gamma : [0, 1] \to \mathbb{C}$, we have $f \circ \gamma : [0, 1] \to \mathbb{C}$ and

$$\int_\Gamma f \equiv \int_{t=0}^1 f(\gamma(t))\, \gamma'(t)\, \mathrm{d}t ,$$

it is clear how to interpret

$$⨍_\Gamma f(z)\, \mathrm{d}z .$$

Continuing Example 7.53 above, we see that

$$\begin{aligned}
⨍_{-1}^1 \frac{1}{x^3} \mathrm{d}x &\equiv \lim_{\varepsilon \to 0^+} \left[\int_{-1}^{-\varepsilon} \frac{1}{x^3} \mathrm{d}x + \int_\varepsilon^1 \frac{1}{x^3} \mathrm{d}x \right] \\
&= \lim_{\varepsilon \to 0^+} \left[\left(-\frac{1}{2x^2} \right) \Big|_{-1}^{-\varepsilon} + \left(-\frac{1}{2x^2} \right) \Big|_\varepsilon^1 \right] \\
&= -\frac{1}{2} \lim_{\varepsilon \to 0^+} \left[\varepsilon^{-2} - 1 + 1 - \varepsilon^{-2} \right] \\
&= 0 .
\end{aligned}$$

which is what we would expect from the integral if it hadn't been for its singularity.

It is tempting to calculate also

$$\begin{aligned}\fint_{-1}^{1}\frac{1}{x}\mathrm{d}x &= \lim_{\varepsilon\to0^+}\left[\int_{-1}^{-\varepsilon}\frac{1}{x}\mathrm{d}x+\int_{\varepsilon}^{1}\frac{1}{x}\mathrm{d}x\right]\\ &= \lim_{\varepsilon\to0^+}\left[\log(x)|_{-1}^{-\varepsilon}+\log(x)|_{\varepsilon}^{1}\right]\\ &= \lim_{\varepsilon\to0^+}\left[\log(-\varepsilon)-\log(-1)+\log(1)-\log(\varepsilon)\right]\\ &= \lim_{\varepsilon\to0^+}\left[\log(-\varepsilon)-\log(-1)-\log(\varepsilon)\right]\end{aligned}$$

At this stage you might protest and say log is not defined on the negative real axis. However, let us interpret this as the complex logarithm, in which case, for $x>0$,

$$\begin{aligned}\log(-x) &= \log(|x|)+\mathrm{i}\arg(-x)\\ &= \log(|x|)+\mathrm{i}\pi+2\pi\mathrm{i}n\end{aligned}$$

for some $n\in\mathbb{Z}$ (recall arg is multi-valued). If we take the same choice for both negative logs we find the usual formula from high school

$$\int_{-a}^{-b}\frac{1}{x}\mathrm{d}x = \log(|b|)-\log(|a|)\,.$$

We conclude

$$\begin{aligned}\fint_{-1}^{1}\frac{1}{x}\mathrm{d}x &= \lim_{\varepsilon\to0^+}\left[\log(\varepsilon)-\log(\varepsilon)\right]\\ &= 0\end{aligned}$$

as expected from an odd function.

In summary, Cauchy's ingenuity here is in insisting that the limit $\varepsilon\to0^+$ be taken only *after* performing the two integrals, and, moreover, that the same small ε is used to approach the singularity in both integrals.

Example 7.55 (The sinc integral revisited)**.** Consider now again the integral $\int_{-\infty}^{\infty}\operatorname{sinc}(x)\,\mathrm{d}x$ which was studied in Example 6.39. That calculation there actually involved Cauchy's principal value without giving it a name, since we interpreted

$$\int_{-\infty}^{\infty}\operatorname{sinc}(x)\,\mathrm{d}x = \lim_{R\to\infty,\varepsilon\to0^+}\lim_{\varepsilon\to0^+}\left[\int_{-\infty}^{-\varepsilon}\operatorname{sinc}(x)\,\mathrm{d}x+\int_{\varepsilon}^{\infty}\operatorname{sinc}(x)\,\mathrm{d}x\right]\equiv\fint_{-\infty}^{\infty}\operatorname{sinc}\,.$$

Let us systemize the calculation which appeared in Example 6.39 in the following

Lemma 7.56. *If Γ is a simple closed CCW contour, and if $f:\Omega\setminus\{z_0\}\to\mathbb{C}$ is meromorphic with a* simple *pole at $z_0\in\Gamma$ such that Γ which passes through z_0 infinitesimally in a straight line, then*

$$\fint_{\Gamma}f = \pi\mathrm{i}\,\mathrm{residue}_{z_0}(f)+2\pi\mathrm{i}\sum_{j=1}^{\ell}\mathrm{residue}_{z_j}(f)$$

where $\{z_j\}_{j=1}^{\ell}\subseteq\operatorname{int}(\Gamma)$ are the set of poles in the interior *of Γ.*

Proof. We emphasize the contour involved in $\fint_\Gamma f$ is *not* closed despite Γ being a closed contour. Let $\Gamma_\varepsilon^{\text{line}}$ be the small segment of Γ of length ε about z_0. Then

$$\Gamma_\varepsilon^{\text{p.v.}} := \Gamma\setminus\Gamma_\varepsilon^{\text{line}}$$

and

$$\fint_\Gamma f \equiv \lim_{\varepsilon\to0^+}\int_{\Gamma_\varepsilon^{\text{p.v.}}}f\,.$$

Let $\Gamma_\varepsilon^{\text{semicircle}}$ be a small semicircle which goes around z_0 in radius ε in such a way so that

$$\Gamma_\varepsilon^{\text{p.v.}} \cup \Gamma_\varepsilon^{\text{semicircle}}$$

is a closed CCW contour which does *not* include z_0 (this is a generalization of the indented semicircle contour from Figure 18). Since $\Gamma_\varepsilon^{\text{p.v.}} \cup \Gamma_\varepsilon^{\text{semicircle}}$ is closed CCW, the residue theorem Corollary 7.26 states that

$$\oint_{\Gamma_\varepsilon^{\text{p.v.}} \cup \Gamma_\varepsilon^{\text{semicircle}}} f \quad = \quad 2\pi\mathrm{i}\sum_{j=1}^{\ell} \text{residue}_{z_j}(f) \ .$$

On the other hand by the linearity of the integral we have

$$\int_{\Gamma_\varepsilon^{\text{p.v.}}} f \quad = \quad -\int_{\Gamma_\varepsilon^{\text{semicircle.}}} f + 2\pi\mathrm{i}\sum_{j=1}^{\ell} \text{residue}_{z_j}(f) \ .$$

Let us calculate $\int_{\Gamma_\varepsilon^{\text{semicircle.}}} f$ by parametrizing it with

$$\gamma(\theta) \quad := \quad z_0 + \varepsilon \mathrm{e}^{\mathrm{i}\theta} \qquad (\theta : \pi \to 0) \ .$$

Since f has a *simple* pole at z_0, expanding it in a Laurent series about z_0 we find

$$f(z) \quad = \quad \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \dots$$

for some $\{ a_k \}_{k=-1}^{\infty} \subseteq \mathbb{C}$. Hence we have

$$\begin{aligned} \int_{\Gamma_\varepsilon^{\text{semicircle.}}} f \quad &= \quad \int_{\theta=\pi}^{0} f\left(z_0 + \varepsilon \mathrm{e}^{\mathrm{i}\theta}\right) \varepsilon \mathrm{e}^{\mathrm{i}\theta} \mathrm{i} \mathrm{d}\theta \\ &= \quad \int_{\theta=\pi}^{0} \left[\frac{a_{-1}}{\varepsilon \mathrm{e}^{\mathrm{i}\theta}} + a_0 + a_1 \varepsilon \mathrm{e}^{\mathrm{i}\theta} + \dots\right] \varepsilon \mathrm{e}^{\mathrm{i}\theta} \mathrm{i} \mathrm{d}\theta \\ &= \quad -\mathrm{i}\pi a_{-1} + a_0 \mathcal{O}(\varepsilon) + a_1 \mathcal{O}\left(\varepsilon^2\right) + \dots \end{aligned}$$

where we have used

$$\int_{\Gamma_\varepsilon^{\text{semicircle.}}} (z - z_0)^k \mathrm{d}z = \mathrm{i}\varepsilon^{k+1} \int_{\theta=\pi}^{0} \mathrm{e}^{\beta\theta(k+1)} \mathrm{d}\theta = \delta_{k,-1} \mathrm{i}\pi + \varepsilon^{k+1}(1 - \delta_{k,-1}) \frac{1 + \mathrm{e}^{-\mathrm{i}k\pi}}{k+1} \ . \tag{7.10}$$

But since $a_{-1} = \text{residue}_{z_0}(f)$, we find the result as $\varepsilon \to 0$. $\square$

It is clear from (7.10) that if the pole is of higher order then $\int_{\Gamma_\varepsilon^{\text{semicircle.}}} f$ will include the other terms a_{-k} for $k > 1$, since the contour is only a semicircle, so the lemma from HW4Q6 that says

$$\oint_{\partial B_1(z_0)} (z - z_0)^k \mathrm{d}z \quad = \quad 2\pi\mathrm{i}\delta_{k,-1}$$

does *not* apply. However, we may state the following relaxation:

Lemma 7.57. *If* Γ *is a simple CCW contour and* $f:\Omega\setminus\{z_0\}\to\mathbb{C}$ *is meromorphic with a pole at* $z_0\in\Gamma$ *such that at* z_0*,* f *has the Laurent expansion*

$$f(z) \quad = \quad \sum_{k=\ell}^{0}\frac{a_{-(2k+1)}}{(z-z_0)^{2k+1}}+\textit{analytic part}$$

and if Γ *is a closed CCW contour which passes through* z_0 *infinitesimally in a straight line, then*

$$\oint_\Gamma f \quad = \quad \pi\mathrm{i}\,\mathrm{residue}_{z_0}(f)+2\pi\mathrm{i}\sum_{j=1}^{\ell}\mathrm{residue}_{z_j}(f)$$

where $\{z_j\}_{j=1}^{\ell}\subseteq\mathrm{int}(\Gamma)$ *are the set of poles* in the interior *of* Γ.

Proof. Everything leading up, in the previous proof, up to (7.10) goes through. Now, in (7.10), since $\exp(\mathrm{i}k\pi)=-1$ if k is odd, it is clear that

$$\int_{\Gamma_\varepsilon^{\text{semicircle.}}}\frac{1}{(z-z_0)^{2k+1}}\mathrm{d}z \quad = \quad 0 \qquad (k\geq 1)$$

whereas the even terms do not exist in the Laurent expansion, so again we pick up only the term a_{-1}, the residue; when the powers are positive we get zero not from the integral itself but only thanks to the ε^{k+1} factor and in the $\varepsilon\to 0^+$ limit. ☐

Example 7.58. Invoking the above, Example 6.39 is solved as follows. We start by

$$\begin{aligned} I &:= \int_{-\infty}^{\infty}\mathrm{sinc}(x)\,\mathrm{d}x \\ &= \mathbb{Im}\left\{\int_{-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}x}}{x}\mathrm{d}x\right\} \\ &= \mathbb{Im}\left\{\oint_{-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}z}}{z}\mathrm{d}z\right\}. \end{aligned}$$

Now, since $z\mapsto\frac{\mathrm{e}z^{\mathrm{i}z}}{z}$ decays sufficiently fast on the half-circle of radius R centered at the origin, that arc may be added at no extra cost in order to obtain a closed contour of the upper half circle (just as in Example 6.39). Then, the function $z\mapsto\frac{\mathrm{e}^{\mathrm{i}z}}{z}$ does have a simple pole precisely at the origin, which is a point on the contour, so we apply Lemma 7.56 to get

$$\begin{aligned} I &:= \mathbb{Im}\left\{\oint_{\text{half circle of radius }R}\frac{\mathrm{e}^{\mathrm{i}z}}{z}\mathrm{d}z\right\} \\ &= \mathbb{Im}\left\{\mathrm{i}\pi\,\mathrm{residue}_0\left(z\mapsto\frac{\mathrm{e}^{\mathrm{i}z}}{z}\right)\right\} && \text{(Application of Lemma 7.56)} \\ &= \mathbb{Im}\left\{\mathrm{i}\pi\mathrm{e}^0\right\} \\ &= \pi\,. \end{aligned}$$

Another possible extension of Lemma 7.56 is the following

Lemma 7.59. *If* Γ *is a simple closed CCW contour, and if* $f:\Omega\setminus\{z_0\}\to\mathbb{C}$ *is meromorphic with a* simple *pole at* $z_0\in\Gamma$ *such that* Γ *which passes through* z_0 *infinitesimally in a corner of angle* α*, then*

$$\oint_\Gamma f \quad = \quad \alpha\mathrm{i}\,\mathrm{residue}_{z_0}(f)+\sum_{j=1}^{\ell}\mathrm{residue}_{z_j}(f)$$

where $\{z_j\}_{j=1}^{\ell}\subseteq\mathrm{int}(\Gamma)$ *are the set of poles* in the interior *of* Γ.

Example 7.60. Let $\alpha \in \mathbb{C}$ with $\mathbb{Im}\{\alpha\} > 0$. We claim that

$$\int_{-\infty}^{\infty} \frac{1}{(x-1)(x-\alpha)^2} \mathrm{d}x \quad = \quad \frac{-\mathrm{i}\pi}{(1-\alpha)^2} \,.$$

Proof. Let us close the contour $[-R, R]$ with an upper half circle. This is justified as follows. We need to show that

$$\left| \int_{\theta=0}^{\pi} \frac{1}{(R\mathrm{e}^{\mathrm{i}\theta}-1)(R\mathrm{e}^{\mathrm{i}\theta}-\alpha)^2} R\mathrm{e}^{\mathrm{i}\theta}\mathrm{i}\mathrm{d}\theta \right| \quad \stackrel{R\to\infty}{\longrightarrow} \quad 0\,.$$

To that end, note that

$$\begin{aligned} \left|R\mathrm{e}^{\mathrm{i}\theta}-\alpha\right|^2 &= R^2 + |\alpha|^2 - 2R|\alpha|\cos(\theta - \arg(\alpha)) \\ &\geq R^2 + |\alpha|^2 - 2R|\alpha| \\ &= (R - |\alpha|)^2 \end{aligned}$$

so we may estimate that integral from above with

$$\frac{\pi R}{|R-|\alpha||^2 |R-1|}$$

which indeed converges to zero as $R \to \infty$, regardless of the value of α.

Moreover, since we have a pole on the real line, we need to interpret the integral with Cauchy's principal value. We thus find

$$\begin{aligned} I &:= \int_{-\infty}^{\infty} \frac{1}{(x-1)(x-\alpha)^2} \mathrm{d}x \\ &= \lim_{R\to\infty} \oint_{\text{closed semicircle of radius } R \text{ about origin}} \frac{1}{(z-1)(z-\alpha)^2} \mathrm{d}z\,. \end{aligned}$$

Now since $\mathbb{Im}\{\alpha\} > 0$ and we have a simple pole *on* the contour, we have using Lemma 7.56,

$$I \quad = \quad \mathrm{i}\pi\,\mathrm{residue}_1\left(z \mapsto \frac{1}{(z-1)(z-\alpha)^2}\right) + 2\pi\mathrm{i}\,\mathrm{residue}_\alpha\left(z \mapsto \frac{1}{(z-1)(z-\alpha)^2}\right)\,.$$

The residues are straight forward to calculate:

$$\begin{aligned} \mathrm{residue}_1\left(z \mapsto \frac{1}{(z-1)(z-\alpha)^2}\right) &= \lim_{z\to 1}(z-1)\frac{1}{(z-1)(z-\alpha)^2} \\ &= \frac{1}{(1-\alpha)^2}\,. \end{aligned}$$

and

$$\begin{aligned}\text{residue}_\alpha\left(z\mapsto\frac{1}{(z-1)(z-\alpha)^2}\right) &= \lim_{z\to\alpha}\partial_z(z-\alpha)^2\frac{1}{(z-1)(z-\alpha)^2}\\ &= \lim_{z\to\alpha}\partial_z\frac{1}{(z-1)}\\ &= \lim_{z\to\alpha}-\frac{1}{(z-1)^2}\\ &= -\frac{1}{(\alpha-1)^2}\\ &= -\frac{1}{(1-\alpha)^2}\,.\end{aligned}$$

Collecting everything, we have

$$\begin{aligned}I &= \mathrm{i}\pi\frac{1}{(1-\alpha)^2}+2\pi\mathrm{i}\left[-\frac{1}{(1-\alpha)^2}\right]\\ &= -\mathrm{i}\pi\frac{1}{(1-\alpha)^2}\,.\end{aligned}$$

□

Finally, let us discuss a very useful tool for integrals on the real line, which comes up a lot in quantum mechanics, quantum field theory, etc. It is also related to the more general *Sokhotski–Plemelj theorem* (which we shall skip).

Corollary 7.61 (Kramers-Kronig relations)**.** *Let* $f:\mathbb{C}\to\mathbb{C}$ *be entire such that*

$$\sup_{R>0}\int_{\theta=0}^{\pi}\left|f\left(R\mathrm{e}^{\mathrm{i}\theta}\right)\right|\mathrm{d}\theta<\infty\,.$$

Then for any $0<R\leq\infty$*,*

$$\lim_{\varepsilon\to0^+}\int_{-R}^{R}\frac{f(x)}{x\pm\mathrm{i}\varepsilon}\mathrm{d}x=\mp\mathrm{i}\pi f(0)+⨍_{x=-R}^{R}\frac{f(x)}{x}\mathrm{d}x\,. \tag{7.11}$$

We note that this identity may be proven with *far* less stringent restrictions on f, but we prefer to assume here analyticity to connect it with our previous statements.

Proof. Let us write,

$$\Gamma_R \;:=\; [-R,R]\cup S_R$$

where S_R is the semi-circle in the upper half plane of radius R about the origin and so Γ_R is a simple closed contour. Then by linearity,

$$\int_{-R}^{R}\frac{f(x)}{x\pm\mathrm{i}\varepsilon}\mathrm{d}x \;=\; \oint_{\Gamma_R}\frac{f(z)}{z\pm\mathrm{i}\varepsilon}\mathrm{d}z-\int_{S_R}\frac{f(z)}{z\pm\mathrm{i}\varepsilon}\mathrm{d}z$$

and now thanks to the residue theorem and the fact $f(z)$ is entire, we have the following two identities depending on the sign in the denominator:

$$\begin{aligned}\oint_{\Gamma_R}\frac{f(z)}{z-\mathrm{i}\varepsilon}\mathrm{d}z &= 2\pi\mathrm{i}f(\mathrm{i}\varepsilon)\\ \oint_{\Gamma_R}\frac{f(z)}{z+\mathrm{i}\varepsilon}\mathrm{d}z &= 0\,.\end{aligned}$$

On the other hand,

$$\oint_{x=-R}^{R}\frac{f(x)}{x}\mathrm{d}x = \lim_{\varepsilon\to0^+}\int_{[-R,R]\setminus[-\varepsilon,\varepsilon]}\frac{f(x)}{x}\mathrm{d}x$$

and

$$\int_{[-R,R]\setminus[-\varepsilon,\varepsilon]}\frac{f(x)}{x}\mathrm{d}x = \oint_{\Gamma_R\setminus[-\varepsilon,\varepsilon]}\frac{f(z)}{z}\mathrm{d}z-\int_{S_R}\frac{f(z)}{z}\mathrm{d}z\,.$$

Now we apply Lemma 7.56 on $\oint_{\Gamma_R\setminus[-\varepsilon,\varepsilon]}\frac{f(z)}{z}\mathrm{d}z$ to obtain

$$\lim_{\varepsilon\to0^+}\oint_{\Gamma_R\setminus[-\varepsilon,\varepsilon]}\frac{f(z)}{z}\mathrm{d}z = \mathrm{i}\pi f(0)\,.$$

For $z-\mathrm{i}\varepsilon$, we find the two equations (using the fact that f is continuous at the origin):

$$\begin{aligned}\mathrm{i}\pi f(0) &= \lim_{\varepsilon\to0^+}\int_{[-R,R]\setminus[-\varepsilon,\varepsilon]}\frac{f(x)}{x}\mathrm{d}x+\int_{S_R}\frac{f(z)}{z}\mathrm{d}z\\ 2\pi\mathrm{i}f(0) &= \lim_{\varepsilon\to0^+}\oint_{\Gamma_R}\frac{f(z)}{z-\mathrm{i}\varepsilon}\mathrm{d}z=\lim_{\varepsilon\to0^+}\left[\int_{-R}^{R}\frac{f(x)}{x-\mathrm{i}\varepsilon}\mathrm{d}x+\int_{S_R}\frac{f(z)}{z-\mathrm{i}\varepsilon}\mathrm{d}z\right]\end{aligned}$$

taking the second minus the first, we find

$$\begin{aligned}\mathrm{i}\pi f(0) &= \lim_{\varepsilon\to0^+}\left[\int_{-R}^{R}\frac{f(x)}{x-\mathrm{i}\varepsilon}\mathrm{d}x+\int_{S_R}\frac{f(z)}{z-\mathrm{i}\varepsilon}\mathrm{d}z\right]-\\ &\quad-\lim_{\varepsilon\to0^+}\left[\int_{[-R,R]\setminus[-\varepsilon,\varepsilon]}\frac{f(x)}{x}\mathrm{d}x+\int_{S_R}\frac{f(z)}{z}\mathrm{d}z\right]\\ &= \lim_{\varepsilon\to0^+}\int_{-R}^{R}\frac{f(x)}{x-\mathrm{i}\varepsilon}\mathrm{d}x-\oint_{x=-R}^{R}\frac{f(x)}{x}\mathrm{d}x\\ &\quad+\lim_{\varepsilon\to0^+}\int_{S_R}f(z)\left(\frac{1}{z-\mathrm{i}\varepsilon}-\frac{1}{z}\right)\mathrm{d}z\,.\end{aligned}$$

Now using the identity

$$\frac{1}{z-\mathrm{i}\varepsilon}-\frac{1}{z} = \frac{1}{z-\mathrm{i}\varepsilon}\mathrm{i}\varepsilon\frac{1}{z}$$

we have

$$\begin{aligned}\left|\int_{S_R}f(z)\left(\frac{1}{z-\mathrm{i}\varepsilon}-\frac{1}{z}\right)\mathrm{d}z\right| &= \varepsilon\left|\int_{S_R}\frac{f(z)}{z(z-\mathrm{i}\varepsilon)}\right|\\ &\leq \frac{\varepsilon}{R-\varepsilon}\int_{\theta=0}^{\pi}\left|f\left(R\mathrm{e}^{\mathrm{i}\theta}\right)\right|\mathrm{d}\theta\\ &\leq \frac{\varepsilon}{R-\varepsilon}\sup_{R>0}\int_{\theta=0}^{\pi}\left|f\left(R\mathrm{e}^{\mathrm{i}\theta}\right)\right|\mathrm{d}\theta\,.\end{aligned}$$

Hence we find this goes to zero already at finite R as $\varepsilon\to0^+$ (though $R\to\infty$ doesn't hurt). Moving the principal value term to the other side we find the result.

For the other sign possibility, we get the two equations

$$\begin{aligned} \mathrm{i}\pi f(0) &= \lim_{\varepsilon\to 0^+}\int_{[-R,R]\setminus[-\varepsilon,\varepsilon]}\frac{f(x)}{x}\mathrm{d}x+\int_{S_R}\frac{f(z)}{z}\mathrm{d}z \\ 0 &= \lim_{\varepsilon\to 0^+}\oint_{\Gamma_R}\frac{f(z)}{z+\mathrm{i}\varepsilon}\mathrm{d}z=\lim_{\varepsilon\to 0^+}\left[\int_{-R}^{R}\frac{f(x)}{x+\mathrm{i}\varepsilon}\mathrm{d}x+\int_{S_R}\frac{f(z)}{z+\mathrm{i}\varepsilon}\mathrm{d}z\right] \end{aligned}$$

and now taking the first minus the second, we get

$$\begin{aligned} \mathrm{i}\pi f(0) &= ⨍_{-R}^{R}\frac{f(x)}{x}\mathrm{d}x-\lim_{\varepsilon\to 0^+}\left[\int_{-R}^{R}\frac{f(x)}{x+\mathrm{i}\varepsilon}\mathrm{d}x\right]+ \\ &\quad +\int_{S_R}\frac{f(z)}{z}\mathrm{d}z-\lim_{\varepsilon\to 0^+}\int_{S_R}\frac{f(z)}{z+\mathrm{i}\varepsilon}\mathrm{d}z \end{aligned}$$

so the result follows by the same considerations. □

This result may be summarized *in the distributional sense* (common in physics) as

$$\lim_{\varepsilon\to 0^+}\frac{1}{x\pm\mathrm{i}\varepsilon} \overset{\mathcal{D}}{=} \mp\mathrm{i}\pi\delta(x)+\mathcal{P}\left(\frac{1}{x}\right). \tag{7.12}$$

It is worth mentioning also that one sometimes meets the phrase “Kramers-Kronig relations” when taking real and imaginary parts of the identity (7.11), it particular, we find for real-valued functions,

$$\delta(x)\overset{\mathcal{D}}{=}\lim_{\varepsilon\to 0^+}\frac{1}{\pi}\mathbb{Im}\left\{\frac{1}{x-\mathrm{i}\varepsilon}\right\}\overset{\mathcal{D}}{=}\lim_{\varepsilon\to 0^+}\frac{1}{\pi}\frac{\varepsilon}{x^2+\varepsilon^2} \tag{7.13}$$

and

$$\mathcal{P}\left(\frac{1}{x}\right) \overset{\mathcal{D}}{=} \lim_{\varepsilon\to 0^+}\mathbb{Re}\left\{\frac{1}{x-\mathrm{i}\varepsilon}\right\}\overset{\mathcal{D}}{=}\lim_{\varepsilon\to 0^+}\frac{x}{x^2+\varepsilon^2}\overset{\mathcal{D}}{=}\lim_{\varepsilon\to 0^+}\frac{x^2}{x^2+\varepsilon^2}\frac{1}{x}. \tag{7.14}$$

Again, it must be emphasized these equations are to be understood in the *distributional* sense (which is why we use $\overset{\mathcal{D}}{=}$), i.e.,

$$\begin{aligned} \lim_{\varepsilon\to 0^+} f_\varepsilon &\overset{\mathcal{D}}{=} g \\ &\Updownarrow \\ \lim_{\varepsilon\to 0^+}\int f_\varepsilon h &= \int gh \qquad \forall h \end{aligned}$$

and so, e.g., we really mean by (7.13) that

$$f(0)=\int_{x\in\mathbb{R}}f(x)\,\delta(x)\,\mathrm{d}x = \lim_{\varepsilon\to 0^+}\int_{x\in\mathbb{R}}\frac{1}{\pi}\frac{\varepsilon}{x^2+\varepsilon^2}f(x)\,\mathrm{d}x \qquad (f:\mathbb{R}\to\mathbb{R}).$$

8 Fourier analysis

Complex analysis is very useful for when performing the Fourier series and transform. We review some of these topics, concentrating on results that are proven using complex analysis.

8.1 Fourier series

The Fourier series converts between functions on the unit circle to functions on the integers. We denote the unit circle embedded within the complex plane as

$$\mathbb{S}^1 \equiv \partial B_1(0)\equiv\{\,z\in\mathbb{C}\mid |z|=1\,\}$$

so we are concerned with functions $\psi:\mathbb{S}^1\to\mathbb{C}$. For such functions, we define their Fourier coefficients [Kat04]

$$\hat{\psi}:\mathbb{Z} \to \mathbb{C}$$

by the formula

$$\hat{\psi}(n) \quad := \quad \frac{1}{2\pi}\int_{\theta=0}^{2\pi}\mathrm{e}^{-\mathrm{i}\theta n}\psi(\theta)\,\mathrm{d}\theta \qquad (n\in\mathbb{Z})\ .$$

For $\hat{\psi}$ to exist one needs the integrability of ψ. This relation may *sometimes* be inverted by summing the series:

$$\psi(\theta) \quad = \quad \sum_{n\in\mathbb{Z}}\mathrm{e}^{\mathrm{i}\theta n}\hat{\psi}(n)$$

depending on ψ. We have, for example,

Theorem 8.1 (Jackson's theorem)**.** *If $\psi:\mathbb{S}^1\to\mathbb{C}$ is k-times continuously differentiable and $\psi^{(k)}$ has modulus of continuity (see HW1) ω, then*

$$\left|\psi(\theta)-\sum_{n=-N}^{N}\mathrm{e}^{\mathrm{i}\theta n}\hat{\psi}(n)\right| \quad \leq \quad K\frac{\log(N)}{N^k}\omega\left(\frac{2\pi}{N}\right) \qquad (\theta\in[0,2\pi])\ .$$

for some $K\in(0,\infty)$ independent of ψ, k or N. In particular in that scenario the partial sums converge uniformly *to ψ.*

Let us now take a holomorphic function $f:\Omega\to\mathbb{C}$ in some open connected Ω. Then we know that within some $B_R(z_0)\subseteq\Omega$, there is a convergent Taylor expansion

$$f(z) \quad = \quad \sum_{n=0}^{\infty}a_n(z-z_0)^n \qquad (z\in B_R(z_0))\ .$$

If we restrict to the boundary of $B_r(z_0)$ for any $0<r<R$, i.e., if we define some $\psi:\mathbb{S}^1\to\mathbb{C}$ out of f via

$$\psi(\theta) \quad := \quad f\left(z_0+r\mathrm{e}^{\mathrm{i}\theta}\right) \qquad (\theta\in[0,2\pi])$$

then we find via the Taylor expansion, the following expansion for ψ:

$$\psi(\theta)=\sum_{n=0}^{\infty}a_n r^n\mathrm{e}^{\mathrm{i}n\theta} \tag{8.1}$$

so the Taylor series of a holomorphic function restricted to a circle is apparently the Fourier series with coefficients

$$\left\{\,\hat{\psi}(n)\,\right\}_{n=0}^{\infty}=\{\,a_n r^n\,\}_{n=0}^{\infty}\ ,$$

i.e., no negative terms for $\hat{\psi}$! Using the Fourier inversion formula we know that

$$a_n r^n \quad = \quad \frac{1}{2\pi}\int_{\theta=0}^{\pi}\mathrm{e}^{-\mathrm{i}n\theta}f\left(z_0+r\mathrm{e}^{\mathrm{i}\theta}\right)\mathrm{d}\theta \qquad (n\geq 0)$$

and also, apparently,

$$0 \quad = \quad \frac{1}{2\pi}\int_{\theta=0}^{\pi}\mathrm{e}^{-\mathrm{i}n\theta}f\left(z_0+r\mathrm{e}^{\mathrm{i}\theta}\right)\mathrm{d}\theta \qquad (n<0)\ .$$

This agrees with Theorem 7.12 since

$$\begin{aligned} a_n &= \frac{f^{(n)}(z_0)}{n!} \\ &= \frac{1}{2\pi\mathrm{i}}\oint_{\partial B_r(z_0)}\frac{f(z)}{(z-z_0)^{n+1}}\mathrm{d}z \end{aligned} \qquad \text{(Using Theorem 6.26)}$$

for any $0<r<R$.

Theorem 8.2. *A holomorphic function $f:\Omega\to\mathbb{C}$ restricted to a circle as $\psi:\mathbb{S}^1\to\mathbb{C}$ via (8.1) has all its $n<0$ Fourier series coefficients zero: $\hat{\psi}(n)=0$ for all $n<0$.*

Another curious fact about the Fourier series is the connection between the rate of decay of $n\mapsto\left|\hat{\psi}(n)\right|$ at infinity and the regularity of $\psi:\mathbb{S}^1\to\mathbb{C}$: how many continuous derivatives it has. This goes under the general name of *the Riemann-Lebesgue lemma*.

To understand it we first note that if we take the Fourier coefficients of a polynomial (in $e^{i\theta}$) we find only a finite number of non-zero Fourier coefficients, and the same is true for a meromorphic function with a polynomial analytic part. More generally,

Lemma 8.3 (Riemann-Lebesgue)**.** *Iff $f:\Omega\to\mathbb{C}$ is analytic on some open connected Ω where $B_r(z_0)\subseteq\Omega$ then $\hat{\psi}:\mathbb{Z}\to\mathbb{C}$ (defined via (8.1)) exhibits exponential decay.*

Proof. Using the above, we know that

$$\hat{\psi}(n) = \frac{f^{(n)}(z_0)}{n!}r^n$$

for *any* $0<r<R$ and using Theorem 6.30 we know that

$$\left|f^{(n)}(z_0)\right| \leq \frac{n!}{R^n}\|f\|_{L^\infty(B_R(z_0))}$$

so that we find

$$\begin{aligned}\left|\hat{\psi}(n)\right| &\leq \left(\frac{r}{R}\right)^n\|f\|_{L^\infty(B_R(z_0))}\\ &= \exp\left(-n\left|\log\left(\frac{r}{R}\right)\right|\right)\|f\|_{L^\infty(B_R(z_0))}\,.\end{aligned}$$

Conversely, using the formula

$$\psi(\theta) = \sum_{n\in\mathbb{Z}}e^{i\theta n}\hat{\psi}(n)$$

we see that since $n\mapsto\hat{\psi}(n)$ has exponential decay, the above equation defines an absolutely convergent power series and so an analytic function in an annulus. □

Hence, as long as we have any strictly bigger radius of analyticity for f than the circle ψ is defined on, we will have exponential decay.

What about functions which are only analytic on some annulus of width $2\varepsilon>0$ about $\mathbb{S}^1$?

Lemma 8.4 (Riemann-Lebesgue on annulus)**.** *Let*

$$A_\varepsilon := \{\, z\in\mathbb{C}\mid 1-\varepsilon<|z|<1+\varepsilon\,\}$$

and assume $f:A_\varepsilon\to\mathbb{C}$ is analytic. Then if $\psi:\mathbb{S}^1\to\mathbb{C}$ is defined via

$$\psi(\theta) := f\left(e^{i\theta}\right) \qquad (\theta\in[0,2\pi])$$

then $\hat{\psi}:\mathbb{Z}\to\mathbb{C}$ exhibits exponential decay.

Proof. We have, using the parametrization $z=e^{i\theta}$,

$$\begin{aligned}\hat{\psi}(n) &= \frac{1}{2\pi}\int_{\theta=0}^{2\pi}e^{-in\theta}\psi(\theta)\,d\theta\\ &= \frac{1}{2\pi i}\oint_{z\in\partial B_1(0)}z^{-n-1}f(z)\,dz\,.\end{aligned}$$

Since f is analytic in an annulus, using the Cauchy integral theorem, we may deform the contour of integration $\partial B_1(0)$ to some $\partial B_t(0)$ for some $t \in (1-\varepsilon, 1+\varepsilon)$. Then

$$\begin{aligned}\hat{\psi}(n) &= \frac{1}{2\pi \mathrm{i}} \oint_{z\in\partial B_t(0)} z^{-n-1} f(z)\,\mathrm{d}z \\ &= \frac{1}{2\pi\mathrm{i}} \int_{\theta=0}^{2\pi} \left(t\mathrm{e}^{\mathrm{i}\theta}\right)^{-n-1} f\left(t\mathrm{e}^{\mathrm{i}\theta}\right) t\mathrm{e}^{\mathrm{i}\theta}\mathrm{i}\mathrm{d}\theta\end{aligned}$$

so that

$$\left|\hat{\psi}(n)\right| \leq t^{-n}\|f\|_{L^\infty(A_\varepsilon)} \qquad (n\in\mathbb{Z})\,.$$

Note that necessarily $\|f\|_{L^\infty(A_\varepsilon)} < \infty$ since $f : A_\varepsilon \to \mathbb{C}$ is an analytic (and hence continuous) function on a compact set.

If $n > 0$ pick $t = 1+\varepsilon-\delta$ for some $\delta \in (0,\varepsilon)$ so that

$$\begin{aligned} t^{-n} &= (1+\varepsilon-\delta)^{-n} \\ &= \exp(-n\log(1+\varepsilon-\delta))\,. \end{aligned}$$

Conversely, if $n < 0$ pick $t = 1-\varepsilon+\delta$ to get

$$\begin{aligned} t^{-n} &= t^{|n|} \\ &= \exp(|n|\log(t)) \\ &= \exp(|n|\log(1-\varepsilon+\delta)) \\ &= \exp(-|n|\,|\log(1-\varepsilon+\delta)|)\,. \end{aligned}$$

□

When comparing Lemma 8.3 and Lemma 8.4, we see that for the former, we derive exponential decay at rate

$$\left|\log\left(\frac{r}{R}\right)\right|$$

where r is the radius of the circle on which our function is restricted to and R is the biggest radius of holomorphicity. On the other hand, for Lemma 8.4 the decay rate is essentially the width of the annulus on which we may assume holomorphicity.

We finish this section with two unavoidable facts

Theorem 8.5 (Parseval's theorem)**.** *For any $\psi, \varphi : \mathbb{S}^1 \to \mathbb{C}$ which are square integrable, we have*

$$\langle\psi,\varphi\rangle_{L^2(\mathbb{S}^1)} = 2\pi\left\langle\hat{\psi},\hat{\varphi}\right\rangle_{\ell^2(\mathbb{Z})}\,.$$

Proof. To begin we plug in

$$\begin{aligned}\left\langle\hat{\psi},\hat{\varphi}\right\rangle_{\ell^2(\mathbb{Z})} &\equiv \sum_{n\in\mathbb{Z}} \overline{\hat{\psi}(n)}\hat{\varphi}(n) \\ &= \sum_{n\in\mathbb{Z}} \overline{\frac{1}{2\pi}\int_{\theta=0}^{2\pi} \mathrm{e}^{\mathrm{i}n\theta}\psi(\theta)\,\mathrm{d}\theta} \frac{1}{2\pi}\int_{\tilde{\theta}=0}^{2\pi} \mathrm{e}^{\mathrm{i}n\tilde{\theta}}\varphi\left(\tilde{\theta}\right)\mathrm{d}\tilde{\theta} \\ &= \frac{1}{4\pi^2}\sum_{n\in\mathbb{Z}}\int_{\theta,\tilde{\theta}=0}^{2\pi} \mathrm{d}\theta\mathrm{d}\tilde{\theta}\overline{\psi(\theta)}\varphi\left(\tilde{\theta}\right)\mathrm{e}^{\mathrm{i}n\left(\tilde{\theta}-\theta\right)}\,. && \text{(Fubini since } \psi,\varphi\in L^1)\end{aligned}$$

At this stage, we will invoke HW5Q5, namely Abel summation to regularize our expression and make it absolutely integrable. Abel says that if $\{\,a_n\,\}_n \subseteq \mathbb{C}$ is such that $\sum_n a_n$ converges, then

$$\lim_{\varepsilon\to 0^+}\sum_n \mathrm{e}^{-\varepsilon|n|}a_n = \sum_n a_n\,.$$

Since we know our series converges (otherwise the first expression makes no sense) we may write

$$\begin{aligned}\left\langle\hat{\psi},\hat{\varphi}\right\rangle_{\ell^{2}(\mathbb{Z})} &= \frac{1}{4\pi^{2}}\lim_{\varepsilon\to0^{+}}\sum_{n\in\mathbb{Z}}\mathrm{e}^{-\varepsilon|n|}\int_{\theta,\tilde{\theta}=0}^{2\pi}\mathrm{d}\theta\mathrm{d}\tilde{\theta}\overline{\psi\left(\theta\right)}\varphi\left(\tilde{\theta}\right)\mathrm{e}^{\mathrm{i}n\left(\tilde{\theta}-\theta\right)}\\ &= \frac{1}{4\pi^{2}}\lim_{\varepsilon\to0^{+}}\sum_{n\in\mathbb{Z}}\int_{\theta,\tilde{\theta}=0}^{2\pi}\mathrm{d}\theta\mathrm{d}\tilde{\theta}\overline{\psi\left(\theta\right)}\varphi\left(\tilde{\theta}\right)\mathrm{e}^{\mathrm{i}n\left(\tilde{\theta}-\theta\right)-\varepsilon|n|}\,.\end{aligned}$$

We now invoke Fubini's theorem Theorem D.5 to exchange $\sum_{n\in\mathbb{Z}}$ and $\int_{\theta,\tilde{\theta}=0}^{2\pi}$ since the integrand is absolutely integrable. We thus obtain

$$\left\langle\hat{\psi},\hat{\varphi}\right\rangle_{\ell^{2}(\mathbb{Z})} = \frac{1}{4\pi^{2}}\lim_{\varepsilon\to0^{+}}\int_{\theta,\tilde{\theta}=0}^{2\pi}\mathrm{d}\theta\mathrm{d}\tilde{\theta}\overline{\psi\left(\theta\right)}\varphi\left(\tilde{\theta}\right)\sum_{n\in\mathbb{Z}}\mathrm{e}^{\mathrm{i}n\left(\tilde{\theta}-\theta\right)-\varepsilon|n|}\,.$$

The sum may be carried out explicitly using the geometric series formula (A.2) to obtain

$$\begin{aligned}\sum_{n\in\mathbb{Z}}\mathrm{e}^{\mathrm{i}n\left(\tilde{\theta}-\theta\right)-\varepsilon|n|} &= \frac{\sinh\left(\varepsilon\right)}{\cosh\left(\varepsilon\right)-\cos\left(\tilde{\theta}-\theta\right)}\\ &\overset{\varepsilon\ll1}{\approx} 2\pi\frac{1}{\pi}\frac{\varepsilon}{\varepsilon^{2}+2\left(1-\cos\left(\tilde{\theta}-\theta\right)\right)}\,.\end{aligned}$$

We now invoke the Krammers-Kronig relation (7.11), in particular, (7.13), to obtain

$$\begin{aligned}\left\langle\hat{\psi},\hat{\varphi}\right\rangle_{\ell^{2}(\mathbb{Z})} &= \frac{1}{2\pi}\int_{\theta,\tilde{\theta}=0}^{2\pi}\mathrm{d}\theta\mathrm{d}\tilde{\theta}\overline{\psi\left(\theta\right)}\varphi\left(\tilde{\theta}\right)\delta\left(\sqrt{2\left(1-\cos\left(\tilde{\theta}-\theta\right)\right)}\right)\\ &= \frac{1}{2\pi}\int_{\theta,\tilde{\theta}=0}^{2\pi}\mathrm{d}\theta\mathrm{d}\tilde{\theta}\overline{\psi\left(\theta\right)}\varphi\left(\tilde{\theta}\right)\delta\left(\tilde{\theta}-\theta\right)\end{aligned}$$

where in the last line we have used $\delta\left(f\left(x\right)\right)=\frac{\delta(x)}{|f'(0)|}$. We find

$$\begin{aligned}\left\langle\hat{\psi},\hat{\varphi}\right\rangle_{\ell^{2}(\mathbb{Z})} &= \frac{1}{2\pi}\int_{\theta=0}^{2\pi}\mathrm{d}\theta\overline{\psi\left(\theta\right)}\varphi\left(\theta\right)\\ &\equiv \frac{1}{2\pi}\left\langle\psi,\varphi\right\rangle_{L^{2}}\end{aligned}$$

and hence the claim. □

We may think of any $\psi:\mathbb{S}^{1}\to\mathbb{C}$ as defining a probability measure $\mathbb{P}_{\psi}$ through its expectation value on functions

$$f \quad\mapsto\quad \frac{\int_{\theta=0}^{2\pi}f\left(\theta\right)\left|\psi\left(\theta\right)\right|^{2}\mathrm{d}\theta}{\int_{\theta=0}^{2\pi}\left|\psi\left(\theta\right)\right|^{2}\mathrm{d}\theta}\equiv\mathbb{E}_{\psi}\left[f\right]\qquad\left(f:\mathbb{S}^{1}\to\mathbb{C}\right)\,.$$

In this regard the following uncertainty principle (which is the same one from quantum mechanics) says that the product of variances of $\mathbb{P}_{\psi}$ and that of $\mathbb{P}_{\hat{\psi}}$ cannot be too small, with:

$$\mathbb{Var}_{\psi}\left[\theta\right] \equiv \mathbb{E}_{\psi}\left[\theta^{2}\right]-\left(\mathbb{E}_{\psi}\left[\theta\right]\right)^{2}=\mathbb{E}_{\psi}\left[\left(\theta-\mathbb{E}_{\psi}\left[\theta\right]\right)^{2}\right]\,.$$

Another, equivalent way to define the variance is via

$$\mathbb{Var}_{\psi}\left[\theta\right] := \inf_{t\in\mathbb{R}}\mathbb{E}_{\psi}\left[\left(\theta-t\right)^{2}\right]\,.$$

Since

$$\hat{\psi}:\mathbb{Z} \to \mathbb{C}$$

defines a probability measure on $\mathbb{Z}$, it may happen that $\mathbb{E}_{\hat{\psi}}\left[n\right]\notin\mathbb{Z}$. Let us nonetheless define

$$\widetilde{\mathbb{Var}}_{\hat{\psi}}\left[n\right] := \inf_{m\in\mathbb{Z}}\mathbb{E}_{\hat{\psi}}\left[\left(n-m\right)^{2}\right]\,.$$

Theorem 8.6 (The uncertainty principle for the Fourier series)**.** *If $\psi:\mathbb{S}^1\to\mathbb{C}$ is sufficiently smooth so that*

$$\left|\hat{\psi}(n)\right|^2|n| \text{ is summable} \tag{8.2}$$

then we have

$$1\leq\mathbb{E}_\psi\left[\theta^2\right]\left(\sqrt{\mathbb{E}_{\hat{\psi}}\left[n^2\right]}+\sqrt{\mathbb{E}_{\hat{\psi}}\left[(n+1)^2\right]}\right)^2. \tag{8.3}$$

Proof. We have the summation by parts formula (HW5Q4)

$$\begin{aligned}\sum_{n\in\mathbb{Z}}a_n &= \sum_{n\in\mathbb{Z}}a_n(n+1-n)\\ &= \sum_{n\in\mathbb{Z}}a_n(n+1)-\sum_{n\in\mathbb{Z}}a_n n\\ &= \sum_{n\in\mathbb{Z}}a_{n-1}n-\sum_{n\in\mathbb{Z}}a_n n\\ &= -\sum_{n\in\mathbb{Z}}(a_n-a_{n-1})n\end{aligned}$$

which relies on $\sum_{n\in\mathbb{Z}}|a_n n|<\infty$ ((8.2)) so we get, with $a_n=\left|\hat{\psi}(n)\right|^2$,

$$\begin{aligned}\sum_{n\in\mathbb{Z}}\left|\hat{\psi}(n)\right|^2 &= -\sum_{n\in\mathbb{Z}}\left(\left|\hat{\psi}(n)\right|^2-\left|\hat{\psi}(n-1)\right|^2\right)n\\ &= -\sum_{n\in\mathbb{Z}}\left[\left(\overline{\hat{\psi}(n)-\hat{\psi}(n-1)}\right)\hat{\psi}(n)+\overline{\hat{\psi}(n-1)}\left(\hat{\psi}(n)-\hat{\psi}(n-1)\right)\right]n\\ &= -\sum_{n\in\mathbb{Z}}\left(\overline{\hat{\psi}(n)-\hat{\psi}(n-1)}\right)\hat{\psi}(n)n-\sum_{n\in\mathbb{Z}}\overline{\hat{\psi}(n-1)}n\left(\hat{\psi}(n)-\hat{\psi}(n-1)\right)\\ &\leq \sum_{n\in\mathbb{Z}}\left|\hat{\psi}(n)-\hat{\psi}(n-1)\right|\left(\left|\hat{\psi}(n)\right||n|\right)+\sum_{n\in\mathbb{Z}}\left(\left|\hat{\psi}(n-1)\right||n|\right)\left|\hat{\psi}(n)-\hat{\psi}(n-1)\right|\\ &\overset{\text{Cauchy-Schwarz}}{\leq} \sqrt{\sum_{n\in\mathbb{Z}}\left|\hat{\psi}(n)-\hat{\psi}(n-1)\right|^2}\sqrt{\sum_{n\in\mathbb{Z}}\left|\hat{\psi}(n)\right|^2n^2}+\sqrt{\sum_{n\in\mathbb{Z}}\left|\hat{\psi}(n-1)\right|^2n^2}\sqrt{\sum_{n\in\mathbb{Z}}\left|\hat{\psi}(n)-\hat{\psi}(n-1)\right|^2}\\ &= \sqrt{\sum_{n\in\mathbb{Z}}\left|\hat{\psi}(n)-\hat{\psi}(n-1)\right|^2}\left(\sqrt{\sum_{n\in\mathbb{Z}}\left|\hat{\psi}(n)\right|^2n^2}+\sqrt{\sum_{n\in\mathbb{Z}}\left|\hat{\psi}(n)\right|^2(n+1)^2}\right)\\ &= \sqrt{\sum_{n\in\mathbb{Z}}\left|\hat{\psi}(n)-\hat{\psi}(n-1)\right|^2}\left(\sqrt{\int_{t=0}^1\mathrm{d}t\partial_t\sum_{n\in\mathbb{Z}}\left|\hat{\psi}(n)\right|^2(n+t)^2}+2\sqrt{\sum_{n\in\mathbb{Z}}\left|\hat{\psi}(n)\right|^2n^2}\right)\end{aligned}$$

Now we calculate

$$\begin{aligned}\hat{\psi}(n)-\hat{\psi}(n-1) &\equiv \frac{1}{2\pi}\int_{\theta=0}^{2\pi}\mathrm{e}^{-\mathrm{i}n\theta}\psi(\theta)\,\mathrm{d}\theta-\frac{1}{2\pi}\int_{\theta=0}^{2\pi}\mathrm{e}^{-\mathrm{i}(n-1)\theta}\psi(\theta)\,\mathrm{d}\theta\\ &= \frac{1}{2\pi}\int_{\theta=0}^{2\pi}\mathrm{e}^{-\mathrm{i}n\theta}\left(1-\mathrm{e}^{\mathrm{i}\theta}\right)\psi(\theta)\,\mathrm{d}\theta\end{aligned}$$

and using Parseval above, Theorem 8.5, we get, with $\varphi(\theta):=\left(1-\mathrm{e}^{\mathrm{i}\theta}\right)\psi(\theta)$,

$$\sum_{n\in\mathbb{Z}}\left|\hat{\psi}(n)-\hat{\psi}(n-1)\right|^2=\frac{1}{2\pi}\int_{\theta=0}^{2\pi}|\varphi(\theta)|^2\,\mathrm{d}\theta=\frac{1}{2\pi}\int_0^{2\pi}\left|1-\mathrm{e}^{\mathrm{i}\theta}\right|^2|\psi(\theta)|^2\,\mathrm{d}\theta\,.$$

Thinking of $\left|1-\mathrm{e}^{\mathrm{i}\theta}\right|$ as the distance between the points 1 and $\mathrm{e}^{\mathrm{i}\theta}$ on the unit circle, it is geometrically clear that $\left|1-\mathrm{e}^{\mathrm{i}\theta}\right| \leq |\theta|$. Equivalently, more algebraically,

$$\begin{aligned}\left|1-\mathrm{e}^{\mathrm{i}\theta}\right|^2 &= 2-2\cos(\theta) \\ &= 2(1-\cos(\theta)) \\ &= 4\sin\left(\frac{\theta}{2}\right)^2 \\ &\leq \theta^2 \qquad (\theta\in\mathbb{R}) .\end{aligned}$$

We thus find

$$\left(\sum_{n\in\mathbb{Z}}\left|\hat{\psi}(n)\right|^2\right)^2 \leq \left(\frac{1}{2\pi}\int_{\theta=0}^{2\pi}\theta^2|\psi(\theta)|^2\mathrm{d}\theta\right)\left(\sqrt{\sum_{n\in\mathbb{Z}}\left|\hat{\psi}(n)\right|^2 n^2}+\sqrt{\sum_{n\in\mathbb{Z}}\left|\hat{\psi}(n)\right|^2(n+1)^2}\right)^2 \tag{8.4}$$

and hence upon division by the LHS,

$$1 \leq \left(\frac{\int_{\theta=0}^{2\pi}\theta^2|\psi(\theta)|^2\mathrm{d}\theta}{\int_{\theta=0}^{2\pi}|\psi(\theta)|^2\mathrm{d}\theta}\right)\left(\sqrt{\frac{\sum_{n\in\mathbb{Z}}\left|\hat{\psi}(n)\right|^2 n^2}{\sum_{n\in\mathbb{Z}}\left|\hat{\psi}(n)\right|^2}}+\sqrt{\frac{\sum_{n\in\mathbb{Z}}\left|\hat{\psi}(n)\right|^2(n+1)^2}{\sum_{n\in\mathbb{Z}}\left|\hat{\psi}(n)\right|^2}}\right)^2$$

or in probabilistic notation,

$$1\leq\mathbb{E}_\psi\left[\theta^2\right]\left(\sqrt{\mathbb{E}_{\hat{\psi}}\left[n^2\right]}+\sqrt{\mathbb{E}_{\hat{\psi}}\left[(n+1)^2\right]}\right)^2 .$$

□

8.1.1 Some quantum mechanics [extra]

There are two ways to understand the Fourier series in quantum mechanics. The first one is that $\hat{\psi}:\mathbb{Z}\to\mathbb{C}$ is the wave function of a particle on a discrete one dimensional lattice, and then it just so happens that its momentum space is the circle $\mathbb{S}^1$ (and not $\mathbb{R}$). So $\hat{\psi}:\mathbb{Z}\to\mathbb{C}$ is the real-space representation of the wave-function and $\psi:\mathbb{S}^1\to\mathbb{C}$ is the momentum-space representation of the same wave-function. The quantity

$$\inf_{m\in\mathbb{Z}}\mathbb{E}_{\hat{\psi}}\left[(n-m)^2\right]$$

may be interpreted as the uncertainty in position measurement (since multiplication by n is like an application of the position operator) and

$$\mathbb{Var}_\psi[\theta]$$

as the uncertainty in momentum measurement.

Momentum and position may coexist in the same integral: we just saw above that an application of $\mathrm{i}\partial_\theta$ on ψ is tantamount to multiplication by n. Thus

$$\begin{aligned}[\mathrm{i}\partial_\theta,\theta]f(\theta) &= \mathrm{i}\partial_\theta\theta f(\theta)-\theta\mathrm{i}\partial_\theta f(\theta) \\ &= \mathrm{i}f(\theta)\end{aligned}$$

so that

$$[\mathrm{i}\partial_\theta,\theta] = \mathrm{i}\mathbb{1} .$$

We thus recover the more usual phrasing of the uncertainty principle in quantum mechanics with the RHS of (8.3) as the expectation value of the commutator $\mathrm{i}\mathbb{1}$ of position $\mathrm{i}\partial_\theta$ and momentum θ:

$$\sqrt{\frac{\pi}{32}}\left|\mathbb{E}_\psi\left[[\mathrm{i}\partial_\theta,\theta]\right]\right|^2 .$$

Conversely, $\psi:\mathbb{S}^1\to\mathbb{C}$ may be viewed as the wave function of a particle constrained to move on a circle (ring), and so $\hat{\psi}:\mathbb{Z}\to\mathbb{C}$ is its *angular momentum representation*, which is now quantized.

8.2 Fourier transform

We have the Hilbert space

$$L^2(\mathbb{R}\to\mathbb{C}) \quad := \quad \left\{ f:\mathbb{R}\to\mathbb{C} \,\middle|\, \int_{\mathbb{R}} |f|^2 < \infty \right\}$$

with a sesquilinear inner product

$$\langle \psi, \varphi \rangle_{L^2} \quad \equiv \quad \int_{x\in\mathbb{R}} \overline{\psi(x)}\varphi(x)\,\mathrm{d}x\,.$$

On it, and we define for $f \in L^2$, the Fourier transform

$$\mathscr{F} : L^2(\mathbb{R}) \quad \to \quad L^2(\mathbb{R})$$

via

$$\mathscr{F}(f)(\xi) \equiv \hat{f}(\xi) \quad := \quad \int_{x\in\mathbb{R}} \mathrm{e}^{-2\pi \mathrm{i} x\xi} f(x)\,\mathrm{d}x \qquad (\xi\in\mathbb{R})\,. \tag{8.5}$$

Strictly speaking, one first defines this for functions f which are in

$$L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \quad \subseteq \quad L^2(\mathbb{R})$$

and then extends $\mathscr{F}$ via an approximation argument. We will not pursue such a procedure here. In fact we will only consider $\mathscr{F}$ as defined on a much smaller space $\mathscr{A}_\varepsilon$ to be defined right below.

There is a Fourier inversion theorem (analogous to Theorem 8.1) which guarantees that for a wide class of f one may go back, i.e.,

$$f(x) = \int_{\xi\in\mathbb{R}} \mathrm{e}^{+2\pi\mathrm{i}x\xi}\hat{f}(\xi)\,\mathrm{d}\xi \qquad (x\in\mathbb{R})\,.$$

To prove theorems on $\mathscr{F}$ using complex analysis, unsurprisingly, we'll need analyticity, i.e., to assume that our functions

$$f:\mathbb{R} \quad \to \quad \mathbb{C}$$

actually extend analytically to some strip of width $2\varepsilon > 0$ centered on the real axis:

$$S_\varepsilon := \{\, z\in\mathbb{C} \mid \mathbb{Im}\{z\} \in (-\varepsilon,\varepsilon) \,\}\,.$$

We thus define a class of analytic functions which are uniformly L^1 (absolutely integrable) on the strip:

$$\mathscr{A}_\varepsilon := \left\{ f: S_\varepsilon \to \mathbb{C} \,\middle|\, f \text{ analytic and } \sup_{y\in(-\varepsilon,\varepsilon)} \int_{x\in\mathbb{R}} |f(x+\mathrm{i}y)|\,\mathrm{d}x < \infty \right\}. \tag{8.6}$$

Example 8.7 (The Cauchy distribution)**.** If $f: S_\varepsilon \to \mathbb{C}$ is analytic and obeys

$$|f(z)| \quad \leq \quad \frac{A}{1+\mathbb{Re}\{z\}^2} \qquad (z\in S_\varepsilon)$$

then $f\in\mathscr{A}_\varepsilon$. Indeed, $\int_{x\in\mathbb{R}} \frac{1}{1+x^2}\mathrm{d}x < \infty$. By the way, it is instructive to calculate the Fourier transform of what is called the Cauchy distribution:

$$\int_{x\in\mathbb{R}} \frac{\mathrm{e}^{-2\pi\mathrm{i}x\xi}}{1+x^2}\mathrm{d}x \qquad (\xi\in\mathbb{R})\,.$$

Since $z\mapsto \frac{\mathrm{e}^{-2\pi\mathrm{i}z\xi}}{1+z^2}$ is meromorphic, we use the residue theorem. There are two poles, at $z=\pm\mathrm{i}$. If we close the contour of integration with an arc (going up, or down, to be determined later), then on the arc we have

$$\left| \int_{\theta=0}^{\pi} \frac{\mathrm{e}^{-2\pi\mathrm{i}\xi\left(R\mathrm{e}^{\mathrm{i}\theta}\right)}}{1+R^2\mathrm{e}^{2\mathrm{i}\theta}} R\mathrm{e}^{\mathrm{i}\theta}\mathrm{i}\mathrm{d}\theta \right| \quad \leq \quad \frac{R}{R^2-1}\int_{\theta=0}^{\pi} \mathrm{e}^{2\pi\xi R\sin(\theta)}\mathrm{d}\theta\,.$$

This latter expression dictates if we should use the upper or lower arc, depending on the sign of ξ. At any rate we see that for the correct arc we get at least $\frac{1}{R}$ decay (actually exponential if we use Jordan's lemma Lemma A.1). Hence this arc may indeed be added free of charge. Hence, for $\xi > 0$, e.g., we take the lower arc, in which case it is the pole at $z=-\mathrm{i}$ which gets included in the residue theorem. Because we close the contour with the arc below, the contour is actually *clockwise* which earns an extra minus sign with the residue theorem. Thus

$$\begin{aligned}\int_{x\in\mathbb{R}}\frac{\mathrm{e}^{-2\pi\mathrm{i}x\xi}}{1+x^2}\mathrm{d}x &= \oint_{\text{Clockwise arc below}}\frac{\mathrm{e}^{-2\pi\mathrm{i}z\xi}}{1+z^2}\mathrm{d}z\\ &= -2\pi\mathrm{i}\,\mathrm{residue}_{-\mathrm{i}}\left(z\mapsto\frac{\mathrm{e}^{-2\pi\mathrm{i}z\xi}}{1+z^2}\right)\\ &= -2\pi\mathrm{i}\,\mathrm{residue}_{-\mathrm{i}}\left(z\mapsto\frac{\mathrm{e}^{-2\pi\mathrm{i}z\xi}}{(z-\mathrm{i})(z+\mathrm{i})}\right)\\ &= -2\pi\mathrm{i}\frac{\mathrm{e}^{-2\pi\mathrm{i}(-\mathrm{i})\xi}}{-2\mathrm{i}}\\ &= \pi\mathrm{e}^{-2\pi\xi}\,.\end{aligned}$$

An analogous calculation shows that if $\xi<0$ one obtains $\pi\mathrm{e}^{2\pi\xi}$ so that

$$\int_{x\in\mathbb{R}}\frac{\mathrm{e}^{-2\pi\mathrm{i}x\xi}}{1+x^2}\mathrm{d}x = \pi\mathrm{e}^{-2\pi|\xi|}\qquad(\xi\in\mathbb{R})\,.$$

Example 8.8 (The Gaussian)**.** Define $f(z):=\mathrm{e}^{-\pi z^2}$. Then

$$\begin{aligned}\int_{x\in\mathbb{R}}|f(x+\mathrm{i}y)|\,\mathrm{d}x &= \int_{x\in\mathbb{R}}\mathrm{e}^{-\pi\left(x^2-y^2\right)}\mathrm{d}x\\ &= \mathrm{e}^{\pi y^2}\\ &\leq \mathrm{e}^{\pi\varepsilon^2}\,. &&(|y|\leq\varepsilon)\end{aligned}$$

In Example 6.36 we calculated the Fourier transform of the Gaussian and found

$$\int_{x\in\mathbb{R}}\mathrm{e}^{-2\pi\mathrm{i}\xi x-\pi x^2}\mathrm{d}x=\mathrm{e}^{-\pi\xi^2}\qquad(\xi\in\mathbb{R})\,.$$

Example 8.9. Define $f(z):=\frac{1}{\pi}\frac{\delta}{\delta^2+z^2}$ for some $\delta>0$. This is the Poisson kernel from below (9.2). This function is analytic so long as $z^2+\delta^2\neq0$, i.e., when $z=\pm\mathrm{i}\delta$. Hence it belongs to $\mathcal{A}_\varepsilon$ with $\varepsilon<\delta$. The L^1 estimate is

$$\begin{aligned}\int_{x\in\mathbb{R}}|f(x+\mathrm{i}y)|\,\mathrm{d}x &= \int_{x\in\mathbb{R}}\left|\frac{1}{\pi}\frac{\delta}{\delta^2+x^2-y^2+2\mathrm{i}xy}\right|\mathrm{d}x\\ &= \frac{\delta}{\pi}\int_{x\in\mathbb{R}}\frac{1}{\sqrt{\left(\delta^2+x^2-y^2\right)^2+4x^2y^2}}\mathrm{d}x\\ &\leq \frac{\delta}{\pi}\int_{x\in\mathbb{R}}\frac{1}{\delta^2+x^2-y^2}\mathrm{d}x\,.\end{aligned}$$

Now, since $|y|<\varepsilon$, we have

$$\delta^2-y^2 > \delta^2-\varepsilon^2$$

and hence

$$\begin{aligned}\int_{x\in\mathbb{R}}|f(x+\mathrm{i}y)|\,\mathrm{d}x &\leq \frac{\delta}{\pi}\int_{x\in\mathbb{R}}\frac{1}{\delta^2-\varepsilon^2+x^2}\mathrm{d}x\\ &= \frac{\delta}{\sqrt{\delta^2-\varepsilon^2}}\,.\end{aligned}$$

Example 8.10 (The delta function)**.** We have the delta function which is certainly not analytic. In fact it's not even *a function* but rather a distribution (or a measure). Ignoring that for a minute we ask what is its Fourier transform and obtain, formally

$$\int_{x\in\mathbb{R}} \mathrm{e}^{-2\pi\mathrm{i}\xi x}\delta(x-a)\,\mathrm{d}x \quad = \quad \mathrm{e}^{-2\pi\mathrm{i}\xi a}$$

and in particular for $a=0$ we get

$$(\mathcal{F}(\delta))(\xi) \quad = \quad 1 \qquad (\xi\in\mathbb{R})$$

the constant 1 function. So this example indeed shows that if a function is very irregular (as the delta function is) its Fourier transform does *not* decay at infinity.

Lemma 8.11 (A Riemann-Lebesgue type lemma)**.** *If $f\in\mathcal{A}_\varepsilon$ then for all $\delta\in[0,\varepsilon)$,*

$$\left|\hat{f}(\xi)\right| \quad \leq \quad B\mathrm{e}^{-2\pi\delta|\xi|} \qquad (\xi\in\mathbb{R})\,.$$

In particular, $\mathcal{F}(\mathcal{A}_\varepsilon)\subseteq L^2$.

Proof. We have

$$\hat{f}(\xi) \quad \equiv \quad \int_{x\in\mathbb{R}} \mathrm{e}^{-2\pi\mathrm{i}x\xi} f(x)\,\mathrm{d}x\,.$$

Consider a vertical leg, from $-R-\mathrm{i}\delta$ to $-R$:

$$\begin{aligned}
\left|\int_{-R-\mathrm{i}\delta}^{-R} \underbrace{f(z)\,\mathrm{e}^{-2\pi\mathrm{i}z\xi}}_{=:g(z)}\mathrm{d}z\right| \quad &= \quad \left|\int_0^\delta f(-R-\mathrm{i}y)\,\mathrm{e}^{-2\pi\mathrm{i}\xi(-R-\mathrm{i}y)}(-\mathrm{i})\,\mathrm{d}y\right| \\
&\leq \quad \int_0^\delta \mathrm{e}^{-2\pi\xi y}\left|f(-R-\mathrm{i}y)\right|\mathrm{d}y\,.
\end{aligned}$$

Integrate this over R to get

$$\begin{aligned}
\int_{R=0}^\infty \left|\int_{-R-\mathrm{i}\delta}^{-R} g(z)\,\mathrm{d}z\right|\mathrm{d}R \quad &\leq \quad \int_{R=0}^\infty \mathrm{e}^{-2\pi\xi y}\int_0^\delta \left|f(-R-\mathrm{i}y)\right|\mathrm{d}y\mathrm{d}R \\
&\overset{\text{Fubini}}{\leq} \quad \int_0^\delta \mathrm{e}^{-2\pi\xi y}\int_{R=0}^\infty \left|f(-R-\mathrm{i}y)\right|\mathrm{d}R\mathrm{d}y \\
&\leq \quad \int_0^\delta \mathrm{e}^{-2\pi\xi y}\int_{R=-\infty}^\infty \left|f(-R-\mathrm{i}y)\right|\mathrm{d}R\mathrm{d}y \\
&\leq \quad \delta\sup_{y\in(0,\delta)}\int_{R=-\infty}^\infty \left|f(-R-\mathrm{i}y)\right|\mathrm{d}R \\
&\leq \quad \delta\sup_{y\in(-\varepsilon,\varepsilon)}\int_{R=-\infty}^\infty \left|f(-R-\mathrm{i}y)\right|\mathrm{d}R
\end{aligned}$$

Since $f\in\mathcal{A}_\varepsilon$, this latter integral is finite, and thus,

$$R \quad \mapsto \quad \left|\int_{-R-\mathrm{i}\delta}^{-R} g(z)\,\mathrm{d}z\right|$$

is integrable at infinity. Avoiding the question whether that means $R\mapsto\left|\int_{-R-\mathrm{i}\delta}^{-R} g(z)\,\mathrm{d}z\right|$ must vanish at infinity, we *can* necessarily find a subsequence of R's for which it *will* decay. Ignoring this slight complication in notation (which would have amounted to replacing $\lim_{R\to\infty}$ with $\lim_{N\to\infty}$ of $\{ R_N \}_N$) we learn that

$$\lim_{R\to\infty}\left|\int_{-R-\mathrm{i}\delta}^{-R} g(z)\,\mathrm{d}z\right| = 0\,.$$

As a result, we may replace the integral of g over $[-R,R]$ with an integral on $[-R-\mathrm{i}\delta, R-\mathrm{i}\delta]$ to get

$$\int_{-R}^{R} g = \int_{-R-\mathrm{i}\delta}^{R-\mathrm{i}\delta} g$$

which is equivalent to

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{e}^{-2\pi\mathrm{i}x\xi}\mathrm{d}x = \int_{-\infty}^{\infty} f(x-\mathrm{i}\delta)\,\mathrm{e}^{-2\pi\mathrm{i}(x-\mathrm{i}\delta)\xi}\mathrm{d}x\,. \tag{8.7}$$

We thus find

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x-\mathrm{i}\delta)\,\mathrm{e}^{-2\pi\mathrm{i}(x-\mathrm{i}\delta)\xi}\mathrm{d}x \tag{8.8}$$

and hence

$$\begin{aligned}\left|\hat{f}(\xi)\right| &\leq \mathrm{e}^{-2\pi\delta\xi}\int_{-\infty}^{\infty}|f(x-\mathrm{i}\delta)|\,\mathrm{d}x\\ &\leq \mathrm{e}^{-2\pi\delta\xi}\sup_{|\delta|<\varepsilon}\int_{-\infty}^{\infty}|f(x-\mathrm{i}\delta)|\,\mathrm{d}x\,.\end{aligned}$$

Now if $\xi > 0$, we get the desired exponential decay. If $\xi < 0$ one may alternatively take the contour *above* the real axis. □

This principle should be pushed to the extreme with an example: imagine $\hat{f}$ is not just exponentially decaying, but actually having compact support within some interval $[-L, L]$ say. What kind of behavior would that correspond to for f? We will see down below this corresponds to f being an entire function such that

$$\lim_{z\to\infty}\frac{1}{|z|}\log(|f(z)|) < \infty\,.$$

8.2.1 The Fourier inversion formula

In this section we shall need the following representation of the reciprocal $z \mapsto \frac{1}{z}$ function, its Laplace transform presentation:

Lemma 8.12. *For any $z \in \mathbb{C}$ with $\mathbb{Re}\{z\} > 0$, we have*

$$\int_{t=0}^{\infty}\mathrm{e}^{-zt}\mathrm{d}t = \frac{1}{z}\,.$$

Proof. We plug in the limit definition of the integral as

$$\begin{aligned}\int_{t=0}^{\infty}\mathrm{e}^{-zt}\mathrm{d}t &\equiv \lim_{R\to\infty}\int_{t=0}^{R}\mathrm{e}^{-zt}\mathrm{d}t\\ &= \lim_{R\to\infty}\left.\frac{\mathrm{e}^{-zt}}{-z}\right|_{t=0}^{R}\\ &= \frac{1}{z}\left(1-\lim_{R\to\infty}\mathrm{e}^{-zR}\right)\,.\end{aligned}$$

This latter limit converges to zero precisely since $\mathbb{Re}\{z\} > 0$. □

We now tend to the question of which functions f are equal to the Fourier inverse of their Fourier transform, i.e.:

Theorem 8.13 (Fourier inversion formula). *Let $\varepsilon > 0$. Then on $\mathcal{A}_\varepsilon$, $\mathcal{F}$ has a left-inverse. I.e., there is an inverse Fourier transform $\mathcal{F}^{-1} : \mathcal{F}(\mathcal{A}_\varepsilon) \to \mathcal{A}_\varepsilon$ given by Section 8.2 such that*

$$\mathcal{F}^{-1}\mathcal{F} \quad = \quad \mathbb{1}_{\mathcal{A}_\varepsilon} \, .$$

More explicitly, defining $\hat{f}$ out of f as in Section 8.2, one has the identity

$$f(x) = \int_{\xi\in\mathbb{R}} \mathrm{e}^{2\pi\mathrm{i}\xi x} \hat{f}(\xi)\,\mathrm{d}\xi \qquad (x \in \mathbb{R}) \, . \tag{8.9}$$

Proof. Let us study the integral in (8.9) by separating it into positive and negative parts. We thus write

$$f(x) = \int_{\xi>0} \mathrm{e}^{2\pi\mathrm{i}\xi x} \hat{f}(\xi)\,\mathrm{d}\xi + \int_{\xi<0} \mathrm{e}^{2\pi\mathrm{i}\xi x} \hat{f}(\xi)\,\mathrm{d}\xi \, . \tag{8.10}$$

For the first term, since $\xi > 0$, we may write using (8.7)

$$\hat{f}(\xi) \quad = \quad \int_{-\infty}^{\infty} f(x - \mathrm{i}\delta)\,\mathrm{e}^{-2\pi\mathrm{i}(x-\mathrm{i}\delta)\xi}\mathrm{d}x$$

where $\delta \in (0, \varepsilon)$. Plugging it into the first term in (8.10) to get

$$\int_{\xi>0} \mathrm{e}^{2\pi\mathrm{i}\xi x} \hat{f}(\xi)\,\mathrm{d}\xi \quad = \quad \int_{\xi>0} \mathrm{e}^{2\pi\mathrm{i}\xi x} \int_{-\infty}^{\infty} f(\tilde{x} - \mathrm{i}\delta)\,\mathrm{e}^{-2\pi\mathrm{i}(\tilde{x}-\mathrm{i}\delta)\xi}\mathrm{d}\tilde{x}\mathrm{d}\xi \, .$$

Thanks to the term $\mathrm{e}^{-2\pi\xi\delta}$, the double integral is absolutely integrable, so that Theorem D.5 applies and we may exchange the orders of integration. We find

$$\int_{\xi>0} \mathrm{e}^{2\pi\mathrm{i}\xi x} \hat{f}(\xi)\,\mathrm{d}\xi \quad = \quad \int_{-\infty}^{\infty} \mathrm{d}\tilde{x} f(\tilde{x} - \mathrm{i}\delta) \int_{\xi>0} \mathrm{e}^{-2\pi\mathrm{i}((\tilde{x}-x)-\mathrm{i}\delta)\xi}\mathrm{d}\xi \, .$$

We apply Lemma 8.12 to the inner integral to get

$$\begin{aligned} \int_{\xi>0} \mathrm{e}^{-2\pi\mathrm{i}((\tilde{x}-x)-\mathrm{i}\delta)\xi}\mathrm{d}\xi \quad &= \quad \int_{\xi>0} \mathrm{e}^{-2\pi(\delta+\mathrm{i}(\tilde{x}-x))\xi}\mathrm{d}\xi \\ &= \quad \frac{1}{2\pi(\delta + \mathrm{i}(\tilde{x} - x))} \, . \end{aligned}$$

Plug this back into the first expression in (8.10) to get

$$\begin{aligned} \int_{\xi>0} \mathrm{e}^{2\pi\mathrm{i}\xi x} \hat{f}(\xi)\,\mathrm{d}\xi \quad &= \quad \int_{-\infty}^{\infty} \mathrm{d}\tilde{x} f(\tilde{x} - \mathrm{i}\delta) \frac{1}{2\pi(\delta + \mathrm{i}(\tilde{x} - x))} \\ &= \quad \frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \mathrm{d}\tilde{x} f(\tilde{x} - \mathrm{i}\delta) \frac{1}{\tilde{x} - \mathrm{i}\delta - x} \, . \end{aligned}$$

Similarly we have for the second term

$$\begin{aligned} \int_{\xi<0} \mathrm{e}^{2\pi\mathrm{i}\xi x} \hat{f}(\xi)\,\mathrm{d}\xi &= \int_{\xi<0} \mathrm{e}^{2\pi\mathrm{i}\xi x} \int_{-\infty}^{\infty} f(\tilde{x} + \mathrm{i}\delta)\,\mathrm{e}^{-2\pi\mathrm{i}(\tilde{x}+\mathrm{i}\delta)\xi}\mathrm{d}\tilde{x}\mathrm{d}\xi \\ &\overset{\text{Fubini}}{=} \int_{-\infty}^{\infty} f(\tilde{x} + \mathrm{i}\delta) \int_{\xi<0} \mathrm{e}^{2\pi(\mathrm{i}(x-\tilde{x})+\delta)\xi}\mathrm{d}\xi\mathrm{d}\tilde{x} \\ &= \int_{-\infty}^{\infty} f(\tilde{x} + \mathrm{i}\delta) \frac{1}{2\pi(\mathrm{i}(x - \tilde{x}) + \delta)}\mathrm{d}\tilde{x} && \text{(Lemma 8.12)} \\ &= -\frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} f(\tilde{x} + \mathrm{i}\delta) \frac{1}{\tilde{x} + \mathrm{i}\delta - x}\mathrm{d}\tilde{x} \, . \end{aligned}$$

adding the two together we see that we would find a closed contour if we could add vertical legs, which we indeed can, precisely thanks to the same proof as in Lemma 8.11. Hence the residue theorem yields the reside at the pole, which is at $\tilde{z} = x$ and we find our result. ☐

Using the same technique as above, it should be clear that also a Parseval-type theorem holds, i.e.,

Theorem 8.14 (Plancherel)**.** *If $f, g \in \mathcal{A}_\varepsilon$ and moreover, f, g are uniformly in L^2, in the sense that*

$$\sup_{y\in(-\varepsilon,\varepsilon)} \int_{x\in\mathbb{R}} |f(x+\mathrm{i}y)|^2 \, \mathrm{d}x \quad < \quad \infty$$

then

$$\langle \mathcal{F}f, \mathcal{F}g\rangle_{L^2} \quad = \quad \langle f, g\rangle_{L^2} \, .$$

Proof. The proof is very similar to the one presented above, but we produce it nonetheless. We have

$$\langle \mathcal{F}f, \mathcal{F}g\rangle_{L^2(\mathbb{R})} \quad = \quad \int_{\xi\in\mathbb{R}} \overline{(\mathcal{F}f)(\xi)} \, (\mathcal{F}g)(\xi) \, \mathrm{d}\xi \, .$$

Since our functions f and g belong in $\mathcal{A}_\varepsilon$, we have thanks to Lemma 8.11 that this integral is absolutely convergent. Moreover, applying dividing the ξ integral as above into its positive and negative sections, we have

$$\langle \mathcal{F}f, \mathcal{F}g\rangle_{L^2(\mathbb{R})} \quad = \quad \int_{\xi>0} \overline{(\mathcal{F}f)(\xi)} \, (\mathcal{F}g)(\xi) \, \mathrm{d}\xi + \int_{\xi<0} \overline{(\mathcal{F}f)(\xi)} \, (\mathcal{F}g)(\xi) \, \mathrm{d}\xi$$

and on each of these, we may write using Section 8.2,

$$\begin{aligned} & \int_{\xi>0} \overline{(\mathcal{F}f)(\xi)} \, (\mathcal{F}g)(\xi) \, \mathrm{d}\xi \\ = \quad & \int_{\xi>0} \overline{\left(\int_{-\infty}^{\infty} f(x-\mathrm{i}\delta) \, \mathrm{e}^{-2\pi\mathrm{i}(x-\mathrm{i}\delta)\xi} \mathrm{d}x\right)} \left(\int_{-\infty}^{\infty} g(\tilde{x}-\mathrm{i}\delta) \, \mathrm{e}^{-2\pi\mathrm{i}(\tilde{x}-\mathrm{i}\delta)\xi} \mathrm{d}\tilde{x}\right) \mathrm{d}\xi \, . \end{aligned}$$

This last integral is absolutely integrable, thanks to the assumption on $f, g \in \mathcal{A}_\varepsilon$ and the shift in the contour which yields factors of $\mathrm{e}^{-2\pi\delta|\xi|}$. We may thus use Fubini Theorem D.5 to exchange order of integrals and get

$$\begin{aligned} & \int_{\xi>0} \overline{(\mathcal{F}f)(\xi)} \, (\mathcal{F}g)(\xi) \, \mathrm{d}\xi \\ & = \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}\tilde{x} \overline{f(x-\mathrm{i}\delta)} g(\tilde{x}-\mathrm{i}\delta) \int_{\xi>0} \mathrm{e}^{2\pi[\mathrm{i}(x-\tilde{x})-2\delta]\xi} \mathrm{d}\xi \\ & = \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}\tilde{x} \overline{f(x-\mathrm{i}\delta)} g(\tilde{x}-\mathrm{i}\delta) \frac{1}{2\pi[2\delta-\mathrm{i}(x-\tilde{x})]} \, . && \text{(Lemma 8.12)} \\ & = \frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}\tilde{x} \overline{f(x-\mathrm{i}\delta)} g(\tilde{x}-\mathrm{i}\delta) \frac{1}{\tilde{x}-x-2\mathrm{i}\delta} \\ & = \frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}\tilde{x} \overline{f(x-\mathrm{i}\delta)} g(\tilde{x}-\mathrm{i}\delta) \frac{1}{(\tilde{x}-\mathrm{i}\delta)-(x-\mathrm{i}\delta)-2\mathrm{i}\delta} \end{aligned}$$

For the other part we find

$$
\begin{aligned}
&\int_{\xi<0} \overline{(\mathcal{F}f)(\xi)}\,(\mathcal{F}g)(\xi)\,\mathrm{d}\xi \\
&= \int_{-\infty}^{\infty}\mathrm{d}x\int_{-\infty}^{\infty}\mathrm{d}\tilde{x}\overline{f(x+\mathrm{i}\delta)}g(\tilde{x}+\mathrm{i}\delta)\int_{\xi<0}\mathrm{e}^{2\pi[\mathrm{i}(x-\tilde{x})+2\delta]\xi}\mathrm{d}\xi \\
&= \int_{-\infty}^{\infty}\mathrm{d}x\int_{-\infty}^{\infty}\mathrm{d}\tilde{x}\overline{f(x+\mathrm{i}\delta)}g(\tilde{x}+\mathrm{i}\delta)\,\frac{1}{2\pi\left[\mathrm{i}(x-\tilde{x})+2\delta\right]} \\
&= -\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\mathrm{d}x\int_{-\infty}^{\infty}\mathrm{d}\tilde{x}\overline{f(x+\mathrm{i}\delta)}g(\tilde{x}+\mathrm{i}\delta)\,\frac{1}{\tilde{x}-x+2\mathrm{i}\delta} \\
&= -\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\mathrm{d}x\int_{-\infty}^{\infty}\mathrm{d}\tilde{x}\overline{f(x+\mathrm{i}\delta)}g(\tilde{x}+\mathrm{i}\delta)\,\frac{1}{(\tilde{x}+\mathrm{i}\delta)-(x+\mathrm{i}\delta)+2\mathrm{i}\delta}
\end{aligned}
$$

Adding the two pieces together we find

$$
\begin{aligned}
\langle\mathcal{F}f,\mathcal{F}g\rangle_{L^2(\mathbb{R})} &= \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\mathrm{d}x\int_{-\infty}^{\infty}\mathrm{d}\tilde{x}\overline{f(x-\mathrm{i}\delta)}g(\tilde{x}-\mathrm{i}\delta)\,\frac{1}{\tilde{x}-x-2\mathrm{i}\delta}+ \\
&\quad -\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\mathrm{d}x\int_{-\infty}^{\infty}\mathrm{d}\tilde{x}\overline{f(x+\mathrm{i}\delta)}g(\tilde{x}+\mathrm{i}\delta)\,\frac{1}{\tilde{x}-x+2\mathrm{i}\delta} \\
&= \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\mathrm{d}x\int_{-\infty}^{\infty}\mathrm{d}\tilde{x}\overline{f(x-\mathrm{i}\delta)}g(\tilde{x}-\mathrm{i}\delta)\,\frac{1}{(\tilde{x}-\mathrm{i}\delta)-(x+\mathrm{i}\delta)}+ \\
&\quad -\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\mathrm{d}x\int_{-\infty}^{\infty}\mathrm{d}\tilde{x}\overline{f(x+\mathrm{i}\delta)}g(\tilde{x}+\mathrm{i}\delta)\,\frac{1}{\tilde{x}+\mathrm{i}\delta-(x-\mathrm{i}\delta)} \\
&= \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\mathrm{d}x\int_{-\infty}^{\infty}\mathrm{d}\tilde{x}\overline{f(x-\mathrm{i}\delta)}g(\tilde{x}-\mathrm{i}\delta)\,\frac{1}{(\tilde{x}-\mathrm{i}\delta)-(x+\mathrm{i}\delta)}+ \\
&\quad -\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\mathrm{d}x\int_{-\infty}^{\infty}\mathrm{d}\tilde{x}\overline{f(x-\mathrm{i}\delta)}g(\tilde{x}+\mathrm{i}\delta)\,\frac{1}{(\tilde{x}+\mathrm{i}\delta)-(x+\mathrm{i}\delta)}+ \\
&\quad +\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\mathrm{d}x\int_{-\infty}^{\infty}\mathrm{d}\tilde{x}\overline{f(x-\mathrm{i}\delta)}g(\tilde{x}+\mathrm{i}\delta)\,\frac{1}{(\tilde{x}+\mathrm{i}\delta)-(x+\mathrm{i}\delta)}+ \\
&\quad -\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\mathrm{d}x\int_{-\infty}^{\infty}\mathrm{d}\tilde{x}\overline{f(x+\mathrm{i}\delta)}g(\tilde{x}+\mathrm{i}\delta)\,\frac{1}{\tilde{x}+\mathrm{i}\delta-(x-\mathrm{i}\delta)}\,.
\end{aligned}
$$

Adding the vertical legs as before to each pair of integrals, we integrate first $\tilde{z}$ on the first two lines. The pole actually sits *on* the $\tilde{z}$ contour, since it is at $\tilde{z}=x+\mathrm{i}\delta$ which is on the contour. As such, using Lemma 7.56 we find

$$
\frac{1}{2}\int_{-\infty}^{\infty}\mathrm{d}x\overline{f(x-\mathrm{i}\delta)}g(x+\mathrm{i}\delta)\,.
$$

We rewrite the last two lines as

$$
\begin{aligned}
&\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\mathrm{d}x\int_{-\infty}^{\infty}\mathrm{d}\tilde{x}\overline{f(x-\mathrm{i}\delta)}g(\tilde{x}+\mathrm{i}\delta)\,\frac{1}{(\tilde{x}+\mathrm{i}\delta)-(x+\mathrm{i}\delta)}+ \\
&-\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\mathrm{d}x\int_{-\infty}^{\infty}\mathrm{d}\tilde{x}\overline{f(x+\mathrm{i}\delta)}g(\tilde{x}+\mathrm{i}\delta)\,\frac{1}{\tilde{x}+\mathrm{i}\delta-(x-\mathrm{i}\delta)} \\
=\;&\overline{-\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\mathrm{d}x\int_{-\infty}^{\infty}\mathrm{d}\tilde{x}f(x-\mathrm{i}\delta)\,\overline{g(\tilde{x}+\mathrm{i}\delta)}\,\frac{1}{(\tilde{x}-\mathrm{i}\delta)-(x-\mathrm{i}\delta)}} \\
&\overline{\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\mathrm{d}x\int_{-\infty}^{\infty}\mathrm{d}\tilde{x}f(x+\mathrm{i}\delta)\,\overline{g(\tilde{x}+\mathrm{i}\delta)}\,\frac{1}{\tilde{x}-\mathrm{i}\delta-(x+\mathrm{i}\delta)}}\,.
\end{aligned}
$$

Performing now the x integral first, adding two vertical legs as before, and using the residue theorem, the pole is at

$z=\tilde{x}-\mathrm{i}\delta$, so again using Lemma 7.56 we find

$$\overline{\frac{1}{2}\int_{-\infty}^{\infty}\mathrm{d}\tilde{x}f\left(\tilde{x}-\mathrm{i}\delta\right)\overline{g\left(\tilde{x}+\mathrm{i}\delta\right)}}=\frac{1}{2}\int_{-\infty}^{\infty}\mathrm{d}\tilde{x}\overline{f\left(\tilde{x}-\mathrm{i}\delta\right)}g\left(\tilde{x}+\mathrm{i}\delta\right)\,.$$

Adding the two together we find

$$\langle\mathcal{F}f,\mathcal{F}g\rangle_{L^2(\mathbb{R})} \quad = \quad \int_{-\infty}^{\infty}\overline{f\left(x-\mathrm{i}\delta\right)}g\left(x+\mathrm{i}\delta\right)\mathrm{d}x\,.$$

We now take the limit $\delta\to 0$ into the integral. This is allowed since f,g are analytic, so their Taylor series allow us to estimate the linear error term in the linear approximation. Essentially,

$$\begin{aligned}
\left|g\left(x+\mathrm{i}\delta\right)-g\left(x\right)\right| &= \left|\sum_{n=0}^{\infty}\frac{1}{n!}g^{(n)}\left(x\right)\left(\mathrm{i}\delta\right)^n\right| \\
&= \left|\sum_{n=1}^{\infty}\frac{1}{n!}\frac{n!}{2\pi\mathrm{i}}\oint_{\partial B_\eta(x)}\frac{g\left(z\right)}{\left(z-x\right)^{n+1}}\mathrm{d}z\left(\mathrm{i}\delta\right)^n\right| \\
&\leq \left|\sum_{n=1}^{\infty}\frac{1}{2\pi}\oint_{\partial B_\eta(x)}\frac{g\left(z\right)}{\left(z-x\right)^{n+1}}\mathrm{d}z\left(\mathrm{i}\delta\right)^n\right| \\
&\leq \int_{\theta=0}^{2\pi}\left|g\left(x+\eta\mathrm{e}^{\mathrm{i}\theta}\right)\right|\mathrm{d}\theta\sum_{n=1}^{\infty}\frac{1}{\eta^n}\delta^n \\
&= \int_{\theta=0}^{2\pi}\left|g\left(x+\eta\mathrm{e}^{\mathrm{i}\theta}\right)\right|\mathrm{d}\theta\frac{\delta}{\eta-\delta}
\end{aligned}$$

which works as long as $\eta>\delta$. We thus find, for any $0<\delta<\eta<\varepsilon$,

$$\begin{aligned}
&\left|\int_{-\infty}^{\infty}\overline{f\left(x-\mathrm{i}\delta\right)}g\left(x+\mathrm{i}\delta\right)\mathrm{d}x-\int_{-\infty}^{\infty}\overline{f\left(x\right)}g\left(x\right)\mathrm{d}x\right| \\
\leq\;& \left|\int_{-\infty}^{\infty}\overline{f\left(x-\mathrm{i}\delta\right)}g\left(x+\mathrm{i}\delta\right)\mathrm{d}x-\int_{-\infty}^{\infty}\overline{f\left(x-\mathrm{i}\delta\right)}g\left(x\right)\mathrm{d}x\right|+ \\
&+\left|\int_{-\infty}^{\infty}\overline{f\left(x-\mathrm{i}\delta\right)}g\left(x\right)\mathrm{d}x-\int_{-\infty}^{\infty}\overline{f\left(x\right)}g\left(x\right)\mathrm{d}x\right| \\
\leq\;& \left|\int_{-\infty}^{\infty}\left|f\left(x-\mathrm{i}\delta\right)\right|\int_{\theta=0}^{2\pi}\left|g\left(x+\eta\mathrm{e}^{\mathrm{i}\theta}\right)\right|\mathrm{d}\theta\mathrm{d}x\right|\frac{\delta}{\eta-\delta}+ \\
&+\left|\int_{-\infty}^{\infty}\int_{\theta=0}^{2\pi}\left|f\left(x+\eta\mathrm{e}^{\mathrm{i}\theta}\right)\right|\mathrm{d}\theta\left|g\left(x\right)\right|\mathrm{d}x\right|\frac{\delta}{\eta-\delta} \\
\leq\;& \left|\int_{-\infty}^{\infty}\left|f\left(x\right)\right|\int_{\theta=0}^{2\pi}\left|g\left(x+\eta\mathrm{e}^{\mathrm{i}\theta}\right)\right|\mathrm{d}\theta\mathrm{d}x\right|\frac{\delta}{\eta-\delta}+ \\
&+\int_{-\infty}^{\infty}\int_{\theta=0}^{2\pi}\left|f\left(x+\eta\mathrm{e}^{\mathrm{i}\theta}\right)\right|\mathrm{d}\theta\int_{\theta=0}^{2\pi}\left|g\left(x+\eta\mathrm{e}^{\mathrm{i}\theta}\right)\right|\mathrm{d}\theta\mathrm{d}x\left(\frac{\delta}{\eta-\delta}\right)^2 \\
&+\int_{-\infty}^{\infty}\int_{\theta=0}^{2\pi}\left|f\left(x+\eta\mathrm{e}^{\mathrm{i}\theta}\right)\right|\mathrm{d}\theta\left|g\left(x\right)\right|\mathrm{d}x\frac{\delta}{\eta-\delta}\,.
\end{aligned}$$

Since all these integrals are finite by assumption on $f,g\in L^2$, we may safely take the limit $\delta\to 0^+$. □

Theorem 8.15 (Convolution theorem)**.** *If $f,g\in L^2\in\mathcal{A}_\varepsilon$ then the transform of the convolution is the product of the transforms, i.e.,*

$$\mathcal{F}\left(f*g\right)=\left(\mathcal{F}f\right)\left(\mathcal{F}g\right)\,.$$

Similarly, the transform of the product is the convolution of the transforms:

$$\mathcal{F}\left(fg\right) \quad = \quad \left(\mathcal{F}f\right)*\left(\mathcal{F}g\right)\,.$$

Proof. Recall that

$$(f*g)(x) \equiv \int_{y\in\mathbb{R}} f(y)\, g(x-y)\, \mathrm{d}y\,.$$

Then

$$\begin{aligned}(\mathcal{F}(f*g))(\xi) &\equiv \int_{x\in\mathbb{R}} \mathrm{e}^{-2\pi\mathrm{i}\xi x}(f*g)(x)\,\mathrm{d}x\\ &= \int_{x\in\mathbb{R}} \mathrm{e}^{-2\pi\mathrm{i}\xi x}\int_{y\in\mathbb{R}} f(y)\, g(x-y)\,\mathrm{d}y\mathrm{d}x\\ &\overset{x\mapsto x+y}{=} \int_{x\in\mathbb{R}} \mathrm{e}^{-2\pi\mathrm{i}\xi(x+y)}\int_{y\in\mathbb{R}} f(y)\, g(x)\,\mathrm{d}y\mathrm{d}x\\ &= ((\mathcal{F}f)(\xi))\,((\mathcal{F}g)(\xi))\,.\end{aligned}$$

The usage of Fubini's theorem here was justified by $f,g\in L^2$ and the Cauchy-Schwarz inequality.

Similarly,

$$\begin{aligned}[(\mathcal{F}f)*(\mathcal{F}g)](p) &= \int_{q\in\mathbb{R}}(\mathcal{F}f)(q)\,(\mathcal{F}g)(p-q)\,\mathrm{d}q\\ &= \int_{q\in\mathbb{R}}\left(\int_{x\in\mathbb{R}}\mathrm{e}^{-2\pi\mathrm{i}qx}f(x)\,\mathrm{d}x\right)\left(\int_{y\in\mathbb{R}}\mathrm{e}^{-2\pi\mathrm{i}(p-q)y}g(y)\,\mathrm{d}y\right)\mathrm{d}q\,.\end{aligned}$$

Now in principle we separate the integral in $q>0$ and $q<0$, as well as $p-q>0$ and $p-q<0$, i.e., into three different sections. In each of these sections we know how to shift the integral for the Fourier transform so that it is absolutely convergent (just as in the proofs above). Once there is absolute convergence we may do the q integral first. Avoiding these technicalities here (since they are not that interesting) we would find the rigorous version of the following heuristic maneuver:

$$\begin{aligned}[(\mathcal{F}f)*(\mathcal{F}g)](p) &= \int_{x\in\mathbb{R}}\mathrm{d}x\int_{y\in\mathbb{R}}\mathrm{d}y g(y)\, f(x)\,\mathrm{e}^{-2\pi\mathrm{i}py}\underbrace{\int_{q\in\mathbb{R}}\mathrm{e}^{-2\pi\mathrm{i}q(x-y)}\mathrm{d}q}_{=\delta(x-y)}\\ &= \int_{x\in\mathbb{R}}\mathrm{d}x g(x)\, f(x)\,\mathrm{e}^{-2\pi\mathrm{i}px}\,.\end{aligned}$$

□

8.2.2 The Poisson summation formula

Theorem 8.16 (The Poisson summation formula)**.** *For a function $f\in\mathcal{A}_\varepsilon$ we have*

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}}\hat{f}(n)\,.$$

Proof. Consider the function

$$z \mapsto \frac{1}{\mathrm{e}^{2\pi\mathrm{i}z}-1}$$

which has simple poles on $\mathbb{Z}$, each with residue $\frac{1}{2\pi\mathrm{i}}$. Indeed, we verify this with

$$\lim_{z\to n}(z-n)\frac{1}{\mathrm{e}^{2\pi\mathrm{i}z}-1} = \lim_{z\to n}\frac{z-n}{\mathrm{e}^{2\pi\mathrm{i}z}-1}$$

We recognize the last prelimit as the reciprocal of the definition of the derivative of $z\mapsto\mathrm{e}^{2\pi\mathrm{i}z}$, and hence we may write

$$\begin{aligned}\lim_{z\to n}\left(z-n\right)\frac{1}{\mathrm{e}^{2\pi\mathrm{i}z}-1} &= \frac{1}{\partial_z|_{z=n}\exp\left(2\pi\mathrm{i}z\right)}\\ &= \frac{1}{2\pi\mathrm{i}}\,.\end{aligned}$$

For any $\delta\in(0,\varepsilon)$, let Γ_N be the closed rectangular contour vertices at

$$-N-\frac{1}{2}-\mathrm{i}\delta, N+\frac{1}{2}-\mathrm{i}\delta, N+\frac{1}{2}+\mathrm{i}\delta, -N-\frac{1}{2}+\mathrm{i}\delta\,.$$

Within Γ_N, the function $z\mapsto\frac{f(z)}{\mathrm{e}^{2\pi\mathrm{i}z}-1}$ is meromorphic and there are no poles on Γ. Hence the residue theorem implies

$$\oint_{\Gamma_N}\frac{f\left(z\right)}{\mathrm{e}^{2\pi\mathrm{i}z}-1}\mathrm{d}z = \sum_{n=-N}^{N}f\left(n\right)\,.$$

In the limit $N\to\infty$, the vertical lines vanish since $f\in\mathcal{A}_\varepsilon$, as in Lemma 8.11. Hence

$$\begin{aligned}\sum_{n\in\mathbb{Z}}f\left(n\right) &= -\int_{x\in\mathbb{R}}\frac{f\left(x+\mathrm{i}\delta\right)}{\mathrm{e}^{2\pi\mathrm{i}(x+\mathrm{i}\delta)}-1}\mathrm{d}x+\int_{x\in\mathbb{R}}\frac{f\left(x-\mathrm{i}\delta\right)}{\mathrm{e}^{2\pi\mathrm{i}(x-\mathrm{i}\delta)}-1}\mathrm{d}x\\ &= \int_{z\in\Gamma_N^{\mathrm{lower}}}\frac{f\left(z\right)}{\mathrm{e}^{2\pi\mathrm{i}z}-1}\mathrm{d}z-\int_{z\in\Gamma_N^{\mathrm{upper}}}\frac{f\left(z\right)}{\mathrm{e}^{2\pi\mathrm{i}z}-1}\mathrm{d}z\,.\end{aligned}$$

Now, on $\Gamma_N^{\mathrm{upper}}$, since $\left|\mathrm{e}^{2\pi\mathrm{i}z}\right|<1$,

$$\frac{1}{\mathrm{e}^{2\pi\mathrm{i}z}-1} = -\sum_{n=0}^{\infty}\mathrm{e}^{2\pi\mathrm{i}nz}$$

where on $\Gamma_N^{\mathrm{lower}}$, since $\left|\mathrm{e}^{2\pi\mathrm{i}z}\right|>1$, we get

$$\frac{1}{\mathrm{e}^{2\pi\mathrm{i}z}-1} = \mathrm{e}^{-2\pi\mathrm{i}z}\sum_{n=0}^{\infty}\mathrm{e}^{-2\pi\mathrm{i}zn}\,.$$

Together we find

$$\sum_{n\in\mathbb{Z}}f\left(n\right) = \int_{z\in\Gamma_N^{\mathrm{lower}}}f\left(z\right)\mathrm{e}^{-2\pi\mathrm{i}z}\sum_{n=0}^{\infty}\mathrm{e}^{-2\pi\mathrm{i}zn}\mathrm{d}z+\int_{z\in\Gamma_N^{\mathrm{upper}}}f\left(z\right)\sum_{n=0}^{\infty}\mathrm{e}^{2\pi\mathrm{i}nz}\mathrm{d}z\,.$$

We are allowed to exchange the integration and sums, since both are absolutely convergent (thanks to working with z off the real axis). Hence for the first term, e.g.,

$$\begin{aligned}\text{1st term} &= \int_{z\in\Gamma_N^{\mathrm{lower}}}f\left(z\right)\mathrm{e}^{-2\pi\mathrm{i}z}\sum_{n=0}^{\infty}\mathrm{e}^{-2\pi\mathrm{i}zn}\mathrm{d}z\\ &= \sum_{n=0}^{\infty}\int_{z\in\Gamma_N^{\mathrm{lower}}}f\left(z\right)\mathrm{e}^{-2\pi\mathrm{i}z(n+1)}\mathrm{d}z\\ &= \sum_{n=0}^{\infty}\int_{x=-\infty}^{\infty}f\left(x\right)\mathrm{e}^{-2\pi\mathrm{i}x(n+1)}\mathrm{d}x \qquad ((8.7))\\ &= \sum_{n=0}^{\infty}\hat{f}\left(n+1\right)\,.\end{aligned}$$

For the other term, we have

$$\begin{aligned}\text{2nd term} &= \int_{z\in\Gamma_N^{\text{upper}}} f(z)\sum_{n=0}^{\infty}\mathrm{e}^{2\pi i n z}\mathrm{d}z \\ &= \sum_{n=0}^{\infty}\int_{z\in\Gamma_N^{\text{upper}}} f(z)\,\mathrm{e}^{2\pi i n z}\mathrm{d}z \\ &= \sum_{n=0}^{\infty}\int_{x=-\infty}^{\infty} f(x)\,\mathrm{e}^{2\pi i n x}\mathrm{d}x \qquad ((8.7)) \\ &= \sum_{n=0}^{\infty}\hat{f}(-n)\end{aligned}$$

and our statement follows. □

8.3 The Paley-Wiener theorem

In this section, we do *not* assume that $f \in \mathcal{A}_\varepsilon$, but rather *derive* this out of decay properties of $\hat{f}$.

Theorem 8.17 (Converse to Riemann-Lebesgue)**.** *Let $\hat{f} : \mathbb{R} \to \mathbb{C}$ be a function such that there exist $A, a \in (0,\infty)$ with*

$$\left|\hat{f}(\xi)\right| \leq A\mathrm{e}^{-2\pi a|\xi|} \qquad (\xi \in \mathbb{R}) \ .$$

We may then define $f : \mathbb{R} \to \mathbb{C}$ via

$$f(x) \quad := \quad \int_{\xi\in\mathbb{R}} \mathrm{e}^{2\pi i\xi x}\hat{f}(\xi)\,\mathrm{d}\xi \qquad (x \in \mathbb{R}) \ .$$

Then, for such $\hat{f}$ and f, $f = g|_{\mathbb{R}}$ for some analytic $g : S_\varepsilon \to \mathbb{C}$ for some $\varepsilon \in (0, a)$.

Theorem 8.18 (Paley-Wiener)**.** *Let $f : \mathbb{R} \to \mathbb{C}$ be continuous and obey, for some $A \in (0,\infty)$,*

$$|f(x)| \quad \leq \quad \frac{A}{1+x^2} \qquad (x \in \mathbb{R}) \ .$$

Then, for given $L \in (0,\infty)$, the following two conditions are equivalent:

1. *f extends to an entire function $g : \mathbb{C} \to \mathbb{C}$ for which there exists $B \in (0,\infty)$*

$$|g(z)| \leq B\mathrm{e}^{2\pi L|z|} \qquad (z \in \mathbb{C}) \ .$$

2. *$\hat{f}$ is supported within $[-L, L]$.*

9 Conformal maps

9.1 Conformal equivalence

Definition 9.1 (conformal map)**.** Let $U, V \in \text{Open}(\mathbb{C})$. A map $f : U \to V$ which is both holomorphic and bijective (injective and surjective) is called *conformal*. When such a map exists between $U \to V$, these two sets are called *conformally equivalent*.

Sometimes one reads the term *biholomorphic* though here it shall be avoided since it may be confused with a function $\mathbb{C}^2 \to \mathbb{C}$ which is holomorphic in each argument separately.

Lemma 9.2. *The following statements are true:*

1. *Any holomorphic f which is injective has $f' \neq 0$.*
2. *Any invertible holomorphic f with $f' \neq 0$ has a* holomorphic *inverse.*

From this lemma, we learn that in the setup of Definition 9.1, $f'\neq0$. In fact all that is necessary is that f is injective. Since f is injective, it has an inverse f^{-1} on its range. $f'\neq0$ implies that that inverse f^{-1} is also holomorphic.

We thus recover the *structural fact* that if f is conformal, it is a *holomorphic-structure-preserving bijection*.

Proof of Lemma 9.2. We follow [SS03]. Assume otherwise and generate a contradiction. So let's assume $\exists z_0\in U : f'(z_0)=0$. Write the Taylor series for f about z_0:

$$f(z) \quad = \quad f(z_0)+\underbrace{f'(z_0)}_{=0}(z-z_0)+\frac{1}{2}f^{(2)}(z_0)(z-z_0)^2+\dots\,.$$

Now, let $k\geq2$ be the smallest integer so that $f^{(k)}(z_0)\neq0$ (if there is no such finite number we have f as the constant function which is not injective and hence a contradiction). Then we may write

$$f(z)-f(z_0) \quad = \quad a(z-z_0)^k(1+g(z))$$

for some $a\neq0$ and g analytic which converges to zero as $z\to z_0$. For any $w\neq0$, the solutions $z\in\mathbb{C}$ of the equation

$$a(z-z_0)^k-w=0$$

are k distinct ones given by:

$$z-z_0 \quad \in \quad \left|\frac{w}{a}\right|^{\frac{1}{k}}\left\{\exp\left(\mathrm{i}\frac{1}{k}\operatorname{Arg}\left(\frac{w}{a}\right)\right),\dots,\exp\left(\mathrm{i}\frac{1}{k}\left(\operatorname{Arg}\left(\frac{w}{a}\right)+(k-1)2\pi\right)\right)\right\}$$

(we didn't really *need* the precise expression).

Let us now define the polynomial $p:\mathbb{C}\to\mathbb{C}$ given by

$$p(z) \quad := \quad a(z-z_0)^k-w\,.$$

If, for any $\varepsilon>0$, $w\in B_{\left(\frac{1}{2}\varepsilon\right)^k|a|}(0)\setminus\{0\}$, then the k roots of p are in $B_{\frac{1}{2}\varepsilon}(z_0)$ by the above, so p is a holomorphic function that has k roots within $B_\varepsilon(z_0)$ and none on the boundary circle.

We want to boost this statement to the function $z\mapsto f(z)-f(z_0)-w$ using Corollary 7.44 (Rouché). To do so, we need to estimate the difference for $z\in B_\varepsilon(z_0)$:

$$\begin{aligned}|p(z)| &\overset{!}{>} |f(z)-f(z_0)-w-p(z)|\\ &= \left|a(z-z_0)^k g(z)\right|\end{aligned}$$

so we need

$$\begin{aligned}\left|a(z-z_0)^k g(z)\right| &\overset{!}{<} \left|a(z-z_0)^k-w\right| \qquad (z\in B_\varepsilon(z_0))\,.\\ &\Uparrow\\ \left|a(z-z_0)^k\right||g(z)| &< \left|a(z-z_0)^k\right|-|w|\\ &\Updownarrow\\ |w| &< \left|a(z-z_0)^k\right|(1-|g(z)|)\\ &\Updownarrow\\ \left(\frac{1}{2}\varepsilon\right)^k|a| &< |a|\,\varepsilon^k(1-|g(z)|)\\ &\Updownarrow\\ 2^{-k} &< 1-|g(z)| \qquad (z\in B_\varepsilon(z_0))\,.\end{aligned}$$

Since $g\to0$ as $z\to z_0$, pick ε small enough so that $|g(z)|<1-2^{-k}$ for $z\in B_\varepsilon(z_0)$.

For such ε, we now find via Corollary 7.44 that also

$$z \quad \mapsto \quad f(z) - f(z_0) - w$$

has $k \geq 2$ zeros $z_1, \dots, z_k \in B_\varepsilon(z_0)$, i.e.,

$$f(z_j) = f(z_0) - w \qquad (j = 1, \dots, k) \; .$$

In particular,

$$f(z_j) \quad = \quad f(z_l) \qquad (j, l = 1, \dots, k) \; .$$

If there are two zeros z_j, z_l which are distinct, then we are finished since this implies f is not injective. Assume then all zeros are identical actually identical, i.e., f has a kth order zero at $z_\star := z_1 = \dots = z_k$. This means that it must be possible to write

$$f(z) - f(z_0) - w \quad = \quad (z - z_\star)^k h(z)$$

for some non-vanishing holomorphic h, via Lemma 7.20. Take now the derivative w.r.t. z of the above equation to get

$$\begin{aligned} f'(z) \quad &= \quad k(z - z_\star)^{k-1} h(z) + (z - z_\star)^k h'(z) \\ &\downarrow \\ f'(z_\star) \quad &= \quad 0 \, . \end{aligned}$$

However, it is impossible that $f'(z_\star) = 0$: $z_\star \in B_\varepsilon(z_0)$ and since $f'(z_0) = 0$, f is analytic, then so is f', and we know the zeros of analytic functions do not accumulate, some disc in which $f' \neq 0$ except at its center, and pick $\varepsilon > 0$ within that disc.

Hence there must be some distinct roots and so f is not injective, a contradiction.

For the second statement in the claim, we want to show that if $f' \neq 0$ then f^{-1} is also holomorphic. First, we know by injectivity that

$$f^{-1} : \mathrm{im}(f) \quad \to \quad U$$

is defined. To show it is holomorphic, let us consider the pre-limit

$$\frac{f^{-1}(w) - f^{-1}(w_0)}{w - w_0} \, .$$

with $w, w_0 \in \mathrm{im}(f)$. Let $z, z_0 \in U$ such that $f(z) = w$, $f(z_0) = w_0$. Then that prelimit is

$$\begin{aligned} \frac{z - z_0}{f(z) - f(z_0)} \quad &= \quad \frac{1}{\left(\frac{f(z) - f(z_0)}{z - z_0}\right)} \\ &\overset{z \to z_0}{\longrightarrow} \quad \frac{1}{f'(z_0)} \, . \end{aligned}$$

so that the limit for $\left(f^{-1}\right)'$ exists too, since $f' \neq 0$. $\square$

Corollary 9.3. *U, V are conformally equivalent iff they are holomorphically bijective and the inverse is also holomorphic.*

Proof. One direction is stronger than the definition and the other one is the lemma we've just proven. $\square$

Example 9.4 (The unit disc and the upper half plane)**.** Let the upper half plane

$$\mathbb{H} \quad := \quad \{ z \in \mathbb{C} \mid \mathbb{Im}\{z\} > 0 \} \; .$$

We claim that

$$\mathbb{H} \overset{\text{conformal}}{\cong} B_1(0) \, .$$

Proof. Define $f : \mathbb{H} \to \mathbb{C}$ via

$$f(z) \quad := \quad \frac{\mathrm{i} - z}{\mathrm{i} + z} \, .$$

We claim this is the desired conformal map. Since $\mathbb{Im}\{z\} > 0$, $\mathrm{i} + z \neq 0$ so f is certainly holomorphic. Next, since

$$|z + \mathrm{i}| \quad > \quad |z - \mathrm{i}| \qquad (z \in \mathbb{H})$$

we have $|f(z)| < 1$ so $\mathrm{im}(f) \subseteq B_1(0)$. Define $g : B_1(0) \to \mathbb{C}$ via

$$g(w) \quad := \quad \mathrm{i}\frac{1-w}{1+w} \, .$$

We want to show $\mathrm{im}(g) \subseteq \mathbb{H}$. To that end, write $w = x + \mathrm{i}y$ and calculate

$$\begin{aligned}
\mathbb{Im}\{g(w)\} \quad &= \quad \mathbb{Im}\left\{\mathrm{i}\frac{1-w}{1+w}\right\} \\
&= \quad \mathbb{Re}\left\{\frac{1-w}{1+w}\right\} \\
&= \quad \mathbb{Re}\left\{\frac{(1-w)\left(\overline{1+w}\right)}{|1+w|^2}\right\} \\
&= \quad \frac{1}{|1+w|^2}\,\mathbb{Re}\left\{1 - w + \overline{w} - |w|^2\right\} \\
&= \quad \frac{1}{|1+w|^2}\,\mathbb{Re}\left\{1 - |w|^2 - 2\mathrm{i}\,\mathbb{Im}\{w\}\right\} \\
&= \quad \frac{1-|w|^2}{|1+w|^2} \\
&> \quad 0 \, .
\end{aligned}$$

Let $w \in B_1(0)$. Then

$$\begin{aligned}
f(g(w)) \quad &= \quad \frac{\mathrm{i} - g(w)}{\mathrm{i} + g(w)} \\
&= \quad \frac{\mathrm{i} - \mathrm{i}\frac{1-w}{1+w}}{\mathrm{i} + \mathrm{i}\frac{1-w}{1+w}} \\
&= \quad \frac{1 + w - 1 + w}{1 + w + 1 - w} \\
&= \quad \frac{2w}{2} \\
&= \quad w \, .
\end{aligned}$$

So f is surjective. A similar calculation ($g(f(z)) = \cdots = z$ for $z \in \mathbb{H}$) shows f is injective. So f is a holomorphic bijection indeed. Note that the boundary line of $\mathbb{H}$ (the real axis) is mapped onto the unit circle $\mathbb{S}^1$. □

We now come to how one *usually* hears about "conformal" maps, namely, that they preserve angles. Recall the angle α between two vectors $u, v \in \mathbb{R}^n$ is given by

$$\cos(\alpha) \quad = \quad \frac{\langle u, v \rangle_{\mathbb{R}^n}}{\|u\|\|v\|} \, .$$

Hence, we make the

Definition 9.5. A function $f:\mathbb{R}^n\to\mathbb{R}^n$ (or an open subset to an open subset of $\mathbb{R}^n$) preserves angles iff its linearization preserves angles

$$\frac{\langle\mathcal{D}fu,\mathcal{D}fv\rangle}{\|\mathcal{D}fu\|\|\mathcal{D}fv\|} = \frac{\langle u,v\rangle}{\|u\|\|v\|}$$

(i.e., if the Jacobian matrix $|\mathcal{D}f|^2\equiv(\mathcal{D}f)^*\mathcal{D}f$ is constant (non-zero) diagonal, i.e., if $\mathcal{D}f$ is orthogonal up to a scaling) iff for any two curve $\gamma_1,\gamma_2:[0,1]\to\mathbb{R}^n$, the tangents $\gamma_1',\gamma_2':[0,1]\to\mathbb{R}^n$, which get mapped to $(\mathcal{D}f\circ\gamma_j)\gamma_j'$, $j=1,2$ have the same angle between them after the transformation, i.e.,

$$\frac{\langle(\mathcal{D}f\circ\gamma_1)\gamma_1',(\mathcal{D}f\circ\gamma_2)\gamma_2'\rangle_{\mathbb{R}^n}}{\|(\mathcal{D}f\circ\gamma_1)\gamma_1'\|\|(\mathcal{D}f\circ\gamma_2)\gamma_2'\|} \overset{!}{=} \frac{\langle\gamma_1',\gamma_2'\rangle_{\mathbb{R}^n}}{\|\gamma_1'\|\|\gamma_2'\|}\,.$$

So much for $\mathbb{R}^n$. Specifying to $n=2$ which is our case of interest, we see that a function $f:\mathbb{C}\to\mathbb{C}$ preserves angles iff, using (4.1),

$$\mathcal{D}f=\begin{bmatrix}\partial_x f_R & \partial_y f_R\\ \partial_x f_I & \partial_y f_I\end{bmatrix}$$

is orthogonal up to scaling, i.e.,

$$\mathcal{D}f = g\begin{bmatrix}\cos(\alpha) & -\sin(\alpha)\\ \sin(\alpha) & \cos(\alpha)\end{bmatrix}$$

for some function $g\neq 0$. But examining (4.3) which says

$$\det(\mathcal{D}F) = (\partial_x f_R)^2+(\partial_x f_I)^2=|f'|^2$$

we see that for f to be angle preserving is precisely the condition that f is holomorphic with $f'\neq 0$, which is guaranteed by Lemma 9.2.

Corollary 9.6. *Conformal maps are angle-preserving (the converse may fail if somehow we have a map that has $f'\neq 0$ yet it is not injective).*

Example 9.7. The map $\mathbb{C}\setminus\{0\}\ni z\mapsto z^2$ is holomorphic and its derivative $\mathbb{C}\setminus\{0\}\ni z\mapsto z$ is never zero. However, it is *not* injective. If, however, we restricted the domain of the map further, we could make it injective so as to make it conformal.

9.2 Further examples of conformal equivalences

Example 9.8 (Translations and dilations)**.** These maps, $z\mapsto wz$ for $w\neq 0$ and $z\mapsto z+w$ map $\mathbb{C}\to\mathbb{C}$ conformally.

Example 9.9. If $\alpha\in(0,2)$, define

$$f:\mathbb{H} \to \{\, z\in\mathbb{C}\mid 0<\operatorname{Arg}(w)<\alpha\pi\,\}$$

via

$$z \mapsto z^\alpha=|z|^\alpha\,\mathrm{e}^{\mathrm{i}\alpha\widetilde{\operatorname{Arg}}(z)}$$

where $\widetilde{\operatorname{Arg}}(z)$ is the branch of the logarithm with branch cut on the *positive* real axis. Its inverse is

$$z \mapsto z^{\frac{1}{\alpha}}=|z|^{\frac{1}{\alpha}}\,\mathrm{e}^{\mathrm{i}\frac{1}{\alpha}\widetilde{\operatorname{Arg}'}(z)}$$

where $\widetilde{\operatorname{Arg}'}(z)$ is chosen so that $\widetilde{\operatorname{Arg}'}(z)\in(0,\alpha\pi)$.

Example 9.10. Define the upper half disc as

$$U \quad := \quad \{ \, z \in B_1(0) \mid \mathbb{Im}\{z\} > 0 \, \} \; .$$

A function

$$f : U \to \{ \, z \in \mathbb{C} \mid \mathbb{Im}\{z\} , \mathbb{Re}\{z\} > 0 \, \}$$

to the first quadrant is defined via

$$z \quad \mapsto \quad \frac{1+z}{1-z} \; .$$

It is certainly holomorphic since it has a pole at $z = 1$ which is not in its domain. Simple algebraic calculations show that

1. It is well-defined (i.e. $\operatorname{im}(f) \subseteq \{ \, z \in \mathbb{C} \mid \mathbb{Im}\{z\} , \mathbb{Re}\{z\} > 0 \, \}$).
2. The inverse is $z \mapsto \frac{z-1}{z+1}$ well-defined too.

Example 9.11. Let $\operatorname{Log}_{-\mathrm{i}}$ be the branch of the logarithm obtained by deleting the negative imaginary axis, and define the horizontal strip of width π:

$$S_\pi \quad := \quad \{ \, z \in \mathbb{C} \mid \mathbb{Im}\{z\} \in (0, \pi) \, \} \; .$$

Define

$$f : \mathbb{H} \quad \to \quad S_\pi$$

via

$$z \quad \mapsto \quad \operatorname{Log}_{-\mathrm{i}}(z) \; .$$

Then clearly f is holomorphic since $\mathbb{H}$ avoids the branch cut, and lands within S_π since its domain is the upper half plane with arguments in $(0, \pi)$. The inverse of f is exp.

Example 9.12. If we take the example above and restrict its domain to the upper half unit disc, what should its co-domain be to obtain a conformal equivalence?

Example 9.13 (Jukowski map)**.** Define

$$f : \{ \, z \in B_1(0) \mid \mathbb{Im}\{z\} > 0 \, \} \quad \to \quad \mathbb{H}$$

via

$$f(z) \quad := \quad -\frac{1}{2}\left(z + \frac{1}{z}\right) \; .$$

This map is clearly holomorphic, and also conformal (though it a bit tedious to write down its inverse).

We refer the reader to the appendix of [BC13] for more examples of conformal maps.

9.3 The Laplace equation on the disc

The Laplace equation is the following "boundary value problem" (called a *Dirichlet problem*): Let $\Omega \in \operatorname{Open}(\mathbb{C})$. Find $u : \Omega \to \mathbb{R}$ such that

$$\begin{aligned} -\Delta u &= 0 \\ \lim_{z \to \partial\Omega} u &= f \end{aligned}$$

for some given known $f : \partial\Omega \to \mathbb{R}$and $-\Delta \equiv \partial_x^2 + \partial_y^2$ is the Laplacian from Section 4.4. In principle there is some fine print on just what kind of boundary values f are allowed (i.e. regularity conditions, here continuity of f suffices), there is a question of existence of a solution in whichever regularity class etc etc. We avoid all these questions here.

Lemma 9.14. *If $\Omega = B_1(0)$ then the solution of the Laplace equation is given by*

$$u(r,\theta) = \sum_{n\in\mathbb{Z}} a_n r^{|n|} \mathrm{e}^{\mathrm{i}n\theta}$$

with

$$a_n = \frac{1}{2\pi}\int_{\theta=0}^{2\pi} f(\theta)\,\mathrm{e}^{-\mathrm{i}n\theta}\mathrm{d}\theta\,.$$

Proof. First we argue that if such a u exists then it must be unique. Indeed, since u is, by definition, harmonic, it is the real part of some analytic function on Ω with prescribed boundary values. So essentially this would be an analytic continuation, from $\partial\Omega$ to Ω of f. But analytic continuations are unique Definition 7.19, so there is only one such possible solution, should it exist (another way to argue is to establish that $-\Delta$ with the prescribed boundary conditions is self-adjoint on $L^2(\Omega)$, and we would be trying to prove that its zero energy eigenvector is simple: this is beyond our scope).

Next, an arbitrary analytic function may be written in terms of its Taylor series about zero as

$$g(z) = \sum_{n=0}^{\infty} c_n z^n$$

for some $\{ c_n \}_n$, and since u would be the real part of it, we get (with the polar representation $z = r\mathrm{e}^{\mathrm{i}\theta}$)

$$\begin{aligned} u(z) &= \sum_{n=0}^{\infty} \mathbb{Re}\{c_n z^n\} \\ &= \sum_{n=0}^{\infty} \mathbb{Re}\left\{c_n r^n \mathrm{e}^{\mathrm{i}n\theta}\right\} \\ &= \sum_{n=0}^{\infty} \frac{1}{2}\left(c_n r^n \mathrm{e}^{\mathrm{i}n\theta} + \overline{a_n r^n \mathrm{e}^{\mathrm{i}n\theta}}\right) \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{2} c_n r^n \mathrm{e}^{\mathrm{i}n\theta} + \frac{1}{2} r^n \overline{c_n} \mathrm{e}^{-\mathrm{i}n\theta}\right) \\ &=: \sum_{n\in\mathbb{Z}} a_n r^{|n|} \mathrm{e}^{\mathrm{i}n\theta} \end{aligned}$$

with

$$a_n := \begin{cases} \frac{1}{2} c_n & n \geq 0 \\ \frac{1}{2}\overline{c_{-n}} & n < 0 \end{cases}. \tag{9.1}$$

Hence by Proposition 4.15,

$$u(r,\theta) = \sum_{n\in\mathbb{Z}} a_n r^{|n|} \mathrm{e}^{\mathrm{i}n\theta}$$

is an arbitrary harmonic function, for some choice of $\{ a_n \}_{n\in\mathbb{Z}}$. Now, we make sure we match the boundary is obtained as follows:

$$u(1,\theta) = \sum_{n\in\mathbb{Z}} a_n \mathrm{e}^{\mathrm{i}n\theta}$$

is a Fourier series, which we constrain to equal the function f. So now it is clear (by the Fourier convergence theorem above, see Theorem 8.1) that if f has a convergent Fourier series, there is a solution u, with

$$a_n = \frac{1}{2\pi}\int_{\theta=0}^{2\pi} f(\theta)\,\mathrm{e}^{-\mathrm{i}n\theta}\mathrm{d}\theta\,.$$

$\square$

Actually more explicitly we may write

$$\begin{aligned}
u(r,\theta) &= \sum_{n\in\mathbb{Z}} \frac{1}{2\pi}\int_{\tilde{\theta}=0}^{2\pi} f\left(\tilde{\theta}\right)\mathrm{e}^{-\mathrm{i}n\tilde{\theta}}\mathrm{d}\tilde{\theta} r^{|n|}\mathrm{e}^{\mathrm{i}n\theta} \\
&= \frac{1}{2\pi}\int_{\tilde{\theta}=0}^{2\pi}\mathrm{d}\tilde{\theta} f\left(\tilde{\theta}\right)\underbrace{\sum_{n\in\mathbb{Z}}\mathrm{e}^{-\mathrm{i}n\left(\tilde{\theta}-\theta\right)}r^{|n|}}_{=:2\pi P\left(r,\tilde{\theta}-\theta\right)} \\
&= \int_{\tilde{\theta}=0}^{2\pi}\mathrm{d}\tilde{\theta} f\left(\tilde{\theta}\right) P\left(r,\tilde{\theta}-\theta\right)
\end{aligned}$$

and $P\left(r,\tilde{\theta}-\theta\right)$ is the famous "*Poisson kernel*". For this integral to make sense the continuity of f is required.

Definition 9.15 (Poisson kernel)**.** The Poisson kernel is the function

$$P(r,\theta) \quad := \quad \frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\mathrm{e}^{\mathrm{i}n\theta}r^{|n|} = \frac{1}{2\pi}\frac{1-r^2}{\left|1-r\mathrm{e}^{\mathrm{i}\theta}\right|^2}\,.$$

Let us verify that last equality:

$$\begin{aligned}
\sum_{n\in\mathbb{Z}}\mathrm{e}^{\mathrm{i}n\theta}r^{|n|} &= \sum_{n=0}^{\infty}\mathrm{e}^{\mathrm{i}n\theta}r^n + \sum_{n=1}^{\infty}\mathrm{e}^{-\mathrm{i}n\theta}r^n \\
&= \frac{1}{1-r\mathrm{e}^{\mathrm{i}\theta}} + \frac{r\mathrm{e}^{-\mathrm{i}\theta}}{1-r\mathrm{e}^{-\mathrm{i}\theta}} \\
&= \frac{1-r\mathrm{e}^{-\mathrm{i}\theta}+r\mathrm{e}^{-\mathrm{i}\theta}\left(1-r\mathrm{e}^{\mathrm{i}\theta}\right)}{\left|1-r\mathrm{e}^{\mathrm{i}\theta}\right|^2} \\
&= \frac{1-r^2}{\left|1-r\mathrm{e}^{\mathrm{i}\theta}\right|^2}\,.
\end{aligned}$$

In conclusion: we see that

$$P(z) = \frac{1-|z|^2}{|1-z|^2} = 1+2\,\mathbb{R}\mathrm{e}\left\{\sum_{n=1}^{\infty}z^n\right\} = \mathbb{R}\mathrm{e}\left\{\frac{1+z}{1-z}\right\}$$

plays a role in the following process: if we are given the values of a (putative) harmonic function on the boundary of the disc via $f:\mathbb{S}^1\to\mathbb{R}$, we may use P to generate *the unique harmonic extension of that function to the entire disc* via:

$$u(r,\theta) \quad = \quad \int_{\tilde{\theta}=0}^{2\pi}\mathrm{d}\tilde{\theta} f\left(\tilde{\theta}\right) P\left(r,\tilde{\theta}-\theta\right)\,.$$

9.4 Transferring harmonic solutions via conformal maps

Lemma 9.16. *Let $u:V\to\mathbb{R}$ be harmonic for some $V\in\mathrm{Open}(\mathbb{C})$ and $f:U\to V$ holomorphic. Then $u\circ f:U\to\mathbb{R}$ is harmonic.*

Proof. Using Section 4.4, there is some $\psi:V\to\mathbb{C}$ analytic such that $u=\mathbb{R}\mathrm{e}\{\psi\}$. Then, $u\circ f=\mathbb{R}\mathrm{e}\{\psi\circ f\}$. But $\psi\circ f$ is analytic, so $u\circ f$ is the real part of an analytic, and hence harmonic. $\square$

With this tool at hand, we may readily write down the solution of the Dirichlet problem on $\mathbb{H}$ via our known solution on $B_1(0)$.

Corollary 9.17. *More generally, once we have a conformal equivalence of U and V which restricted to a continuous bijection $\partial U \to \partial V$, solving the Dirichlet problem on V implies a solution of the Dirichlet problem on U.*

The following example is somewhat construed since it is certainly possible to solve the Dirichlet problem directly on the upper half plane explicitly. It would be more illuminating with domains which are not amenable to a direct solution, but still, it is good to use this method where we know the solution in advance. Here in this case the direct solution may be obtained using the Fourier transform.

Example 9.18. Let us try to solve the Dirichlet problem

$$\begin{aligned} -\Delta u &= 0 \\ u|_{\partial\mathbb{H}} &= g \end{aligned}$$

for some $g : \partial\mathbb{H} \to \mathbb{R}$ (the boundary of the half plane is the real axis). Using Example 9.4 we have a conformal equivalence

$$\begin{aligned} c : \mathbb{H} &\to B_1(0) \\ z &\mapsto \frac{\mathrm{i} - z}{\mathrm{i} + z} \end{aligned}$$

with

$$\begin{aligned} c^{-1} : B_1(0) &\to \mathbb{H} \\ w &\mapsto \mathrm{i}\frac{1-w}{1+w}\,. \end{aligned}$$

Note that it restricts to the boundary as follows:

$$\begin{aligned} d : \mathbb{R} &\to \mathbb{S}^1 \\ x &\mapsto \frac{\mathrm{i} - x}{\mathrm{i} + x} \end{aligned}$$

(this is called the Cayley transform) and its inverse is

$$\begin{aligned} d^{-1} : \mathbb{S}^1 &\to \mathbb{R} \\ \mathrm{e}^{\mathrm{i}\theta} &\mapsto \mathrm{i}\frac{1-\mathrm{e}^{\mathrm{i}\theta}}{1+\mathrm{e}^{\mathrm{i}\theta}} \\ &= \tan\left(\frac{\theta}{2}\right) \end{aligned}$$

Hence $g \circ d^{-1} : \partial B_1(0) \to \mathbb{R}$ is a boundary value for the following Dirichlet problem on the unit disc:

$$\begin{aligned} -\Delta v &= 0 \\ v|_{\partial B_1(0)} &= g \circ d^{-1} =: f\,. \end{aligned}$$

Now, we have the explicit harmonic solution v on the disc from above given by

$$\begin{aligned} v : B_1(0) &\to \mathbb{R} \\ (r, \theta) &\mapsto \int_{\tilde{\theta}=-\pi}^{\pi} \mathrm{d}\tilde{\theta} f\left(\tilde{\theta}\right) P\left(r, \tilde{\theta} - \theta\right) \end{aligned}$$

where

$$P(z) = \frac{1}{2\pi} \mathbb{Re}\left\{\frac{1+z}{1-z}\right\}\,.$$

Explicitly,

$$v(z) = \int_{\tilde{\theta}=-\pi}^{\pi} \mathrm{d}\tilde{\theta} f\left(\tilde{\theta}\right) \frac{1}{2\pi} \mathbb{Re}\left\{ \frac{1+z\mathrm{e}^{-\mathrm{i}\tilde{\theta}}}{1-z\mathrm{e}^{-\mathrm{i}\tilde{\theta}}} \right\} .$$

Mapping this back to $\mathbb{H}$ we get

$$\begin{aligned}
u(z) &\equiv (v \circ c)(z) \\
&= \int_{\tilde{\theta}=-\pi}^{\pi} \mathrm{d}\tilde{\theta} f\left(\tilde{\theta}\right) \frac{1}{2\pi} \mathbb{Re}\left\{ \frac{1+c(z)\mathrm{e}^{-\mathrm{i}\tilde{\theta}}}{1-c(z)\mathrm{e}^{-\mathrm{i}\tilde{\theta}}} \right\} \\
&= \int_{\tilde{\theta}=-\pi}^{\pi} \mathrm{d}\tilde{\theta} g\left(\tan\left(\frac{\tilde{\theta}}{2}\right)\right) \frac{1}{2\pi} \mathbb{Re}\left\{ \frac{1+c(z)\mathrm{e}^{-\mathrm{i}\tilde{\theta}}}{1-c(z)\mathrm{e}^{-\mathrm{i}\tilde{\theta}}} \right\} \\
&\overset{t:=\tan\left(\frac{\theta}{2}\right)}{=} \int_{t=-\infty}^{\infty} \mathrm{d}t g(t) \frac{2}{1+t^2} \frac{1}{2\pi} \mathbb{Re}\left\{ \frac{1+c(z)\mathrm{e}^{-\mathrm{i}2\arctan(t)}}{1-c(z)\mathrm{e}^{-\mathrm{i}2\arctan(t)}} \right\}
\end{aligned}$$

where in the last step we made a change of variable $t := \tan\left(\frac{\theta}{2}\right)$ so that $\mathrm{d}t = \frac{1}{2\cos\left(\frac{\theta}{2}\right)^2}\mathrm{d}\theta$. We have

$$\begin{aligned}
\frac{1+c(z)\mathrm{e}^{-\mathrm{i}2\arctan(t)}}{1-c(z)\mathrm{e}^{-\mathrm{i}2\arctan(t)}} &= \frac{1+\frac{\mathrm{i}-z}{\mathrm{i}+z}\mathrm{e}^{-\mathrm{i}2\arctan(t)}}{1-\frac{\mathrm{i}-z}{\mathrm{i}+z}\mathrm{e}^{-\mathrm{i}2\arctan(t)}} \\
&= \mathrm{i}\frac{1+tz}{-t+z} \\
&= \mathrm{i}\frac{1+tx+\mathrm{i}ty}{-t+x+\mathrm{i}y} \\
&= \mathrm{i}\frac{1+tx+\mathrm{i}ty}{-t+x+\mathrm{i}y}\frac{-t+x-\mathrm{i}y}{-t+x-\mathrm{i}y} \\
&= \mathrm{i}\frac{1+tx+\mathrm{i}ty}{(x-t)^2+y^2} \\
&= \frac{\left(1+t^2\right)y+\mathrm{i}(\star)}{(x-t)^2+y^2}
\end{aligned}$$

where $\star$ is some unimportant real expression. Thus,

$$u(z) = \int_{t=-\infty}^{\infty} \mathrm{d}t g(t) \frac{1}{\pi} \frac{y}{(x-t)^2+y^2} .$$

We identify this last factor as *the Poisson kernel on* $\mathbb{H}$:

$$P_{\mathbb{H}}(x,y) := \frac{1}{\pi}\frac{y}{x^2+y^2} . \tag{9.2}$$

Using the Krammers-Kronig relation (7.11) we have that

$$\lim_{y\to 0^+} \int_{t=-\infty}^{\infty} \mathrm{d}t g(t) P_{\mathbb{H}}(x-t,y) = g(x)$$

so that the harmonic u we've constructed really does take the prescribed boundary values g.

9.5 The Riemann Mapping Theorem

We will not prove the following theorem but cannot avoid pointing out that it exists. For a full discussion see [SS03]. More concrete formulas may be obtained for polygons using *the Schwarz-Christoffel integral*, which we do not pursue further here.

Theorem 9.19 (Riemann mapping)**.** *If $\Omega \subsetneq \mathbb{C}$ is non-empty and simply-connected (as in Definition 4.21) then for any $z_0 \in \Omega$ there exists a unique conformal equivalence $F : \Omega \to B_1(0)$ such that*

$$\begin{aligned} F(z_0) &= 0 \\ F'(z_0) &> 0 . \end{aligned}$$

9.6 [extra] Complex analysis and conformal maps in fluid mechanics

To learn more about the application of complex analysis in fluid mechanics, we refer the interested readers to [CM00, Chapter 2.1]. Here we merely sketch what notions that we've studied would be relevant.

The motion of a two dimensional fluid is described by a *velocity field*, i.e., a map

$$V : \mathbb{R}^2 \to \mathbb{R}^2$$

which specifies the velocity of an infinitesimal element of fluid at every point in space. This vector field V is called *a flow*. In principle this map should depend on time as well (i.e. for every time instant there is a flow) in order to describe the dynamics.

Definition 9.20 (Incompressibility)**.** A flow is called incompressible iff

$$\operatorname{div}(V) \equiv \partial_1 V_1 + \partial_2 V_2 = 0 .$$

Definition 9.21 (Irrotationality)**.** A flow is called *irrotational* iff

$$\operatorname{curl}(V) \equiv \partial_1 V_2 - \partial_2 V_1 = 0 .$$

Back in Lemma 4.23 if $\operatorname{curl}(V) = 0$ and we define V on a simply-connected domain $\Omega \subseteq \mathbb{R}^2$ then we can find a scalar field $G : \Omega \to \mathbb{R}$ such that

$$V = \operatorname{grad}(G) .$$

Hence irrotational flows have scalar gradients. Moreover, since the divergence of the gradient is the Laplacian, if V is furthermore incompressible, we find that $-\Delta G = 0$, i.e., that G is harmonic. Now, harmonic functions $\Omega \to \mathbb{R}$ are known to be the real part of analytic functions. Let us call that analytic function f so that $f_R = G$. We then have (by Section 4.4

$$V = \overline{f'} .$$

The analytic function $f : \mathbb{C} \to \mathbb{C}$ is called *the complex velocity potential*. Explicitly, we have

$$V = \begin{bmatrix} \partial_x f_R \\ \partial_y f_R \end{bmatrix} .$$

Conformal maps may help us find the flow in complicated geometric situations by mapping them to simpler scenarios. Some examples follow [TODO: continue...].

10 The Laplace method and the method of steepest descent

In this section we describe an old, powerful and trusted approximation method, which is necessary for problems that cannot be exactly solved–most problems. Initially, it is merely a chapter in real analysis–this is the Laplace approximation. However, through a straightforward application of the ideas around the Cauchy integral formula, we obtain an extremely powerful complex extension, the steepest descent method. Both of these approximations are frequently used in physics and in mathematics.

This chapter belongs inside a vast topic in mathematics termed "asymptotic analysis" or "asymptotic expansions". We do not really enter into that field or elaborate on it. Instead, we just contend ourselves with the following remark:

Let $f : \mathbb{R} \to \mathbb{R}$ be some complicated function (usually defined via an integral) and we are interested to understand its behavior at infinity, anticipating that it actually decays to zero at infinity. So calculating something like

$$\lim_{x \to \infty} f(x)$$

is useless. Instead, we are more interested in *how quickly* f decays to zero at infinity. We say that we can identify the dominant behavior if we can find another, explicit $g:\mathbb{R}\to\mathbb{R}$ such that

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = 1$$

and in that case we would intuitively understand that

$$f(x)\approx g(x)+\text{stuff that decays faster to zero}\qquad (x\to\infty)\,.$$

10.1 The Laplace approximation

Let $g:\mathbb{R}^n\to\mathbb{C}$ be and $f:\mathbb{R}^n\to\mathbb{R}$ be given. We define the integral

$$I(\lambda):=\int_{x\in\mathbb{R}}\mathrm{e}^{-\lambda f(x)}g(x)\,\mathrm{d}x\,. \tag{10.1}$$

Our goal is to understand the behavior of $I(\lambda)$ as $\lambda\to\infty$.

If f were a second degree polynomial of the form

$$f(x) = \frac{1}{2}\omega^2(x-x_0)^2+c$$

with $\omega,c\in\mathbb{R}$ and g were a polynomial, say, even a constant $g(x)=g_0\in\mathbb{C}$, we'd be able to calculate the integral exactly, since it would then be Gaussian:

$$\begin{aligned} I(\lambda) &= g_0\int_{x\in\mathbb{R}}\mathrm{e}^{-\lambda\left[\frac{1}{2}\omega^2(x-x_0)^2+c\right]}\mathrm{d}x \\ &= g_0\mathrm{e}^{-\lambda c}\int_{x\in\mathbb{R}}\mathrm{e}^{-\frac{1}{2}\lambda\omega^2(x-x_0)^2}\mathrm{d}x \\ &= g_0\mathrm{e}^{-\lambda c}\sqrt{\frac{2\pi}{\lambda\omega^2}}\,. \end{aligned}$$

We see that as $\lambda\to\infty$, this expression is governed by the value of f at its minimum, and behaves as

$$I(\lambda) \sim \text{const}\times\frac{1}{\sqrt{\lambda}}\mathrm{e}^{-\lambda f(x_0)}\,.$$

Laplace's idea is the following: If f has a unique minimum then near it, it also vanishes quadratically and so, at least as $\lambda\to\infty$, the behavior of the integral should still be Gaussian. All other contributions decay to zero *faster* than the Gaussian contribution. This idea is formalized in the following

Theorem 10.1 (Laplace approximation)**.** *For some fixed $n\in\mathbb{N}$, let $f:\mathbb{R}^n\to\mathbb{R}$ and $g:\mathbb{R}^n\to\mathbb{C}$ be given. Assume that f has a continuous second derivative (Hessian matrix) $\mathbb{H}f:\mathbb{R}^n\to\mathrm{Mat}_{n\times n}(\mathbb{R})$ at x_0 and that g is continuous and non-vanishing at x_0; assume further that there is some point $x_0\in\mathbb{R}^n$ where f has a* unique *minimum; in particular,*

$$\begin{aligned} (\nabla f)(x_0) &= 0 \\ (\mathbb{H}f)(x_0) &> 0\,. \end{aligned}$$

Finally, assume that there exists some $\lambda_\star>0$ for which

$$\int_{x\in\mathbb{R}^n}\mathrm{e}^{-\lambda_\star f(x)}g(x)\,\mathrm{d}x<\infty\,. \tag{10.2}$$

Then

$$\lim_{\lambda\to\infty}\frac{\int_{x\in\mathbb{R}^n}\mathrm{e}^{-\lambda f(x)}g(x)\,\mathrm{d}x}{\lambda^{-\frac{n}{2}}\mathrm{e}^{-\lambda f(x_0)}} = \frac{g(x_0)}{\sqrt{\det\left(\frac{1}{2\pi}(\mathbb{H}f)(x_0)\right)}}\,. \tag{10.3}$$

Versions of this theorem also exist with:

1. Integration over a compact set instead of $\mathbb{R}^n$; this is easier and one can then drop (10.2). See discussion below in Section 10.1.1.
2. Allow to assume the minimum of f is on the boundary of the compact set instead of in the interior. Different asymptotics may arise then, see Theorem 10.3.
3. Allow a finite or infinite number of *isolated* minima for f. One merely gets the sum of contributions from each minima, assuming all minima are equal.
4. We may instead work with the negative function $-f$ which should be assumed to have a maximum.
5. The same formula holds if the limit $\lambda \to \infty$ is taken so that λ is allowed to be complex, so long as $\mathbb{Re}\{\lambda\} > 0$; this goes under the name of stationary phase approximation. More generally, allowing f to take complex values is the steepest descent method which is our ultimate goal in Section 10.3.

We refer the reader to the general books about asymptotic analysis for these generalizations, most of which we don't don't take up here: the modern [Mil06] or the classic [Erd56]. One should also mention, among others, [BO99, Won14, dB03].

Proof. Since the equation (10.3) is linear in g, we may write

$$g \quad = \quad g_R + \mathrm{i} g_I$$

and

$$g_R \quad = \quad \underbrace{\frac{1}{2}\left(g_R + |g_R|\right)}_{g_1} - \underbrace{\frac{1}{2}\left(|g_R| - g_R\right)}_{g_2}$$

and similarly for g_I. Hence, any arbitrary complex function is written as

$$g \quad = \quad g_1 - g_2 + \mathrm{i} g_3 - \mathrm{i} g_4$$

where $g_1, g_2, g_3, g_4 : \mathbb{R}^n \to [0, \infty)$.

Thus, assume without loss of generality that actually $g \geq 0$.

Next, let us shift the integration $x \mapsto x + x_0$

$$\begin{aligned} I(\lambda) \quad &:= \quad \int_{x\in\mathbb{R}^n} \mathrm{e}^{-\lambda f(x)} g(x)\,\mathrm{d}x \\ &= \quad \mathrm{e}^{-\lambda f(x_0)} \int_{x\in\mathbb{R}^n} \mathrm{e}^{-\lambda[f(x_0+x)-f(x_0)]} g(x_0+x)\,\mathrm{d}x \end{aligned}$$

define now

$$\tilde{f}(x) \quad := \quad f(x_0 + x) - f(x_0)$$

and

$$\tilde{g}(x) \quad := \quad \frac{g(x_0+x)}{g(x_0)}$$

(the case $g(x_0) = 0$ is different and we postpone it to the end), so that our goal is to show that

$$\tilde{I}(\lambda) \quad := \quad \frac{1}{\lambda^{-\frac{n}{2}}} \int_{x\in\mathbb{R}^n} \mathrm{e}^{-\lambda \tilde{f}(x)} \tilde{g}(x)\,\mathrm{d}x$$

behaves like

$$\tilde{I}(\lambda) \quad \to \quad \frac{1}{\sqrt{\det\left(\frac{1}{2\pi}(\mathbb{H}f)(x_0)\right)}} = \frac{\int_{x\in\mathbb{R}^n} \mathrm{e}^{-\frac{\lambda}{2}\langle x, (\mathbb{H}f)(x_0)x\rangle}\mathrm{d}x}{\lambda^{-\frac{n}{2}}}$$

where the last equation is standard Gaussian integration in $\mathbb{R}^n$.

We derive upper and lower bounds on $\tilde{I}(\lambda)$, both of which shall converge to

$$\int_{x\in\mathbb{R}^n} \mathrm{e}^{-\frac{1}{2}\langle x,(\mathbb{H}f)(x_0)x\rangle}\mathrm{d}x\,.$$

Let us begin with lower bounds.

By Taylor's theorem, and the fact that by assumption $(\nabla f)(x_0) = 0$,

$$\tilde{f}(x) \quad = \quad \frac{1}{2}\langle x, (\mathbb{H}f)(x_0)\,x\rangle + \mathcal{O}\left(\|x\|^3\right)\,.$$

Actually there is a better way to write it, using the Lagrange remainder term, with which we get an equality, but the price to pay is that we must replace $(\mathbb{H}f)(x_0)$ by $(\mathbb{H}f)(x_0+tx)$ for some $t\in[0,1]$. We then get

$$\tilde{f}(x) \quad = \quad \frac{1}{2}\langle x, (\mathbb{H}f)(x_0+tx)\,x\rangle \qquad (x\in\mathbb{R}^n)\,.$$

Since we assume that $x\mapsto(\mathbb{H}f)$ is continuous, for any $\varepsilon>0$ there exists some $\delta_\varepsilon>0$ such that if $\|x\|<\delta_\varepsilon$ then

$$\|(\mathbb{H}f)(x_0)-(\mathbb{H}f)(x_0+x)\| < \frac{1}{2}\varepsilon\,.$$

Hence for such x, since

$$\begin{aligned}
|\langle x,(\mathbb{H}f)(x_0+x)\,x\rangle - \langle x,(\mathbb{H}f)(x_0)\,x\rangle| &= |\langle x,((\mathbb{H}f)(x_0+x)-(\mathbb{H}f)(x_0))\,x\rangle| \\
&\overset{\text{Cauchy-Schwarz}}{\leq} \|x\|\|((\mathbb{H}f)(x_0+x)-(\mathbb{H}f)(x_0))\,x\| \\
&\overset{\text{submul. of norm}}{\leq} \|x\|\|(\mathbb{H}f)(x_0+x)-(\mathbb{H}f)(x_0)\|\|x\| \\
&\leq \frac{1}{2}\varepsilon\|x\|^2 \\
&\equiv \frac{1}{2}\langle x,\varepsilon\mathbb{1}x\rangle
\end{aligned}$$

we find for such x,

$$\begin{aligned}
\tilde{f}(x) &= \langle x,(\mathbb{H}f)(x_0+tx)\,x\rangle \\
&\geq \frac{1}{2}\langle x,((\mathbb{H}f)(x_0)-\varepsilon\mathbb{1})\,x\rangle\,.
\end{aligned}$$

Furthermore, by continuity of g at x_0, we can pick that same $\delta_\varepsilon>0$ to also imply that if $\|x\|<\delta_\varepsilon$ then

$$\tilde{g}(x) \quad \geq \quad (1-\varepsilon)\,.$$

As a result,

$$\begin{aligned}
\tilde{I}(\lambda) &\equiv \lambda^{\frac{n}{2}}\int_{x\in\mathbb{R}^n}\mathrm{e}^{-\lambda\tilde{f}(x)}\tilde{g}(x)\,\mathrm{d}x \\
&\geq \lambda^{\frac{n}{2}}\int_{x\in B_{\delta_\varepsilon}(0)}\mathrm{e}^{-\lambda\tilde{f}(x)}\tilde{g}(x)\,\mathrm{d}x && \text{(integral of positive integrand)} \\
&\geq \lambda^{\frac{n}{2}}\int_{x\in B_{\delta_\varepsilon}(0)}\mathrm{e}^{-\lambda\frac{1}{2}\langle x,((\mathbb{H}f)(x_0)-\varepsilon\mathbb{1})x\rangle}(1-\varepsilon)\,\mathrm{d}x \\
&\overset{\sqrt{\lambda}x\mapsto x}{=} (1-\varepsilon)\int_{x\in B_{\sqrt{\lambda}\delta_\varepsilon}(0)}\mathrm{e}^{-\frac{1}{2}\langle x,((\mathbb{H}f)(x_0)-\varepsilon\mathbb{1})x\rangle}\mathrm{d}x \\
&\overset{\lambda\to\infty}{\longrightarrow} (1-\varepsilon)\int_{x\in\mathbb{R}^n}\mathrm{e}^{-\frac{1}{2}\langle x,((\mathbb{H}f)(x_0)-\varepsilon\mathbb{1})x\rangle}\mathrm{d}x \\
&= (1-\varepsilon)\sqrt{\frac{(2\pi)^n}{\det((\mathbb{H}f)(x_0)-\varepsilon\mathbb{1})}}\,.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we find

$$\lim_{\lambda\to\infty} \tilde{I}(\lambda) \quad \geq \quad \sqrt{\frac{(2\pi)^n}{\det\left(\left(\mathbb{H}f\right)(x_0)\right)}}\,.$$

Let us tend now to the upper bound. Since f has a unique maximum at x_0, there must be some strictly positive function

$$\eta(b) \quad := \quad \inf_{x\in B_b(x_0)^c} f(x) - f(x_0) \qquad (b>0)\,.$$

Then splitting the domain of integration, we get:

$$\begin{aligned}\tilde{I}(\lambda) \quad &\equiv \quad \lambda^{\frac{n}{2}}\int_{x\in\mathbb{R}^n} \mathrm{e}^{-\lambda\tilde{f}(x)}\tilde{g}(x)\,\mathrm{d}x \\ &= \quad \lambda^{\frac{n}{2}}\int_{x\in B_{\delta_\varepsilon}(0)} \mathrm{e}^{-\lambda\tilde{f}(x)}\tilde{g}(x)\,\mathrm{d}x + \lambda^{\frac{n}{2}}\int_{x\in B_{\delta_\varepsilon}(0)^c} \mathrm{e}^{-\lambda\tilde{f}(x)}\tilde{g}(x)\,\mathrm{d}x\,.\end{aligned}$$

For the second term, we have on $x \in B_{\delta_\varepsilon}(0)^c$, $\tilde{f}(x) \geq \eta(\delta_\varepsilon)$. Thus,

$$\begin{aligned}\int_{x\in B_{\delta_\varepsilon}(0)^c} \mathrm{e}^{-\lambda\tilde{f}(x)}\tilde{g}(x)\,\mathrm{d}x \quad &= \quad \int_{x\in B_{\delta_\varepsilon}(0)^c} \mathrm{e}^{-\lambda_\star\tilde{f}(x)}\mathrm{e}^{-(\lambda-\lambda_\star)\tilde{f}(x)}\tilde{g}(x)\,\mathrm{d}x \\ &\leq \quad \mathrm{e}^{-(\lambda-\lambda_\star)\eta(\delta_\varepsilon)}\int_{x\in B_{\delta_\varepsilon}(0)^c} \mathrm{e}^{-\lambda_\star\tilde{f}(x)}\tilde{g}(x)\,\mathrm{d}x \\ &\leq \quad \mathrm{e}^{-(\lambda-\lambda_\star)\eta(\delta_\varepsilon)}\mathrm{e}^{\lambda_\star f(x_0)}\frac{1}{g(x_0)}\underbrace{\int_{x\in\mathbb{R}^n} \mathrm{e}^{-\lambda_\star f(x)}g(x)\,\mathrm{d}x}_{\text{finite by assumption}}\,.\end{aligned}$$

Thus as $\lambda\to\infty$ (recall $\lambda_\star$ is fixed and is independent of λ!) we find this expression is in fact converging exponentially fast to zero. This exponential decay will overcome the exploding $\lambda^{\frac{n}{2}}$ term and we get that the whole second term converges to zero.

For the first term, we have by the same considerations as above about continuity of $\mathbb{H}f$ and g, that within $x \in B_{\delta_\varepsilon}(0)$,

$$\begin{aligned}\tilde{f}(x) \quad &\leq \quad \langle x, \left(\left(\mathbb{H}f\right)(x_0) + \varepsilon\mathbb{1}\right)x\rangle \\ \tilde{g}(x) \quad &\leq \quad (1+\varepsilon)\end{aligned}$$

and so that first term is

$$\begin{aligned}\lambda^{\frac{n}{2}}\int_{x\in B_{\delta_\varepsilon}(0)} \mathrm{e}^{-\lambda\tilde{f}(x)}\tilde{g}(x)\,\mathrm{d}x \qquad &\leq \qquad (1+\varepsilon)\,\lambda^{\frac{n}{2}}\int_{x\in B_{\delta_\varepsilon}(0)} \mathrm{e}^{-\lambda\langle x,\left(\left(\mathbb{H}f\right)(x_0)+\varepsilon\mathbb{1}\right)x\rangle}\mathrm{d}x \\ &\overset{\sqrt{\lambda}x\mapsto x}{=} \qquad (1+\varepsilon)\int_{x\in B_{\sqrt{\lambda}\delta_\varepsilon}(0)} \mathrm{e}^{-\langle x,\left(\left(\mathbb{H}f\right)(x_0)+\varepsilon\mathbb{1}\right)x\rangle}\mathrm{d}x \\ &\overset{\lambda\to\infty}{\longrightarrow} \qquad (1+\varepsilon)\sqrt{\frac{(2\pi)^n}{\det\left(\left(\mathbb{H}f\right)(x_0)+\varepsilon\mathbb{1}\right)}}\,.\end{aligned}$$

Again, since $\varepsilon > 0$ was arbitrary, we learn that

$$\lim_{\lambda\to\infty} \tilde{I}(\lambda) \quad \leq \quad \sqrt{\frac{(2\pi)^n}{\det\left(\left(\mathbb{H}f\right)(x_0)\right)}}\,.$$

$\square$

The classic example for the application of this theorem is the Stirling approximation for the Gamma (and hence the factorial) function $\Gamma(n+1) \equiv n!$:

Example 10.2 (Stirling)**.** Using the integral representation (see e.g. HW5Q18)

$$\Gamma(\lambda) \quad = \quad \int_{t=0}^{\infty} t^{\lambda-1}\mathrm{e}^{-t}\mathrm{d}t$$

we may derive large λ asymptotics of the Gamma function:

$$\Gamma(\lambda) \quad \sim \quad \sqrt{\frac{2\pi}{\lambda}}\exp(\lambda\log(\lambda)-\lambda)\,.$$

Note that one may derive this asymptotic formula also for λ complex as long as $|\arg(\lambda)|$ is uniformly bounded away from π. See [SS03, App. A Thm. 2.3].

Proof. We write

$$t^{\lambda-1} \quad = \quad \exp((\lambda-1)\log(t))$$

and if we were to naively apply Theorem 10.1 then we'd be stuck since $f(t)=\log(t)$ has no minimum on $(0,\infty)$. Instead, let us make a change of variable in the integral, $s := \frac{1}{\lambda}t$ (this is a common procedure in such approximations) so that

$$\begin{aligned}\Gamma(\lambda) \quad &= \quad \int_{s=0}^{\infty}(\lambda s)^{\lambda-1}\mathrm{e}^{-\lambda s}\lambda\mathrm{d}s\\ &= \quad \int_{s=0}^{\infty}\mathrm{e}^{(\lambda-1)\log(\lambda s)-\lambda s}\lambda\mathrm{d}s\\ &= \quad \mathrm{e}^{\lambda\log(\lambda)}\int_{s=0}^{\infty}\mathrm{e}^{-\lambda(s-\log(s))}\frac{1}{s}\mathrm{d}s\,.\end{aligned}$$

We thus apply Theorem 10.1 on this last integral with $f(s) := s-\log(s)$ and $g(s)=\frac{1}{s}$. We have

$$f'(s) \quad = \quad 1-\frac{1}{s}$$

which has a zero at $s=1$. The second derivative is

$$f''(s) \quad = \quad \frac{1}{s^2}$$

so the so that this is indeed a minimum. Moreover, $\int_{s=0}^{\infty}\mathrm{e}^{-\lambda(s-\log(s))}\frac{1}{s}\mathrm{d}s$ exists already at all $\lambda>0$. Indeed, for large s the convergence is clear and for small s, we have the integrand behaving as $s\mapsto s^{\lambda-1}$ which is integrable at the origin as long as $\lambda>0$. We may apply our theorem to get

$$\int_{s=0}^{\infty}\mathrm{e}^{-\lambda(s-\log(s))}\frac{1}{s}\mathrm{d}s \quad \sim \quad \sqrt{\frac{2\pi}{\lambda}}\exp(-\lambda)$$

and thus find the result. □

10.1.1 Extremum at endpoints

In the foregoing discussion, we took the integration domain as the entire real axis, or even the whole of $\mathbb{R}^n$. Let us specify again to one dimension, but further assume that we integrate on $[a,b]$ instead of $\mathbb{R}$:

$$I(\lambda) := \int_{x=a}^{b}\mathrm{e}^{-\lambda f(x)}g(x)\,\mathrm{d}x\,.$$

If f has an isolated extremum somewhere at $x_0\in(a,b)$, then the above theorem still applies (in fact its proof is easier since one doesn't need to control integrability).

No critical point at all If f has no extremum on (a,b) then the extreme value theorem states that if f is continuous it must attain its maximum and minimum on $[a,b]$, and hence on the boundaries. As such we have

Theorem 10.3. *If $f:[a,b]\to\mathbb{R}$ is continuously differentiable with $f'\neq 0$, $f(b)\neq 0$, f attains its maximum on a and minimum on b, and if $g:\mathbb{R}\to\mathbb{C}$ is continuous, then*

$$\lim_{\lambda\to\infty}\frac{\int_{x=a}^{b}\mathrm{e}^{-\lambda f(x)}g(x)\,\mathrm{d}x}{\frac{1}{\lambda}\mathrm{e}^{-\lambda f(b)}} \quad := \quad -\frac{g(b)}{f'(b)}\,.$$

If, conversely, $f(a)\neq 0$ and f attains its minimum on a and maximum on b, we have

$$\lim_{\lambda\to\infty}\frac{\int_{x=a}^{b}\mathrm{e}^{-\lambda f(x)}g(x)\,\mathrm{d}x}{\frac{1}{\lambda}\mathrm{e}^{-\lambda f(a)}} \quad := \quad \frac{g(a)}{f'(a)}\,.$$

Proof. Consider the identity which is adapted towards integration by parts,

$$\mathrm{e}^{-\lambda f(x)} = -\frac{1}{\lambda f'(x)}\partial\mathrm{e}^{-\lambda f(x)}\,, \tag{10.4}$$

which makes sense for we assume $f'\neq 0$. Plug it into $I(\lambda)$ to get

$$\begin{aligned}
I(\lambda) &= \int_{x=a}^{b}\left[-\frac{1}{\lambda f'(x)}\partial\mathrm{e}^{-\lambda f(x)}\right]g(x)\,\mathrm{d}x \\
&= \left[-\frac{1}{\lambda f'(x)}\mathrm{e}^{-\lambda f(x)}g(x)\right]\Big|_{x=a}^{b} + \int_{x=a}^{b}\mathrm{e}^{-\lambda f(x)}\partial\frac{1}{\lambda f'(x)}g(x)\,\mathrm{d}x \qquad \text{(Int. by parts)} \\
&= -\frac{1}{\lambda f'(b)}\mathrm{e}^{-\lambda f(b)}g(b) + \frac{1}{\lambda f'(a)}\mathrm{e}^{-\lambda f(a)}g(a) + \frac{1}{\lambda}\int_{x=a}^{b}\mathrm{e}^{-\lambda f(x)}\left[\frac{g'(x)}{f'(x)} - \frac{f''(x)\,g(x)}{f'(x)^2}\right]\mathrm{d}x
\end{aligned}$$

Assume now that f obtains its maximum on a and minimum on b. That means that $\mathrm{e}^{-\lambda f(b)}$ is the largest exponential involved, i.e.,

$$\lim_{\lambda\to\infty}\frac{\mathrm{e}^{-\lambda f(b)}}{\mathrm{e}^{-\lambda f(x)}} \quad = \quad 0 \qquad (x<b)\,.$$

We thus expect that as $\lambda\to\infty$, the dominant behavior for $I(\lambda)$ will come only from the first term:

$$I(\lambda) \quad \approx \quad -\frac{1}{\lambda f'(b)}\mathrm{e}^{-\lambda f(b)}g(b)\,.$$

The second term is easy, but to really justify this we need to show that the remaining integral,

$$\frac{1}{\lambda}\int_{x=a}^{b}\mathrm{e}^{-\lambda f(x)}\underbrace{\left[\frac{g'(x)}{f'(x)} - \frac{f''(x)\,g(x)}{f'(x)^2}\right]}_{=:h(x)}\mathrm{d}x$$

decays faster than the first term, indeed, like $\frac{1}{\lambda^2}\mathrm{e}^{-\lambda f(b)}$. To that end, let us plug in the identity (10.4) to obtain

$$\frac{1}{\lambda}\int_{x=a}^{b}\mathrm{e}^{-\lambda f(x)}h(x)\,\mathrm{d}x \quad = \quad -\frac{1}{\lambda^2 f'(b)}\mathrm{e}^{-\lambda f(b)}h(b) + \frac{1}{\lambda^2 f'(a)}\mathrm{e}^{-\lambda f(a)}h(a) + \frac{1}{\lambda^2}\int_{x=a}^{b}\mathrm{e}^{-\lambda f(x)}\left[\frac{h'(x)}{f'(x)} - \frac{f''(x)\,h(x)}{f'(x)^2}\right]\mathrm{d}x\,.$$

But now, the integral on the RHS may be bounded by $\mathrm{e}^{-\lambda f(b)}$ times a constant independent of λ:

$$\int_{x=a}^{b}\mathrm{e}^{-\lambda f(x)}\left[\frac{h'(x)}{f'(x)} - \frac{f''(x)\,h(x)}{f'(x)^2}\right]\mathrm{d}x \quad \leq \quad \mathrm{e}^{-\lambda f(b)}\int_{x=a}^{b}\left[\frac{h'(x)}{f'(x)} - \frac{f''(x)\,h(x)}{f'(x)^2}\right]\mathrm{d}x\,.$$

The conclusion is that this term is at most of order $\frac{1}{\lambda}\mathrm{e}^{-\lambda f(b)}$. Hence we learn that

$$I(\lambda) \quad = \quad -\frac{1}{\lambda f'(b)}\mathrm{e}^{-\lambda f(b)}g(b) + \frac{1}{\lambda f'(a)}\mathrm{e}^{-\lambda f(a)}g(a) + \mathcal{O}\left(\frac{1}{\lambda^2}\mathrm{e}^{-\lambda f(b)}\right).$$

Dividing by $\frac{1}{\lambda}\mathrm{e}^{-\lambda f(b)}$ and taking the limit $\lambda \to \infty$ yields the first result. The second result follows a similar argument. □

Example 10.4 (The complementary error function)**.** The complementary error function is defined via

$$\mathrm{erfc}(x) \quad := \quad \frac{2}{\sqrt{\pi}}\int_x^\infty \mathrm{e}^{-t^2}\mathrm{d}t\,.$$

We want to have the asymptotics as $x \to \infty$ of this expression, so we try to bring it to the form of one of the integrals we've studied above. To that end, we make a change of variables $s := \frac{1}{x}t$ to get

$$\mathrm{erfc}(x) \quad = \quad \frac{2}{\sqrt{\pi}}x\int_1^\infty \mathrm{e}^{-x^2s^2}\mathrm{d}s\,.$$

Now the Gaussian $s \mapsto \mathrm{e}^{-x^2s^2}$ has a maximum at the boundary of the integral, so applying Theorem 10.3 with $f(s) := s^2$, so $f'(s) = 2s$ we get

$$\int_1^\infty \mathrm{e}^{-x^2s^2}\mathrm{d}s \quad \approx \quad \frac{1}{x^2}\frac{\mathrm{e}^{-x^2f(1)}}{f'(1)} = \frac{1}{2x^2}\mathrm{e}^{-x^2}\,.$$

We find

$$\mathrm{erfc}(x) \quad \approx \quad \frac{2}{\sqrt{\pi}}x\frac{1}{2x^2}\mathrm{e}^{-x^2} = \frac{1}{\sqrt{\pi}}\frac{1}{x}\mathrm{e}^{-x^2}$$

and, again, we mean $\approx$ in the sense that

$$\lim_{x\to\infty}\frac{\mathrm{erfc}(x)}{\frac{1}{\sqrt{\pi}}\frac{1}{x}\mathrm{e}^{-x^2}} \quad = \quad 1\,.$$

Critical point on the boundary Now let us assume that we integrate on $[a,b]$ but $f' \neq 0$ on (a,b) and $f'(a) = 0$ or $f'(b) = 0$. In this case the analysis of Theorem 10.1 applies, except that one must use the identity

$$\int_{x=0}^\infty \mathrm{e}^{-\frac{1}{2}\lambda x^2}\mathrm{d}x \quad = \quad \frac{1}{2}\sqrt{\frac{2\pi}{\lambda}}\,.$$

The end result is, e.g.,

Theorem 10.5. *If $f : [a,b] \to \mathbb{R}$ is continuously differentiable such that f has a unique minimum at c with $f'(c) = 0$, $f''(c) > 0$, for $c = a$ or $c = b$ and $f'(x) \neq 0$ for $x \neq c$; assume further that g is continuous at c with $g(c) \neq 0$. Then*

$$\lim_{\lambda\to\infty}\frac{\int_{x=a}^b \mathrm{e}^{-\lambda f(x)}g(x)\,\mathrm{d}x}{\frac{1}{2}\sqrt{\frac{2\pi}{\lambda f''(c)}}\mathrm{e}^{-\lambda f(c)}} \quad = \quad g(c)\,.$$

10.2 The method of stationary phase

Now we want to deal with the situation where the asymptotic parameter λ may have imaginary part as well, i.e., we want to understand the asymptotics, for

$$\lambda \in \{\, z \in \mathbb{C} \mid \mathbb{Re}\{z\} \geq 0 \,\}$$

as $|\lambda| \to \infty$, of the integral in (10.1)

$$I(\lambda) \quad := \quad \int_{x\in\mathbb{R}^n} \mathrm{e}^{-\lambda f(x)}g(x)\,\mathrm{d}x\,.$$

Theorem 10.6. *The same asymptotic leading order limit as in Theorem 10.1 holds under the same assumptions on f, g if the limit is taken with $\lambda \to \infty$ in the complex plane, as long as $\lambda \in \{ z \in \mathbb{C} \mid \mathbb{Re}\{z\} \geq 0 \}$.*

We will not prove this statement here because it is, in a sense, an intermediate result. We rather proceed directly the case that f is allowed to take complex values, which is the steepest descent approximation.

10.3 The steepest descent or saddle point method

Let us now concentrate on the one-dimensional case, and consider now the case that f has both real and imaginary parts (unlike before, when it had only a real part). I.e., $f : \mathbb{R} \to \mathbb{C}$. This is an inherently different scenario since now as $\lambda \to \infty$, the integral

$$I(\lambda) \quad = \quad \int_{x \in \mathbb{R}} \mathrm{e}^{-\lambda f(x)} g(x) \, \mathrm{d}x$$

oscillates faster and faster as $\lambda \to \infty$.

To proceed, let us assume that our functions $f, g : \mathbb{R} \to \mathbb{C}$ extend to analytic (or meromorphic more generally) functions $\tilde{f}, \tilde{g} : \mathbb{C} \to \mathbb{C}$, and, moreover, that there exists a unique point $z_\star \in \mathbb{C}$ such that

$$\tilde{f}'(z_\star) \quad = \quad 0 \,. \tag{10.5}$$

Thinking about $\tilde{f}_R, \tilde{f}_I : \mathbb{R}^2 \to \mathbb{R}$ as two functions on the plane, this condition implies via (4.5) that

$$\left(\nabla \tilde{f}_R\right)(z_\star) \quad = \quad 0 \,. \tag{10.6}$$

Our goal is to deform the contour of integration, replacing integration over $[-R, R]$ with some other contour $\gamma : [-1, 1] \to \mathbb{C}$, with the requirement that

$$\left(\tilde{f}_I \circ \gamma\right)(t) = \tilde{f}_I(z_\star) \qquad (t \in [-1, 1]) \,. \tag{10.7}$$

If we manage to find such a contour, and deform

$$I_R(\lambda) \quad = \quad \int_{x=-R}^{R} \mathrm{e}^{-\lambda f(x)} g(x) \, \mathrm{d}x$$

to

$$I_\gamma(\lambda) \quad = \quad \int_{t=-1}^{1} \mathrm{e}^{-\lambda \tilde{f}(\gamma(t))} \tilde{g}(\gamma(t)) \gamma'(t) \, \mathrm{d}t$$

then we could apply the Laplace asymptotics. Indeed, then, let us define

$$F(t) \quad := \quad \tilde{f}_R(\gamma(t)) \qquad (t \in [-1, 1]) \,.$$

Calculating

$$F'(0) \quad = \quad \left\langle \left(\nabla \tilde{f}_R\right)(\gamma(0)), \gamma'(0) \right\rangle_{\mathbb{R}^2} = 0$$

where the last equality is via Section 10.3. If

$$F''(0) > 0 \tag{10.8}$$

furthermore, then we may apply Theorem 10.1 on $I_\gamma(\lambda)$:

$$\begin{aligned} I_\gamma(\lambda) \quad &= \quad \mathrm{e}^{-\mathrm{i}\lambda \tilde{f}_I(z_\star)} \int_{t=-1}^{1} \mathrm{e}^{-\lambda F(t)} \tilde{g}(\gamma(t)) \gamma'(t) \, \mathrm{d}t \\ &\approx \quad \mathrm{e}^{-\mathrm{i}\lambda \tilde{f}_I(z_\star)} \mathrm{e}^{-\lambda F(0)} \sqrt{\frac{2\pi}{\lambda F''(0)}} \tilde{g}(\gamma(0)) \gamma'(0) \,. \end{aligned}$$

We claim that

$$|F''(0)| \quad = \quad |f''(z_\star)| \, |\gamma'(0)|^2 \,.$$

Indeed, the chain rule in complex analysis translates to

$$\partial_t\left(\tilde{f}\circ\gamma\right) = \left(\tilde{f}'\circ\gamma\right)\gamma'$$

where on the RHS we consider the multiplication of two complex numbers. Hence

$$\partial_t^2\left(\tilde{f}\circ\gamma\right) = \left(\tilde{f}'\circ\gamma\right)\gamma''+\left(\tilde{f}''\circ\gamma\right)\gamma'^2\,.$$

Now, at $t=0$, $\gamma(0)=z_\star$ and hence $\tilde{f}'(z_\star)=0$. We find

$$\partial_t^2\left(\tilde{f}\circ\gamma\right) = \left(\tilde{f}''\circ\gamma\right)\gamma'\,.$$

On the other hand,

$$\partial_t^2\left(\tilde{f}\circ\gamma\right) = \partial_t^2\left(F+\mathrm{i}\tilde{f}_I\circ\gamma\right)=F''+\mathrm{i}\partial_t\underbrace{\partial_t\tilde{f}_I\circ\gamma}_{=0\text{ by construction}}\,.$$

Hence we find

$$F''(0) = \tilde{f}''(z_\star)\gamma'(0)^2\,.$$

and so the claim. Returning to the result of the Laplace asymptotics for $I_\gamma(\lambda)$ we get

$$I_\gamma(\lambda) \approx \sqrt{\frac{2\pi}{\lambda\left|\tilde{f}''(z_\star)\right|}}\mathrm{e}^{-\lambda\tilde{f}(z_\star)}\tilde{g}(z_\star)\frac{\gamma'(0)}{|\gamma'(0)|}\,.$$

We claim that $\frac{\gamma'(0)}{|\gamma'(0)|}$ is independent of the choice of γ. Indeed, let us, for concreteness, assume that the Taylor expansion of $\tilde{f}$ near $z_\star$ is of the form

$$\tilde{f}(z)=\tilde{f}(z_\star)+\frac{1}{2}\tilde{f}''(z_\star)(z-z_\star)^2+\mathcal{O}\left(|z-z_\star|^3\right) \tag{10.9}$$

and write

$$\frac{1}{2}\tilde{f}''(z_\star)=:a\mathrm{e}^{\mathrm{i}\alpha}$$

for some $a>0$, $\alpha\in[0,2\pi)$. In order to obtain a constant phase and $F''(0)<0$, γ must be of the form $\gamma(t)\approx z_\star+\mathrm{e}^{\mathrm{i}\beta}t$ for $|t|\ll 1$ where

$$\beta:=-\frac{\alpha}{2}+\frac{\pi}{2}\,. \tag{10.10}$$

We find

$$\begin{aligned}\frac{\gamma'(0)}{|\gamma'(0)|} &= \mathrm{e}^{\mathrm{i}\beta}\\ &= \mathrm{e}^{\mathrm{i}\left(-\frac{\alpha}{2}+\frac{\pi}{2}\right)}\\ &= \mathrm{e}^{\mathrm{i}\left(-\frac{\operatorname{Arg}\left(\tilde{f}''(z_\star)\right)}{2}+\frac{\pi}{2}\right)}\\ &= \sqrt{-\frac{\tilde{f}''(z_\star)}{\left|\tilde{f}''(z_\star)\right|}}\,.\end{aligned}$$

Bringing this back to our Laplace asymptotics we recognize we really should have

$$I_\gamma(\lambda) \approx \sqrt{\frac{2\pi}{\lambda\tilde{f}''(z_\star)}}\mathrm{e}^{-\lambda\tilde{f}(z_\star)}\tilde{g}(z_\star)\,.$$

What remains is to understand why γ which obeys: $\gamma(0)=z_\star$ as well as both (10.7) and (10.8) exists. The first constraint would be implied by

$$\partial_t\left(\tilde{f}_I\circ\gamma\right)=0\,.$$

This, however, may be turned into an ODE. Indeed, let $\nabla\tilde{f}_I =: \begin{bmatrix} A \\ B \end{bmatrix}$. Then this equation is

$$(A\circ\gamma)\,\gamma_1' + (B\circ\gamma)\,\gamma_2' = 0\,.$$

If we pick, e.g., $\gamma_2(t) = t$ as a parametrization then we find an ODE for γ_1, which, by Picard-Lindelöf, exists at least locally.

Obeying (10.8) is seen, from (10.9) as the particular phase requirement (10.10). Finally, we have to *assume* that at sufficiently large R, the vertical lines necessary to close $[-R,R]\cup\gamma$ into a closed loop would decay to zero.

We thus obtain

Theorem 10.7 (Steepest descent approximation)**.** *Let $f,g:\mathbb{R}\to\mathbb{C}$ be given such that*

1. *They both extend to analytic (or meromorphic more generally) functions $\tilde{f},\tilde{g}:\mathbb{C}\to\mathbb{C}$.*
2. *There exists a unique $z_\star\in\mathbb{C}$ such that $\tilde{f}'(z_\star)=0$.*
3. *The differential equation for the unknown contour $\gamma:[-1,1]\to\mathbb{C}$ given by*

$$\partial_t\left(\tilde{f}_I\circ\gamma\right) = 0$$

with boundary condition $\gamma(0) = z_\star$ has the property that along lines $L_\pm$ from $\gamma(\pm1)$ to $\pm R$, the integral $\int_{L_\pm}\mathrm{e}^{-\lambda\tilde{f}(z)}\tilde{g}(z)\,\mathrm{d}z$ decays to zero as $R\to\infty$, so the contour may be closed and Cauchy's theorem applied.

4. *The function $\tilde{f}_R\circ\gamma:[-1,1]\to\mathbb{R}$ obeys the conditions for Theorem 10.1.*

Then we have

$$\lim_{\lambda\to\infty}\frac{\int_{x\in\mathbb{R}}\mathrm{e}^{-\lambda f(x)}g(x)\,\mathrm{d}x}{\frac{1}{\sqrt{\lambda}}\mathrm{e}^{-\lambda\tilde{f}(z_\star)}} = \sqrt{\frac{2\pi}{\tilde{f}''(z_\star)}}\tilde{g}(z_\star)\,.$$

Example 10.8 (The oscillatory Gaussian)**.** Our first example is the oscillatory Gaussian:

$$I(\lambda) = \int_{x\in\mathbb{R}}\mathrm{e}^{-\mathrm{i}\lambda x^2}\mathrm{d}x\,.$$

Hence we have $g=1$ and $f(x) = \mathrm{i}x^2$. This is actually an integral that we have performed in Example 6.38 *exactly*. Indeed, with a change of variables $t:=\sqrt{\lambda}x$ we find

$$\begin{aligned}
I(\lambda) &= \int_{x\in\mathbb{R}}\mathrm{e}^{-\mathrm{i}\lambda x^2}\mathrm{d}x \\
&= \frac{1}{\sqrt{\lambda}}\int_{t\in\mathbb{R}}\mathrm{e}^{-\mathrm{i}t^2}\mathrm{d}t \\
&= \frac{1}{\sqrt{\lambda}}\left[\int_{t\in\mathbb{R}}\cos\left(t^2\right)\mathrm{d}t - \mathrm{i}\int_{t\in\mathbb{R}}\sin\left(t^2\right)\mathrm{d}t\right] \\
&= \frac{1}{\sqrt{\lambda}}\left[2\int_{t=0}^{\infty}\cos\left(t^2\right)\mathrm{d}t - 2\mathrm{i}\int_{t=0}^{\infty}\sin\left(t^2\right)\mathrm{d}t\right] \\
&= \frac{1}{\sqrt{\lambda}}\left[2\frac{\sqrt{2\pi}}{4} - 2\mathrm{i}\frac{\sqrt{2\pi}}{4}\right] \\
&= \sqrt{\frac{2\pi}{\lambda}}\frac{1-\mathrm{i}}{2} \\
&= \sqrt{\frac{\pi}{\lambda}}\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}\,.
\end{aligned}$$

If, on the other hand, we wanted to understand it using Theorem 10.7, we may define $g:=1$ and $f(x):=\mathrm{i}x^2$. Then clearly f,g extend to entire functions $\tilde{g}=1$ and $\tilde{f}(z)=\mathrm{i}z^2$. We have $\tilde{f}'(0)=0$ and by (10.10) we find that, with

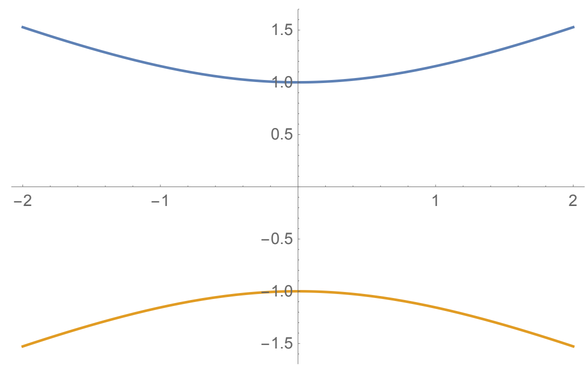

Figure 21: Special paths for the Airy function: the upper and lower hyperbolas.

$\alpha = \frac{\pi}{2}$,

$$\beta = -\frac{\pi}{4} + \frac{\pi}{2} = \frac{\pi}{4}$$

and hence the same result (up to a minus sign? TODO). It is instructive to work out what this contour would be explicitly (this would appear in the HW).

Example 10.9 (The Airy function)**.** The Airy function is defined as a solution to a differential equation that appears in physics (for example in semiclassical (WKB connection formulas) approximation in quantum mechanics). There, it is necessary to work out its asymptotics. It is given by

$$\mathrm{Ai}\left(x\right) := \frac{1}{\pi}\int_{t=0}^{\infty}\cos\left(\frac{t^3}{3}+xt\right)\mathrm{d}t \qquad (x>0)\,.$$

To study the asymptotics as $x\to\infty$, we first rewrite

$$\begin{aligned}\mathrm{Ai}\left(x\right) &= \frac{1}{2\pi}\mathbb{Re}\left\{\int_{t=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\left(\frac{t^3}{3}+xt\right)}\mathrm{d}t\right\}\\ &\overset{t:=\sqrt{x}s}{=} \frac{\sqrt{x}}{2\pi}\mathbb{Re}\left\{\int_{s=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\left(\frac{1}{3}x^{\frac{3}{2}}s^3+x\sqrt{x}s\right)}\mathrm{d}s\right\}\\ &\overset{\lambda:=x^{\frac{3}{2}}}{=} \frac{\lambda^{\frac{1}{3}}}{2\pi}\mathbb{Re}\left\{\int_{s=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\lambda\left(\frac{1}{3}s^3+s\right)}\mathrm{d}s\right\}\,.\end{aligned}$$

To apply Theorem 10.7, we identify $\tilde{f}\left(z\right) := -\mathrm{i}\left(\frac{1}{3}z^3+z\right)$ and $\tilde{g}\left(z\right) := 1$ so we are interested in:

$$I_R\left(\lambda\right) := \int_{z\in[-R,R]}\mathrm{e}^{-\lambda\tilde{f}(z)}\mathrm{d}z\,.$$

First, let us find the critical point: $\tilde{f}'\left(z\right) = -\mathrm{i}\left(z^2+1\right)$ which zeros at $z_\star = \pm\mathrm{i}$. Moreover, we have the imaginary part as

$$\begin{aligned}\tilde{f}_I\left(x+\mathrm{i}y\right) &= \mathbb{Im}\left\{-\mathrm{i}\left(\frac{1}{3}\left(x+\mathrm{i}y\right)^3+\left(x+\mathrm{i}y\right)\right)\right\}\\ &= -x-\frac{1}{3}x^3+xy^2\end{aligned}$$

and the real part is

$$\tilde{f}_R\left(x+\mathrm{i}y\right) = x^2y+y-\frac{1}{3}y^3$$

and so the value of the imaginary part at the critical point is

$$\tilde{f}_I(\pm\mathrm{i}) \quad = \quad 0\,.$$

This yields an implicit equation (a relation between γ_1 and γ_2) for a path $\gamma : [-1,1] \to \mathbb{C}$ which passes through the critical point:

$$-\gamma_1 - \frac{1}{3}\gamma_1^3 + \gamma_1\gamma_2^2 \stackrel{!}{=} \tilde{f}_I(\pm\mathrm{i}) = 0\,.$$

This yields

$$\gamma_1\left(1+\frac{1}{3}\gamma_1^2 - \gamma_2^2\right) \quad = \quad 0\,.$$

One solution is $\gamma_1 = 0$, i.e., γ traverses the imaginary axis. Another solution is

$$\gamma_2 \quad = \quad \pm\sqrt{1+\frac{1}{3}\gamma_1^2}$$

which are depicted in Figure 21. There are now three conceivable possibilities to deform the initial, real contour $[-R,R]$ into. We need to pick the one which will allow us to employ Cauchy's theorem as $R\to\infty$.

1. Option one: go along the imaginary axis. If we do this then we have *two* critical points to take care of. We may, e.g., close the contour along arcs

   $$\gamma(\theta) \quad := \quad R\mathrm{e}^{\mathrm{i}\theta}$$

   with $\theta \in \left[0, \frac{\pi}{2}\right]$ and $\theta \in \left[-\frac{\pi}{2}, -\pi\right]$. The integrals on the arcs should go to zero as $R\to\infty$. Can this be arranged? Consider arc $\theta \in \left[0,\frac{\pi}{2}\right]$ e.g.,

   $$\begin{aligned}
   &\int_{z\in\text{one of the arcs}} \mathrm{e}^{-\lambda\tilde{f}(z)}\mathrm{d}z \\
   =\;& \int_{\theta=0}^{\frac{\pi}{2}} \mathrm{e}^{\mathrm{i}\lambda\left(\frac{1}{3}\left(R\mathrm{e}^{\mathrm{i}\theta}\right)^3 + R\mathrm{e}^{\mathrm{i}\theta}\right)} R\mathrm{e}^{\mathrm{i}\theta}\mathrm{i}\mathrm{d}\theta \\
   =\;& \mathrm{i}R\int_{\theta=0}^{\frac{\pi}{2}} \mathrm{e}^{\mathrm{i}\lambda\left(\frac{1}{3}R^3[\cos(3\theta)+\mathrm{i}\sin(3\theta)] + R\cos(\theta) + \mathrm{i}R\sin(\theta)\right)}\mathrm{e}^{\mathrm{i}\theta}\mathrm{d}\theta \\
   =\;& \mathrm{i}R\int_{\theta=0}^{\frac{\pi}{2}} \mathrm{e}^{\mathrm{i}\lambda R\left(\frac{1}{3}R^2\cos(3\theta)+\cos(\theta)\right)}\mathrm{e}^{-\lambda R\left[R^2\sin(3\theta)+\sin(\theta)\right]}\mathrm{e}^{\mathrm{i}\theta}\mathrm{d}\theta
   \end{aligned}$$

   As $R\to\infty$, it is only the sign of $\sin(3\theta)$ that matters. This, however, will become negative so that the exponent within the integral diverges and there's (probably–this is not a rigorous lower bound) no way to close the contour.

2. Option two: go along the lower parabola. In order to do that we need to add two vertical legs at $x=\pm R$ that will go down until $y = -\sqrt{1+\frac{1}{3}R^2}$. Hence we need to make sure e.g. the following contour integral converges to zero:

   $$\begin{aligned}
   &\int_{z\in\text{right vertical leg}} \mathrm{e}^{-\lambda\tilde{f}(z)}\mathrm{d}z \\
   =\;& \int_{y=0}^{-\sqrt{1+\frac{1}{3}R^2}} \mathrm{e}^{\mathrm{i}\lambda\left(\frac{1}{3}(R+\mathrm{i}y)^3+(R+\mathrm{i}y)\right)}\mathrm{i}\mathrm{d}y \\
   =\;& \int_{y=0}^{-\sqrt{1+\frac{1}{3}R^2}} \mathrm{e}^{-\lambda\left(R^2y+y-\frac{1}{3}y^3\right)}\mathrm{e}^{-\mathrm{i}\lambda\left(-R-\frac{1}{3}R^3+Ry^2\right)}\mathrm{i}\mathrm{d}y\,.
   \end{aligned}$$

   We see again that if $y<0$, as $R\to\infty$, the integrand behaves like $\mathrm{e}^{+\lambda R^3}$ which diverges.

3. The failure of the first two possibilities leads us to hope for the last remaining possibility: going on the upper hyperbola. This indeed turns out to work by the above calculation, which shows that the integrals on the

vertical legs will behave like $\mathrm{e}^{-\lambda R^3}$ and so we can surely add them "for free". The integral for either one of these legs is bounded by:

$$\begin{aligned}&\left|\int_{z\in\text{right vertical leg going up}}\mathrm{e}^{-\lambda\tilde{f}(z)}\mathrm{d}z\right|\\ &\leq\int_{y=0}^{\sqrt{1+\frac{1}{3}R^2}}\mathrm{e}^{-\lambda\left(R^2y+y-\frac{1}{3}y^3\right)}\mathrm{d}y\\ &\overset{\star}{\lesssim}\int_{y=0}^{\sqrt{1+\frac{1}{3}R^2}}\mathrm{e}^{-\frac{8}{9}R^2\lambda y}\mathrm{d}y\\ &\lesssim\frac{1}{R^2}\to 0\end{aligned}$$

where in $\star$ we have used the fact that $y\leq\sqrt{1+\frac{1}{3}R^2}$ within this integral. We thus find, with the parametrization of the upper hyperbola as

$$\gamma(t):=t+\mathrm{i}\sqrt{1+\frac{1}{3}t^2}\qquad(t\in\mathbb{R})$$

$$\begin{aligned}&\int_{s=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\lambda\left(\frac{1}{3}s^3+s\right)}\mathrm{d}s\\ &=\int_{z\in\text{upper hyperbola}}\mathrm{e}^{\mathrm{i}\lambda\left(\frac{1}{3}z^3+z\right)}\mathrm{d}z && \text{(After contour deformation)}\\ &\equiv\int_{t=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\lambda\left(\frac{1}{3}\gamma(t)^3+\gamma(t)\right)}\gamma'(t)\,\mathrm{d}t\\ &=\int_{t=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\lambda\left(\frac{1}{3}\left(t+\mathrm{i}\sqrt{1+\frac{1}{3}t^2}\right)^3+t+\mathrm{i}\sqrt{1+\frac{1}{3}t^2}\right)}\left(1+\mathrm{i}\frac{t}{3\sqrt{1+\frac{1}{2}t^2}}\right)\mathrm{d}t\\ &=\int_{t=-\infty}^{\infty}\mathrm{e}^{-\lambda\frac{2\sqrt{1+\frac{1}{3}t^2}\left(3+4t^2\right)}{9}}\left(1+\mathrm{i}\frac{t}{3\sqrt{1+\frac{1}{2}t^2}}\right)\mathrm{d}t\,.\end{aligned}$$

We now perform Laplace asymptotics on this latter integral, since its exponential function is real. We find the critical point on the real axis is at $t=0$ (where it should be) and hence, with

$$\begin{array}{rcl}f(t)&:=&\dfrac{2\sqrt{1+\frac{1}{3}t^2}\left(3+4t^2\right)}{9}\\ &\downarrow&\\ f''(t=0)&=&2\end{array}$$

and so

$$\begin{aligned}&\int_{t=-\infty}^{\infty}\mathrm{e}^{-\lambda\frac{2\sqrt{1+\frac{1}{3}t^2}\left(3+4t^2\right)}{9}}\left(1+\mathrm{i}\frac{t}{3\sqrt{1+\frac{1}{2}t^2}}\right)\mathrm{d}t\\ \approx\quad&\sqrt{\frac{\pi}{\lambda}}\exp\left(-\frac{2}{3}\lambda\right)\end{aligned}$$

Collecting this together we have

$$\begin{aligned}
\operatorname{Ai}(x) &\approx \frac{\lambda^{\frac{1}{3}}}{2\pi}\,\mathbb{R}\mathrm{e}\left\{\sqrt{\frac{\pi}{\lambda}}\mathrm{e}^{-\frac{2}{3}\lambda}\right\} \\
&= \frac{1}{2\sqrt{\pi}}\lambda^{-\frac{1}{6}}\mathrm{e}^{-\frac{2}{3}\lambda} \\
&\overset{\lambda:=x^{\frac{3}{2}}}{=} \frac{1}{2\sqrt{\pi}}x^{-\frac{1}{4}}\mathrm{e}^{-\frac{2}{3}x^{\frac{3}{2}}}\,.
\end{aligned}$$

A Useful identities and inequalities

1. We have Euler's formula
$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$
and recall that
$$|e^z| = e^x$$
whereas
$$\mathbb{Re}\{e^z\} = e^x\cos(y)\ , \mathbb{Im}\{e^z\} = e^x\sin(y)\ .$$
2. The triangle inequality is
$$|z+w| \leq |z| + |w|$$
3. and the reverse triangle inequality
$$|z+w| \geq ||z| - |w||\ .$$
4. The geometric sum
$$\sum_{n=0}^{N} z^n = \frac{1-z^{N+1}}{1-z} \qquad (z \in \mathbb{C} \setminus \{1\}, N \in \mathbb{N}) \tag{A.1}$$
and series
$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \qquad (|z| < 1)\ . \tag{A.2}$$

Lemma A.1 (Jordan's inequality)**.** *The* sin *function satisfies the upper and lower bounds*
$$\frac{2}{\pi}\alpha \leq \sin(\alpha) \leq \alpha \qquad \left(\alpha \in \left[0, \frac{\pi}{2}\right]\right)\ .$$

Lemma A.2 (Kober's inequality)**.** *We have*
$$1 - \frac{2}{\pi}|x| \leq \cos(x) \leq 1 - \frac{x^2}{\pi} \qquad \left(x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right)\ .$$

Lemma A.3 (Big arc lemma)**.** *We have*
$$\int_{\theta=0}^{\pi} e^{-R\sin(\theta)} d\theta \lesssim \frac{1}{R}$$
as $R \to \infty$.

Proof. We have $\sin(\theta) \geq \frac{2}{\pi}\theta$ if $\theta \in \left[0, \frac{\pi}{2}\right]$ and conversely $\sin(\theta) \geq 2 - \frac{2}{\pi}\theta$ if $\theta \in \left[\frac{\pi}{2}, \pi\right]$ (see Lemma A.1), so that
$$\begin{aligned} \int_{\theta=0}^{\pi} e^{-R\sin(\theta)} d\theta &\leq \int_{\theta=0}^{\frac{\pi}{2}} e^{-R\frac{2}{\pi}\theta} d\theta + \int_{\theta=\frac{\pi}{2}}^{\pi} e^{-R\left(2-\frac{2}{\pi}\theta\right)} d\theta \\ &= \left(-\frac{\pi}{2R}\right)\left(e^{-R} - 1\right) + e^{-2R}\left(\frac{\pi}{2R}\left(e^{2R} - e^{R}\right)\right)\ . \\ &= \frac{\pi}{R}\left(1 - e^{-R}\right) \\ &\lesssim \frac{1}{R}\ . \end{aligned}$$
$\square$

A.1 The supremum and infimum

Any non-empty set S of $\mathbb{R}$ has a supremum $\sup(S)$, which is *the least upper bound*. It is defined as follows. An upper bound on S is a number $u \in \mathbb{R}$ such that $s \leq u$ for any $s \in S$. The *least upper bound* $\sup(S) \in \mathbb{R}$ is a number that is an upper bound on S such that if $u \in \mathbb{R}$ is any upper bound on S, then $\sup(S) \leq u$. It is unique, if it exists.
One of the most important properties of the supremum is its approximation property: For any set $S \subseteq \mathbb{R}$, let $\sup(S) \in \mathbb{R}$ be its supremum. Then for any $\varepsilon > 0$ there exists some $u_\varepsilon \in S$ such that

$$\sup(S) - \varepsilon \quad \leq \quad a_\varepsilon \,.$$

I.e., from the very definition of the supremum, we must be able to get arbitrarily close to it from *within* the set.

B Glossary of mathematical symbols and acronyms

Sometimes it is helpful to include mathematical symbols which can function as valid grammatical parts of sentences. Here is a glossary of some which might appear in the text:

- The bracket $\langle\cdot,\cdot\rangle_V$ means an inner product on the inner product space V. For example,

  $$\langle u,v\rangle_{\mathbb{R}^2} \equiv u_1 v_1 + u_2 v_2 \qquad \left(u,v\in\mathbb{R}^2\right)$$

  and

  $$\langle u,v\rangle_{\mathbb{C}^2} \equiv \overline{u_1} v_1 + \overline{u_2} v_2 \qquad \left(u,v\in\mathbb{C}^2\right)\,.$$

- Sometimes we denote an integral by writing the integrand without its argument. So if $f:\mathbb{R}\to\mathbb{R}$ is a real function, we sometimes in shorthand write

  $$\int_a^b f$$

  when we really mean

  $$\int_{t=a}^b f(t)\,\mathrm{d}t\,.$$

  This type of shorthand notation will actually also apply for contour integrals, in the following sense: if $\gamma:[a,b]\to\mathbb{C}$ is a contour with image set $\Gamma := \mathrm{im}(\gamma)$ and $f:\mathbb{C}\to\mathbb{C}$ is given, then the contour integral of f along γ will be denoted equivalently as

  $$\int_\Gamma f \equiv \int_\Gamma f(z)\,\mathrm{d}z \equiv \int_{t=a}^b f(\gamma(t))\,\gamma'(t)\,\mathrm{d}t$$

  depending on what needs to be emphasized in the context. Sometimes when the contour is clear one simply writes

  $$\int_{z_0}^{z_1} f(z)\,\mathrm{d}z$$

  for an integral along *any* contour from z_0 to z_1.

- iff means “if and only if”, which is also denoted by the symbol $\iff$.
- WLOG means “without loss of generality”.
- CCW means “counter-clockwise” and CW means “clockwise”.
- $\exists$ means “there exists” and $\nexists$ means “there does not exist”. $\exists!$ means “there exists *a unique*”.
- $\forall$ means “for all” or “for any”.
- : (i.e., a colon) may mean “such that”.
- ! means negation, or “not”.
- $\wedge$ means “and” and $\vee$ means “or”.
- $\implies$ means “and so” or “therefore” or “it follows”.

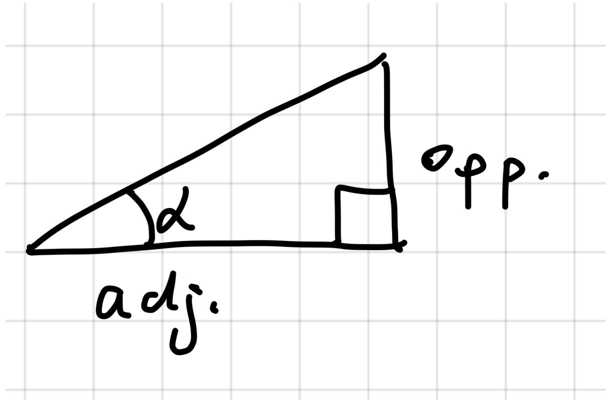

Figure 22: Tangent function.

- $\in$ denotes set inclusion, i.e., $a \in A$ means a is an element of A or a lies in A.
- $\ni$ denotes set inclusion when the set appears first, i.e., $A \ni a$ means A includes a or A contains a.
- Speaking of set inclusion, $A \subseteq B$ means A is contained within B and $A \supseteq B$ means B is contained within A.
- $\varnothing$ is the empty set $\{\ \}$.
- While $=$ means equality, sometimes it is useful to denote types of equality:
  - $a := b$ means "this equation is now the instant when a is defined to equal b".
  - $a \equiv b$ means "at some point above a has been defined to equal b".
  - $a = b$ will then simply mean that the result of some calculation *or* definition stipulates that $a = b$.
  - Concrete example: if we write $\mathrm{i}^2 = -1$ we don't specify anything about *why* this equality is true but writing $\mathrm{i}^2 \equiv -1$ means this is a matter of definition, not calculation, whereas $\mathrm{i}^2 := -1$ is the first time you'll see this definition. So this distinction is meant to help the reader who wonders *why* an equality holds.

B.1 Important sets

1. The unit circle

$$\mathbb{S}^1 \quad \equiv \quad \{\, z \in \mathbb{C} \mid |z| = 1 \,\} \, .$$

2. The (open) upper half plane

$$\mathbb{H} \quad \equiv \quad \{\, z \in \mathbb{C} \mid \mathbb{Im}\{z\} > 0 \,\} \, .$$

C The arctangent function

The arctangent is formally the inverse map to the tangent function tan. The tangent of an angle α may be defined geometrically as the ratio between the opposite edge and the adjacent edge of a right triangle with angle α, see Figure 22.

The issue with this geometric definition is that it only makes sense for $\alpha \in \left[0, \frac{\pi}{2}\right]$ since the sum of angles in a triangle is π. With a convention that the opposite side is negative if α is negative we may extend this to $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. One may show that

$$\tan : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \quad \to \quad \mathbb{R}$$

is a bijection which may be extended to a homeomorphism

$$\tan : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \quad \to \quad \mathbb{R} \cup \{\, \pm\infty \,\} \, ,$$

the latter set being the two-point compactification of $\mathbb{R}$.

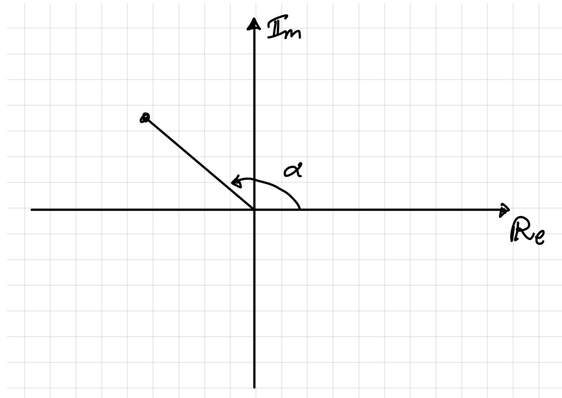

Figure 23: The polar representation of a complex number needs angles beyond $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.

A homeomorphism immediately entails the inverse mapping

$$\arctan : \mathbb{R}\cup\{\pm\infty\} \to \left[-\frac{\pi}{2},\frac{\pi}{2}\right] \tag{C.1}$$

which is also continuous.

This all works well as long as we are talking about right triangles whose adjacent side is positive. On the other hand, when talking about the polar representation of a complex number, we want to denote the angle tended from the positive horizontal axis, and hence we need to assign angles *beyond* $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ (see Figure 23). But since (C.1) is already a homeomorphism, if its range is to be extended it will necessary fail to be a well-defined function. This is a problem that appears still *before* the 2π ambiguity in defining θ within $e^{i\theta}$: From the geometric picture, arctan is defined only in the right-half plane so far. To remedy this we let arctan depend on two arguments instead of one:

Definition C.1 (atan2)**.** The function $\operatorname{atan2} : \mathbb{R}^2\setminus 0 \to (-\pi,\pi]$ is defined as $\operatorname{atan2}(y,x)$ being the angle measured between the horizontal axis and a ray from the origin to a point $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2\setminus\{0\}$. The signs of x and y are used to determine the quadrant of the result and hence possibly shift $\arctan\left(\frac{y}{x}\right)$ by π or $-\pi$ as follows:

$$\operatorname{atan2}(y,x) \quad := \quad \begin{cases} \arctan\left(\frac{y}{x}\right) & x\geq 0 \\ \arctan\left(\frac{y}{x}\right)+\pi & x<0\wedge y\geq 0 \\ \arctan\left(\frac{y}{x}\right)-\pi & x<0\wedge y<0 \end{cases}$$

with the convention $\operatorname{atan2}(y,0)=\pm\frac{\pi}{2}$ for $\frac{y}{|y|}=\pm 1$.

Hence, we see according to the definitions above in the main text, atan2 is identical to the principal argument:

$$\operatorname{Arg}(z) = \operatorname{atan2}(y,x) \in (-\pi,\pi]\ .$$

D Basic facts from real analysis

Here we collect some facts which the reader may leisurely learn in [Rud76, Rud86].

D.1 Convergence theorems for integrals

A sequence of functions $\{g_n\}_{n\in\mathbb{N}}$ (of a real or complex variable) is said to converge *pointwise* to some function g iff for any t,

$$\lim_{n\to\infty} g_n(t) \quad = \quad g(t)\ .$$

The idea is that the rate of convergence of the sequence $\{g_n(t)\}_n$ may depend on t in terrible ways. There are classic examples in which double limits *may not* be exchanged. For instance, consider

$$g_n(t) \quad = \quad \frac{t}{n+t}\ .$$

Then

$$\lim_{n\to\infty} g_n(t) = 0$$

for all t yet

$$\lim_{t\to\infty} g_n(t) = 1$$

so clearly the order of limits *matters*.

Definition D.1. We say that the sequence $\{ g_n \}_n$ converges *pointwise* to some function g iff for any *fixed* t

$$\lim_{n\to\infty} g_n(t) = g(t) .$$

The idea behind this definition is that the rate of convergence (as $n \to \infty$) may very well depend on t. The other type of behavior is when convergence does not depend on t:

Definition D.2. We say that the sequence $\{ g_n \}_n$ converges *uniformly* to some function g iff $\forall \varepsilon > 0 \exists N_\varepsilon \in \mathbb{N} : \forall n \geq N_\varepsilon$,

$$|g_n(t) - g(t)| < \varepsilon \qquad (t \in \mathbb{R}) .$$

I.e., N_ε may depend on ε but not on t.

The following theorem from [Rud76, Theorem 7.16] is the first tool one has to replace limits and integration:

Theorem D.3. *Let $a < b$ be given and consider a sequence $\{ g_n \}_n$ of Riemann integrable functions $[a, b] \to \mathbb{C}$ which converges* uniformly *to some $g : [a, b] \to \mathbb{C}$. Then g is also Riemann integrable and*

$$\lim_{n\to\infty} \int_a^b g_n = \int_a^b g .$$

To emphasize, that the limit of integrals exists is part of the result of the theorem.

The next theorem is not very well known in the theory of Riemann integration, since it is usually introduced in the context of Lebesgue integration on measure theory (where it is called Lebesgue's dominated convergence theorem, see [Rud86, Theorem 1.34]). Be that as it may if we insist to make our material accessible to students who have not yet had measure theory we are lucky to have it presented in [dS10]

Theorem D.4 (Arzela's dominated convergence theorem (1885))**.** *Let $\{ g_n \}_n$ be a sequence of Riemann integrable functions $[a, b] \to \mathbb{R}$ which converges pointwise to some $g : [a, b] \to \mathbb{R}$ which is also assumed to be Riemann integrable. Suppose further that $\{ \|g_n\| \}_n$ is a bounded sequence, i.e.,*

$$\sup_{t\in[a,b], n\in\mathbb{N}} |g_n(t)| < \infty .$$

Then

$$\lim_{n\to\infty} \int_a^b g_n = \int_a^b g .$$

D.2 Fubini's theorem

The next theorem we are concerned with is Fubini's theorem, which roughly speaking allows one to change the order of integration. Its proper discussion is again in a full-fledged course in measure theory (see [Rud86, Theorem 8.8]) but we content ourselves with its statement

Theorem D.5 (Fubini)**.** *Let $f:\mathbb{R}^2\to\mathbb{C}$ be a function of two variables and let $A,B\subseteq\mathbb{R}$ be two subsets. If*

$$\int_{x\in A\times B}|f(x)|\,\mathrm{d}^2x \quad < \quad \infty$$

then

$$\int_{y\in B}\left(\int_{x\in A}f(x,y)\,\mathrm{d}x\right)\mathrm{d}y=\int_{x\in A}\left(\int_{y\in B}f(x,y)\,\mathrm{d}y\right)\mathrm{d}x=\int_{x\in A\times B}f(x)\,\mathrm{d}^2x\,.$$

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