Complex Analysis with Applications
Princeton University MAT330
HW9, Due April 28th 2023

April 27, 2023

1 The Fourier transform

1. Define, for any $y > 0$,

$$f(x) \quad := \quad \frac{1}{\pi} \frac{y}{y^2 + x^2} \qquad (x \in \mathbb{R}) \; .$$

Recall this is actually the Poisson kernel on the half plane! Calculate $\mathcal{F} f$.

2. Let $p : \mathbb{C} \to \mathbb{C}$ be the characteristic polynomial of some matrix (of size at least 2×2), all of whose eigenvalues are *unreal*. Calculate $\mathcal{F} p$ and $\mathcal{F} \frac{1}{p}$ assuming all the eigenvalues are simple, and then calculate $\mathcal{F} p$ and $\mathcal{F} \frac{1}{p}$ allowing for degeneracies.

3. Define

$$f(x) \quad := \quad \frac{1}{\sqrt{|x|}} \qquad (x \in \mathbb{R}) \; .$$

Calculate $\mathcal{F} f$. Is $f \in L^2(\mathbb{R})$?

4. In this question, we consider

$$\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$$

as a unitary operator between Hilbert spaces, and ask what are its eigenvalues. For the purpose of the present question, we may think of an eigenvalue of $\mathcal{F}$ as any number $\lambda \in \mathbb{C}$ such that there exists some $\psi \in L^2$ with

$$\mathcal{F} \psi \quad = \quad \lambda \psi \, .$$

Thus, we are asking which L^2 functions are proportional to their own Fourier transform. Since we know $\mathcal{F}$ is unitary (this is Plancherel), we expect $|\lambda| = 1$ (though to really prove this statement we need a bit more machinery, so we won't use this now). Let us hence take a more hands-on approach.
Consider the function $G(t, x)$ (generating function) defined via

$$G(t, x) \quad := \quad \exp\left(2xt - t^2\right) \qquad (x, t \in \mathbb{R}) \; .$$

Since this function is entire in t, it must have a Taylor expansion in powers of t of the form

$$G(t, x) \quad = \quad \sum_{n=0}^{\infty} \frac{(\partial_t^n G)(0, x)}{n!} t^n \, .$$

The function $x \mapsto (\partial_t^n G)(0, x) =: H_n(x)$ is a polynomial in x, called the Hermite polynomial.
Calculate two possible expressions for the Fourier transform of $x \mapsto \mathrm{e}^{-\frac{1}{2}x^2} G(t, x)$ (one for each term in the Taylor series expansion above, and one directly from $x \mapsto \exp\left(-\frac{1}{2}x^2 + 2xt - t^2\right)$), equate the two, power by power in t, to find

$$x \quad \mapsto \quad \mathrm{e}^{-\frac{1}{2}x^2} H_n(x)$$

are L^2 eigenvectors of $\mathcal{F}$, and calculate the corresponding eigenvalues.

5. $\mathcal{F}^{-1}\mathcal{F} = \mathbb{1}$ always? Let

$$f(x) := x^2$$

and

$$g(x) := \begin{cases} x^2 & x \neq 0 \\ 5 & x = 0\,. \end{cases}$$

Calculate $\mathcal{F}^{-1}\mathcal{F}f$ and $\mathcal{F}^{-1}\mathcal{F}g$.

6. Calculate $\mathcal{F}\chi_{[a,b]}$ where $a < b$ and

$$\chi_S(x) := \begin{cases} 1 & x \in S \\ 0 & x \notin S\,. \end{cases}$$

7. If $f : \mathbb{R} \to \mathbb{C}$ is given, calculate $\mathcal{F}\overline{f}$ and $\mathcal{F}f(\cdot - a)$ for some $a \in \mathbb{R}$.

8. If $f : \mathbb{R} \to \mathbb{C}$ is given to be differentiable and L^1, calculate $\mathcal{F}f'$.

9. Show that if $A : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a linear operator with an integral kernel

$$\mathbb{R}^2 \ni (x,y) \mapsto A(x,y)$$

i.e., it is defined as

$$(Af)(x) \equiv \int_{y\in\mathbb{R}} A(x,y)\, f(y)\,\mathrm{d}y\,,$$

which obeys the following constraint

$$A(x+z, y+z) = A(x,y) \qquad (x,y,z \in \mathbb{R})$$

then

$$\mathcal{F}A\mathcal{F}^{-1} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$$

is a *multiplication operator*, i.e., there exists a function $a : \mathbb{R} \to \mathbb{C}$ such that

$$\left(\mathcal{F}A\mathcal{F}^{-1}f\right)(\xi) = a(\xi)\, f(\xi) \qquad (\xi \in \mathbb{R})\,.$$

Find an expression for a in terms of the integral kernel $A(\cdot,\cdot)$.

10. Without reference to the integral kernel of $-\Delta$ on $L^2(\mathbb{R})$, find directly the function $\mathcal{E} : \mathbb{R} \to \mathbb{R}$ such that

$$\mathcal{F}(-\Delta)\mathcal{F}^{-1}f = \mathcal{E}f\,.$$

11. [extra] Using the above, calculate the integral kernel of $(-\Delta - z\mathbb{1})^{-1}$ for $z \in \mathbb{C}\setminus\mathbb{R}$ and find its boundary values (i.e. the limit of the integral kernel as $z \to \mathbb{R}$).

2 L^p spaces

12. Show that $L^2([0,1]) \subseteq L^1([0,1])$.

13. Show that $x \mapsto \frac{\exp(-x^2)}{\sqrt{x}}$ is in $L^1([0,\infty))$ but not in $L^2([0,\infty))$. Is it in $L^\infty([0,\infty))$?

14. Show that $f(x) = \frac{1}{1+x}$ is in $L^2([0,\infty))$ but not in $L^1([0,\infty))$.

3 The Poisson summation formula

15. Calculate $\frac{1}{\pi}\sum_{n\in\mathbb{Z}} \frac{a}{a^2+n^2}$ for some $a > 0$.

16. Calculate $\sum_{n\in\mathbb{Z}} \frac{1}{(z+n)^\ell}$ where $\ell \in \mathbb{N}_{\geq 2}$ and $z \in \mathbb{C}$ with $\mathbb{Im}\{z\} > 0$.

17. Define

$$f(x) := \sum_{n\in\mathbb{Z}} \mathrm{e}^{-\frac{1}{2}\lambda(n-x)^2} \qquad (x \in \mathbb{R})\,.$$

Find an expression for $\|f\|_{L^\infty(\mathbb{R})}$.