MAY 4 2023

MAT 330 — $\mathbb{C}$-Analysis — HW9 Sample Sol-ns

$\boxed{Q1}$

$$(\mathcal{F}f)(\xi) \equiv \int_{x\in\mathbb{R}} \exp(-2\pi i \xi x) f(x)\, dx \qquad (\xi \in \mathbb{R})$$

$$f(x) := \frac{1}{\pi} \frac{y}{y^2+x^2}$$

$$= \frac{1}{\pi y} \frac{1}{1+(x/y)^2}$$

$$(\mathcal{F}f)(\xi) \equiv \int_{x\in\mathbb{R}} dx\, e^{-2\pi i \xi x} f(x) \equiv \int_{x\in\mathbb{R}} dx\, e^{-2\pi i \xi x} \frac{1}{\pi y} \frac{1}{1+(x/y)^2}$$

$w := x/y$

$dw = \frac{1}{y} dx$

$$\overset{\text{b}}{=} \frac{1}{\pi \cancel{y}} \int_{w\in\mathbb{R}} \exp(-2\pi i \xi y w) \frac{1}{1+w^2} \cancel{y}\, dw$$

Example 8.7

$$\overset{\partial}{=} \frac{1}{\cancel{\pi}} \cancel{\pi} \exp(-2\pi |y\xi|)$$

$$= \exp(-2\pi |y||\xi|).$$

$\boxed{Q2}$

$$p(z) = (z-\lambda_1)(z-\lambda_2)\cdots(z-\lambda_n) \qquad n \geq 2$$

$$\{\lambda_j\}_{j=1}^{n} \subseteq \mathbb{C}\setminus\mathbb{R} \ : \ \lambda_i \neq \lambda_j \quad \forall \ i\neq j.$$

$$\left(\mathcal{F}\frac{1}{p}\right)(\xi) \equiv \int_{x\in\mathbb{R}} \underbrace{\frac{\exp(-2\pi i \xi x)}{(x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_n)}}_{=: f(x)} dx$$

f extends to a meromorphic $\tilde{f}(z)$.

If $\xi > 0$, we may add "free of charge" a lower semicircle w/ arbitrarily large radius, since:

$$\left| \int_{\theta=-\pi}^{0} f(Re^{i\theta})\, Re^{i\theta}\, i\, d\theta \right| =$$

$$= \left| \int_{\theta=-\pi}^{0} \frac{\exp(-2\pi i \xi R e^{i\theta})}{(Re^{i\theta}-\lambda_1)(Re^{i\theta}-\lambda_2)\cdots(Re^{i\theta}-\lambda_n)} Re^{i\theta} i\, d\theta \right|$$

$$\leq \int_{\theta=-\pi}^{0} \frac{\exp(+2\pi\xi R\sin(\theta))}{(R-|\lambda_1|)(R-|\lambda_2|)\cdots(R-|\lambda_n|)} R\, d\theta$$

Big arc Lemma A.

$$\lesssim \frac{R}{(R-|\lambda_1|)\cdots(R-|\lambda_n|)} \frac{1}{R} \xrightarrow{R\to\infty} 0.$$

Similarly if $\xi < 0$ we may add a large upper semi-circle.

Note These are merely upper bounds which allow us to proceed, but if one had chosen closing the semicircle in the other direction one could not have closed the contour.

Hence we find the following:

$$(\mathcal{F}\tfrac{1}{p})(\xi) = 2\pi i \begin{cases} \displaystyle\sum_{j:\,\mathrm{Im}\{\lambda_j\}>0} \mathrm{res}_{\lambda_j}(f) & \xi<0 \\ \displaystyle -\sum_{j:\,\mathrm{Im}\{\lambda_j\}<0} \mathrm{res}_{\lambda_j}(f) & \xi>0. \end{cases}$$

CW contour gets opposite sign!

If $\xi=0$ then we may actually add either one of the semicircles since $n\geq 2$:

$$\left| \int_{\theta=0}^{\pm\pi} f(Re^{i\theta})\, Re^{i\theta}\, i\, d\theta \right| \leq$$

$$\leq \frac{\pi R}{(R-|\lambda_1|)\cdots(R-|\lambda_n|)} \xrightarrow{R\to\infty} 0 .$$

Then we get

$$(\mathcal{F}\tfrac{1}{p})(0) = 2\pi i \sum_{j:\,\mathrm{Im}\{\lambda_j\}>0} \mathrm{res}_{\lambda_j}(z \mapsto (z-\lambda_1)^{-1}\cdots(z-\lambda_n)^{-1})$$

$$= -2\pi i \sum_{j:\,\mathrm{Im}\{\lambda_j\}<0} \mathrm{res}_{\lambda_j}(z \mapsto (z-\lambda_1)^{-1}\cdots(z-\lambda_n)^{-1})$$

Let us calculate these residues:

⊛ if $\xi \neq 0$ and all λ_j's are different:

$$\operatorname{res}_{\lambda_j}(f) = \frac{\exp(-2\pi i \xi \lambda_j)}{\prod_{i\neq j} (\lambda_j - \lambda_i)}$$

⊛ if $\xi = 0$ and all λ_j's are different:

$$\operatorname{res}_{\lambda_j}(f) = \Big[\prod_{i\neq j} (\lambda_j - \lambda_i)\Big]^{-1}$$

⊛ if some λ_j's come w/ higher mul. then we need the more general residue formula:

Say we have

$$f(z) = \frac{\exp(-2\pi i \xi z)}{\prod_{i=1}^{n} (z-\lambda_i)^{\alpha_i}}$$

list of mul.

$\alpha_j \in \mathbb{N}_{\geq 1}$ $\forall j = 1, \dots, n.$

Then

$$\operatorname{res}_{\lambda_j}(f) = \lim_{z\to\lambda_j} \frac{1}{(\alpha_j - 1)!} \partial_z^{\alpha_j - 1} \frac{\exp(-2\pi i \xi z)}{\prod_{\substack{i=1 \\ i\neq j}}^{n} (z-\lambda_i)^{\alpha_i}}$$

$$= \lim_{z \to \lambda_j} \frac{1}{(\alpha_j - 1)!} \sum_{l_1 + \dots + l_n = \alpha_j - 1} \left[\partial_z^{l_j} \exp(-2\pi i \xi z) \right] \times \left[\prod_{\substack{i=1 \\ i \neq j}}^{n} \partial_z^{l_i} (z - \lambda_i)^{-\alpha_i} \right]$$

$$= \frac{1}{(\alpha_j - 1)!} \sum_{l_1 + \dots + l_n = \alpha_j - 1} \left[(-2\pi i \xi)^{l_j} e^{-2\pi i \xi \lambda_j} \right] \times$$

$$\times \prod_{\substack{i=1 \\ i \neq j}}^{n} (-\alpha_i)(-\alpha_i - 1) \cdots (-\alpha_i - l_i)(\lambda_j - \lambda_i)^{-\alpha_i - l_i}$$

and a similar expression if $\xi = 0$.

By the way when $\operatorname{Im}\{\lambda_j\} = 0$ we may still do the Cauchy p.v. integration.

Now, if $p(x) = \sum_{j=0}^{n} a_j x^j$ then to calculate $\mathcal{F} p$ we really only need $\mathcal{F}(x \mapsto x^j)$.

To do so, observe:

If $j=0$,

$$(\mathcal{F}(1))(\xi) \equiv \int_{x\in\mathbb{R}} e^{-2\pi i \xi x}\, dx = ???$$

This integral does NOT make sense since

$$\int_{x\in\mathbb{R}} e^{-2\pi i \xi x}\, dx \equiv \lim_{R\to\infty} \int_{-R}^{R} e^{-2\pi i \xi x}\, dx$$

$$= \lim_{R\to\infty} \left. \frac{e^{-2\pi i \xi x}}{-2\pi i \xi} \right|_{x=-R}^{R}$$

But $\lim_{R\to\infty} e^{\pm 2\pi i \xi R}$ DNE!

To still give meaning to the integral (beyond an L^2 function...) we regularize it, in a very similar manner to the process in which Cauchy principal value is defined:

$\forall\, \varepsilon > 0$, define the ε-regularized Fourier transf. as

$$\left(\mathcal{F}_\varepsilon(f)\right)(\xi) := \int_{x\in\mathbb{R}} \exp(-2\pi i \xi x - \varepsilon|x|)\, f(x)\, dx .$$

Then $$\left(\mathcal{F}_\varepsilon(x\mapsto 1)\right)(\xi) = \int_{x\in\mathbb{R}} \exp(-2\pi i \xi x - \varepsilon|x|)\, dx$$

$$= \int_{x\geq 0} \exp(-2\pi i \xi x - \varepsilon x)\, dx + \int_{x<0} \exp(-2\pi i \xi x + \varepsilon x)\, dx$$

$$= \left.\frac{\exp(-2\pi i \xi x - \varepsilon x)}{-2\pi i \xi - \varepsilon}\right|_{x=0}^{\infty} + \left.\frac{\exp(-2\pi i \xi x + \varepsilon x)}{-2\pi i \xi + \varepsilon}\right|_{x=-\infty}^{0}$$

$$= \frac{1}{2\pi i \xi + \varepsilon} + \frac{1}{-2\pi i \xi + \varepsilon}$$

$$= \frac{-\cancel{2\pi i\xi} + \varepsilon + \cancel{2\pi i \xi} + \varepsilon}{(2\pi\xi)^2 + \varepsilon^2} = \frac{2\varepsilon}{\varepsilon^2 + (2\pi\xi)^2} = \frac{1}{\pi}\,\frac{2\pi\varepsilon}{\varepsilon^2 + (2\pi\xi)^2}$$

$$= 2\pi\, \mathrm{Im}\left\{\frac{1}{\pi}\,\frac{1}{2\pi\xi - i\varepsilon}\right\} \equiv 2\pi\, P_{\mathbb{H}}(2\pi\xi, \varepsilon)$$

Poisson kernel on $\mathbb{H} \equiv \{ z\in\mathbb{C} \mid \mathrm{Im}\{z\}>0\}$

See eq-n (9.2) in lecture notes.

Recall the Krammers-Kronig relation, eq-n (7.12) in the lecture notes and in particular its imaginary part, eq-n (7.13) :

$$\delta(x) = \lim_{\varepsilon \to 0} P_{IH}(x, \varepsilon) \quad .$$

Thus formally we find:

$$\boxed{\lim_{\varepsilon \to 0^+} \big(\mathcal{F}_\varepsilon (x \mapsto 1)\big)(\xi) = 2\pi \, \delta(2\pi \xi)} \quad (*) \quad .$$

This eq-n is to be understood in the distributional sense, just like the Krammers-Kronig relation itself. That is, $(*)$ is equivalent to: for any "nice" function $f : \mathbb{R} \to \mathbb{C}$,

$$\lim_{\varepsilon \to 0^+} \int_{\xi \in \mathbb{R}} \big(\mathcal{F}_\varepsilon (x \mapsto 1)\big)(\xi) \, f(\xi) \, d\xi = \int_{\xi \in \mathbb{R}} f(\xi) \, 2\pi \, \delta(2\pi \xi) \, d\xi .$$

Note the RHS may be simplified using the scaling properties of the δ-function:

Claim: $\alpha\delta(\alpha\,\cdot) = \delta$ for any $\alpha > 0$.

Proof: For any $f: \mathbb{R} \to \mathbb{C}$ which is "nice",

$$\int_{x\in\mathbb{R}} f(x)\;\alpha\,\delta(\alpha x)\,dx \overset{\text{KK rel.}}{=} \lim_{\varepsilon\to 0^+} \int_{x\in\mathbb{R}} dx\, f(x)\,\alpha\,\frac{1}{\pi}\,\frac{\varepsilon^2}{\varepsilon^2+\alpha^2x^2}$$

$$\underset{dy = \alpha dx}{\overset{y := \alpha x}{=}} \lim_{\varepsilon\to 0^+} \int_{y\in\mathbb{R}} f(y/\alpha)\,\frac{1}{\pi}\,\frac{\varepsilon^2}{\varepsilon^2+y^2}\,dy$$

$$\overset{\text{KK. rel.}}{=} f(0/\alpha) = f(0) \overset{\text{KK. rel.}}{=} \lim_{\varepsilon\to 0^+} \int_{x\in\mathbb{R}} f(x)\,\frac{1}{\pi}\,\frac{\varepsilon^2}{\varepsilon^2+x^2}\,dx \qquad \blacksquare$$

Hence we find

$$\boxed{\lim_{\varepsilon\to 0^+} \mathcal{F}_\varepsilon(x \mapsto 1) = \delta} .$$

Cf. Example 8.10: $\mathcal{F}(\delta) = \xi \mapsto 1$.

We need one more "trick" to do FP:

$$\partial_\xi\, e^{-2\pi i \xi x} = (-2\pi i x)\, e^{-2\pi i \xi x}$$

$$\Longrightarrow\; x\, e^{-2\pi i \xi x} = \frac{1}{-2\pi i}\,\partial_\xi\, e^{-2\pi i \xi x}$$

Iteration of this yields:

$$\boxed{x^j e^{-2\pi i \xi x} = \frac{1}{(-2\pi i)^j} \partial_\xi^j e^{-2\pi i \xi x}} \qquad (j \in \mathbb{N}_{\geq 1})$$

Hence $\left(\mathcal{F}_\varepsilon(x \mapsto x^j)\right)(\xi) \equiv \int_{x\in\mathbb{R}} dx\, x^j e^{-2\pi i \xi x - \varepsilon|x|}$

$$= \int_{x\in\mathbb{R}} dx \, \frac{1}{(-2\pi i)^j} \partial_\xi^j e^{-2\pi i \xi x - \varepsilon |x|}$$

$$\overset{\substack{\text{Leibniz}\\ \text{int. rule}}}{=} \frac{1}{(-2\pi i)^j} \partial_\xi^j \int_{x\in\mathbb{R}} e^{-2\pi i \xi x - \varepsilon|x|} \, dx$$

$$\overset{\substack{\text{above}\\ \text{calc.}}}{=} \frac{1}{(-2\pi i)^j} \partial_\xi^j \, 2\pi P_{\mathbb{H}}(2\pi\xi, \varepsilon)$$

We find finally, for $p(x) = \sum_{j=0}^n a_j x^j$,

$$(\mathcal{F}_\varepsilon p)(\xi) = \sum_{j=0}^n a_j \left(\mathcal{F}_\varepsilon(x \mapsto x^j)\right)(\xi)$$

$$= \sum_{j=0}^n a_j \frac{1}{(-2\pi i)^j} \partial_\xi^j \, 2\pi P_{\mathbb{H}}(2\pi\xi, \varepsilon)$$

$$\xrightarrow{\varepsilon \to 0^+} \sum_{j=0}^n a_j \frac{1}{(-2\pi i)^j} \partial_\xi^j \delta(\xi) .$$

So in that sense, $\boxed{\mathcal{F}p = p\left(\frac{\partial_\xi}{-2\pi i}\right)\delta}$.

$\boxed{Q3}$ $f(x) := \frac{1}{\sqrt{|x|}}$ $(x \in \mathbb{R})$.

$\boxed{\text{NOT } L^2 \text{ @ } x=0}$

$$(\mathcal{F}f)(\xi) \equiv \int_{x\in\mathbb{R}} e^{-2\pi i \xi x} \frac{1}{\sqrt{|x|}}\, dx$$

$$= \int_{x\geq 0} e^{-2\pi i \xi x} \frac{1}{\sqrt{x}}\, dx + \int_{x<0} e^{-2\pi i \xi x} \frac{1}{\sqrt{-x}}\, dx$$

$y := \sqrt{x}$

$dy = \frac{1}{2\sqrt{x}} dx$

$$\overset{\curvearrowright}{=} 2\int_{y=0}^{\infty} e^{-2\pi i \xi y^2}\, dy + 2\int_{y=0}^{\infty} e^{+2\pi i \xi y^2}\, dy$$

$$= 4 \int_{y=0}^{\infty} \cos(2\pi \xi y^2)\, dy$$

Note that this last expression is manifestly an even f^n of ξ. Hence we have

$$(\mathcal{F}f)(\xi) = 4 \int_{y=0}^{\infty} \cos(2\pi |\xi| y^2)\, dy$$

$u^2 := 2\pi|\xi|y^2$

$u = \sqrt{2\pi|\xi|}\, y$

$du = \sqrt{2\pi|\xi|}\, dy$

$$\overset{\curvearrowright}{=} \frac{4}{\sqrt{2\pi|\xi|}} \int_{u=0}^{\infty} \cos(u^2)\, du$$

Example 6.38 in lecture notes

$$\stackrel{\text{Example 6.38 in lecture notes}}{=} \frac{4}{\sqrt{2\pi|\xi|}} \frac{\sqrt{2\pi}}{4} = \frac{1}{\sqrt{|\xi|}} .$$

$$\Longrightarrow \quad \boxed{\mathcal{F} f = f} .$$

One may think of f as an eigenvector of the linear operator $\mathcal{F}$ with eigenvalue 1.

However $f \notin L^2$!

Indeed,
$$\|f\|_{L^2}^2 \equiv \int_{x \in \mathbb{R}} |f(x)|^2 \, dx$$
$$= \int_{x \in \mathbb{R}} \frac{1}{|x|} \, dx = \infty$$

(actually both at zero and at ∞).

Q4 L^2 eigenvectors and eigenvalues of $\mathcal{F}$

$$G(t, x) := \exp(+2xt - t^2)$$

is an entire f^n of t, so it has a Taylor expansion about $t=0$:

$$G(t,x) = \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{\left(\partial_t\Big|_{t=0} G(t,x)\right)}_{=: H_n(x)} t^n$$

$=: H_n(x)$ n^{th} Hermite polynomial.

Claim: $\left(\mathcal{F}(x \mapsto e^{-\frac{1}{2}x^2} H_n(x))\right)(\xi) = e^{-\frac{1}{2}\xi^2} H_n(\xi)$

Proof: $\left(\mathcal{F}(x \mapsto e^{-\frac{1}{2}x^2} G(t,x))\right)(\xi) =$

$$= \int_{x\in\mathbb{R}} \exp(-\tfrac{1}{2}x^2 + 2xt - t^2 - 2\pi i \xi x)\, dx$$

Complete the square

$$-\tfrac{1}{2}x^2 + 2xt - 2\pi i \xi x = -\tfrac{1}{2}\left(x^2 - 2x(2t - 2\pi i \xi)\right)$$

$$= -\tfrac{1}{2}\left(x - (2t - 2\pi i \xi)\right)^2 + \tfrac{1}{2}(2t - 2\pi i \xi)^2$$

$$\Rightarrow \left(\mathcal{F}(x \mapsto e^{-\frac{1}{2}x^2} G(t,x))\right)(\xi) =$$

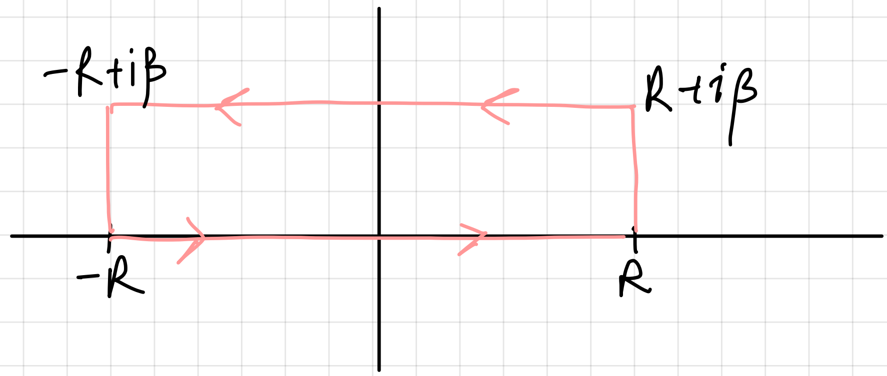
$$= \exp(-t^2 + \tfrac{1}{2}(2t - 2\pi i \xi)^2) \underbrace{\int_{x\in\mathbb{R}} dx \exp(-\tfrac{1}{2}(x - (2t - 2\pi i \xi))^2)}_{(*)\ \sqrt{2\pi}}$$

$$= \sqrt{2\pi} \exp(+t^2 - 4\pi i t \xi - 2\pi^2 \xi^2).$$

(*) | Claim: $\int_{x\in\mathbb{R}} \exp(-\tfrac{1}{2}(x - z)^2)\, dx = \sqrt{2\pi} \quad \forall z \in \mathbb{C}$

Proof: WLOG since $\mathbb{R}+\alpha=\mathbb{R}$ $\forall$ $\alpha\in\mathbb{R}$, we may assume $z=i\beta$ $\exists$ $\beta\in\mathbb{R}$.

Then shift the contour of int as follows:



To do so we must verify the vertical legs vanish as $R\to\infty$:

$$\left|\int_R^{R+i\beta}\exp\left(-\tfrac{1}{2}(z-i\beta)^2\right)dz\right| =$$

$$=\left|\int_0^{\beta}\exp\left(-\tfrac{1}{2}(R+it-i\beta)^2\right)i\,dt\right|$$

$$\leq \underbrace{\exp\left(-\tfrac{1}{2}R^2\right)}_{\to 0}\ \underbrace{\int_0^{\beta}\exp\left(+\tfrac{1}{2}(t-\beta)^2\right)dt}_{\text{Some fixed (w.r.t. } R\text{) constant.}}$$

$$\Rightarrow \int_{x\in\mathbb{R}}\exp\left(-\tfrac{1}{2}(x-z)^2\right)dx=\int_{x\in\mathbb{R}}\exp\left(-\tfrac{1}{2}x^2\right)dx=\sqrt{2\pi}.$$

$\blacksquare$

$$\Rightarrow \left(\mathcal{F}\left(x \mapsto e^{-\frac{1}{2}x^2} G(t,x)\right)\right)(\xi) =$$

$$= \sqrt{2\pi}\, \exp(+t^2 - 4\pi i t\xi - 2\pi^2 \xi^2)$$

Taylor expand in t

$$\overset{?}{=} \sqrt{2\pi}\, \exp(-2\pi^2\xi^2)\, \underbrace{\exp(t^2 - 4\pi i t\xi)}_{G(-it,\, 2\pi\xi)}$$

$$= \sqrt{2\pi}\, \exp(-\tfrac{1}{2}(2\pi\xi)^2)\, G(-it, 2\pi\xi)$$

Taylor expand in t

$$\overset{?}{=} \sqrt{2\pi}\, \exp(-\tfrac{1}{2}(2\pi\xi)^2) \sum_{n=0}^{\infty} \frac{1}{n!} \left[\partial_t^n\big|_{t=0} G(-it, 2\pi\xi)\right] t^n$$

$$\partial_t^n\big|_{t=0} G(-it, 2\pi\xi) = \left[\partial_t^n\big|_{t=0} G(t, 2\pi\xi)\right] (-i)^n$$

$$\equiv H_n(2\pi\xi)\, (-i)^n.$$

On the other hand,

$$\left(\mathcal{F}\left(x \mapsto e^{-\frac{1}{2}x^2} G(t,x)\right)\right)(\xi) =$$

$$= \left(\mathcal{F}\left(x \mapsto e^{-\frac{1}{2}x^2} \sum_{n=0}^{\infty} \frac{1}{n!} H_n(x)\, t^n\right)\right)(\xi) =$$

$\mathcal{F}$ is linear and cont.

$$\overset{?}{=} \sum_{n=0}^{\infty} \frac{1}{n!} t^n \left(\mathcal{F}\left(x \mapsto e^{-\frac{1}{2}x^2} H_n(x)\right)\right)(\xi).$$

Comparing the two power series term by term in t we find:

$$\left(\mathcal{F}\left(x \mapsto e^{-\frac{1}{2}x^2} H_n(x)\right)\right)(\xi) = \sqrt{2\pi}\,(-i)^n \exp\left(-\tfrac{1}{2}(2\pi\xi)^2\right) H_n(2\pi\xi)$$

Define: $\tilde{H}_n(x) := \exp(-\frac{1}{2}x^2) H_n(x)$.

$$(\mathcal{G}f)(\xi) := f\left(\frac{1}{2\pi}\xi\right).$$

We learn that

$$\boxed{\mathcal{G}\circ\mathcal{F}\,\tilde{H}_n = \sqrt{2\pi}\,(-i)^n\,\tilde{H}_n}\,.$$

$\Longrightarrow$ $\tilde{H}_n$ are L^2 eigenvectors of $\mathcal{G}\circ\mathcal{F}$ w/ eigenvalues $\sqrt{2\pi}\,(-i)^n$.

Note if we had used the unitary version of the Fourier transf.

$$(\tilde{\mathcal{F}}f)(\xi) := \frac{1}{\sqrt{2\pi}} \int_{x\in\mathbb{R}} e^{-i\xi x} f(x)\,dx$$

We would've found:

$$\boxed{\tilde{\mathcal{F}}\tilde{H}_n = (-i)^n \tilde{H}_n} \qquad (n\in\mathbb{N}).$$

I.e., the eigenvalues would indeed be of modulus 1.

$\boxed{Q5}$

$$f(x) = x^2$$

$$g(x) = \begin{cases} x^2 & x \neq 0 \\ 5 & x = 0 \end{cases}$$

By $\boxed{Q2}$,

$$(\mathcal{F}f)(\xi) = \frac{1}{(-2\pi i)^2}\delta''(\xi)$$

$$(\mathcal{F}^{-1}\mathcal{F}f)(x) = \int_{\xi\in\mathbb{R}} e^{+2\pi i\xi x}\frac{1}{(-2\pi i)^2}\delta''(\xi)\,d\xi$$

$$\overset{\text{IBP twice}}{\underset{\text{(need } \varepsilon \text{ regularization)}}{=}} x^2.$$

OTOH, since the integral in $\mathcal{F}g$ cannot depend on the value of g at one isolated point, $\mathcal{F}g = \mathcal{F}f$ and hence $(\mathcal{F}^{-1}\mathcal{F}g)(x) = x^2$!

$\boxed{Q6}$ $$(\mathcal{F}\chi_{[a,b]})(\xi) \equiv \int_a^b e^{-2\pi i \xi x}\, dx$$

$$= \frac{1}{-2\pi i \xi} e^{-2\pi i \xi x} \Big|_{x=a}^{b}$$

$$= \frac{e^{-2\pi i \xi a} - e^{-2\pi i \xi b}}{2\pi i \xi} .$$

Note if $a=-R$, $b=R$ we get

$$(\mathcal{F}\chi_{[-R,R]})(\xi) = \frac{\sin(2\pi R \xi)}{\pi \xi}$$

$$= \frac{\sin(2\pi R\xi)}{\sin(\pi \xi)} \frac{\sin(\pi\xi)}{\pi \xi}$$

$$\xrightarrow{R\to\infty} 2\pi \delta(\xi)$$

But we won't show how...

(See "Dirichlet kernel").

$\boxed{Q7}$

$$(\mathcal{F}\bar{f})(\xi) \equiv \int_{x\in\mathbb{R}} e^{-2\pi i \xi x}\, \overline{f(x)}\, dx$$

$$= \overline{\int_{x\in\mathbb{R}} e^{+2\pi i \xi x} f(x)\, dx}$$

$$\equiv \overline{(\mathcal{F}f)(-\xi)}\,.$$

$$\Rightarrow \boxed{\overline{\mathcal{F}\bar{f}} = (\mathcal{F}f)(-\bullet)}\,.$$

$$(\mathcal{F}f(\bullet - a))(\xi) = \int_{x\in\mathbb{R}} e^{-2\pi i \xi x} f(x-a)\, dx$$

$$= \int_{x\in\mathbb{R}} e^{-2\pi i \xi (x+a)} f(x)\, dx$$

$$= e^{-2\pi i \xi a} (\mathcal{F}f)(\xi)$$

$$\Rightarrow \boxed{(\mathcal{F}f(\bullet - a))(\xi) = e^{-2\pi i \xi a} (\mathcal{F}f)(\xi)}\,.$$

$\boxed{Q8}$

$$(\mathcal{F}f')(\xi) = \int_{x\in\mathbb{R}} e^{-2\pi i \xi x} f'(x)\, dx$$

$$\underset{\text{w/ } \varepsilon \text{ regularization}}{\overset{\text{IBP}}{=}} - \int_{x\in\mathbb{R}} (\partial_x e^{-2\pi i \xi x}) f(x)\, dx$$

$$= -\int_{x\in\mathbb{R}} (-2\pi i\xi)\, e^{-2\pi i \xi x} f(x)\,dx$$

$$= 2\pi i\xi\, (\mathcal{F} f)(\xi)$$

$$\Longrightarrow \boxed{\mathcal{F} f' = (2\pi i \,\bullet\,)\, \mathcal{F} f} .$$

$\boxed{\text{Q9}}$ Assume $A : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ has an integral kernel s.t.

$$A(x,y) = A(x+z, y+z) \qquad \forall\ x, y, z \in \mathbb{R}.$$

Claim: $(\mathcal{F} A \mathcal{F}^{-1} f)(\xi) = a(\xi) f(\xi)$

where $a(\xi) := \int_{x\in\mathbb{R}} e^{-2\pi i \xi x} A(x,0)\,dx.$

Proof: $(\mathcal{F} A \mathcal{F}^{-1} f)(\xi) \equiv \int_{x\in\mathbb{R}} e^{-2\pi i\xi x} (A\mathcal{F}^{-1} f)(x)\,dx$

$$\equiv \int_{x\in\mathbb{R}} e^{-2\pi i \xi x}\left[\int_{y\in\mathbb{R}} A(x,y)\,(\mathcal{F}^{-1} f)(y)\,dy\right] dx$$

replace x,y int. ⊛ $\overset{?}{=} \int_{y\in\mathbb{R}} \Big(\underbrace{\int_{x\in\mathbb{R}} e^{-2\pi i\xi x} \underbrace{A(x,y)}_{=A(x-y,0)} dx}_{\text{shift } x\mapsto x+y} \Big) (\mathcal{F}^{-1}f)(y)\,dy$

$$\int_{x\in\mathbb{R}} e^{-2\pi i\xi(x+y)} A(x,0)\,dx =$$

$$= e^{-2\pi i\xi y} \int_{x\in\mathbb{R}} e^{-2\pi i\xi x} A(x,0)\,dx$$

$$= e^{-2\pi i\xi y}\, a(\xi)$$

$$= a(\xi) \underbrace{\int_{y\in\mathbb{R}} e^{-2\pi i\xi y} (\mathcal{F}^{-1}f)(y)\,dy}_{= (\mathcal{F}\mathcal{F}^{-1}f)(\xi) = f(\xi).}$$

Justify ⊛: integral converges absolutely if $f\in L^2$ and

$$\sup_{y\in\mathbb{R}} \int_{x\in\mathbb{R}} |A(x,y)|\,dx < \infty$$

which is the case if $A\in L^{1,\infty}$. ∎

$\boxed{\text{Q 10}}$

$$\left(\mathcal{F}(-\Delta)\mathcal{F}^{-1}f\right)(\xi) \equiv \int_{x\in\mathbb{R}} e^{-2\pi i\xi x}\,(-\Delta\mathcal{F}^{-1}f)(x)\,dx$$

$$\equiv \int_{x\in\mathbb{R}} e^{-2\pi i\xi x}\left[-(\mathcal{F}^{-1}f)''(x)\right]dx$$

IBP twice w/ ε-reg.

$$\overset{?}{=} -\int_{x\in\mathbb{R}} \underbrace{\left(\partial_x^2\, e^{-2\pi i\xi x}\right)}_{(-2\pi i\xi)^2 e^{-2\pi i\xi x}}(\mathcal{F}^{-1}f)(x)\,dx$$

$$= (2\pi\xi)^2 \underbrace{\int_{x\in\mathbb{R}} e^{-2\pi i\xi x}(\mathcal{F}^{-1}f)(x)\,dx}_{(\mathcal{F}\mathcal{F}^{-1}f)(\xi)}$$

$$= 4\pi^2\xi^2 f(\xi)$$

$\Longrightarrow$ $\boxed{\mathcal{F}(-\Delta)\mathcal{F}^{-1} = \text{mul. by } \mathcal{E}(\xi) \quad \text{where } \mathcal{E}(\xi) := 4\pi^2\xi^2.}$

$\boxed{\text{Q11}}$

$$R(z) := (-\Delta - z\mathbb{1})^{-1}$$

$$\mathcal{F}R(z)\mathcal{F}^{-1} = \text{mul. by } \xi \mapsto (4\pi^2\xi^2 - z)^{-1}.$$

Hence the integral kernel is

$$(R(z))(x,y) = \int_{\xi\in\mathbb{R}} e^{+2\pi i \xi (x-y)} (\hat{R}(z))(\xi)\, d\xi$$

$$= \int_{\xi\in\mathbb{R}} e^{2\pi i \xi(x-y)} \left(4\pi^2\xi^2 - z\right)^{-1} d\xi$$

$$4\pi^2\xi^2 - z = 4\pi^2\left(\xi - \frac{\sqrt{z}}{2\pi}\right)\left(\xi + \frac{\sqrt{z}}{2\pi}\right)$$

We are interested in real $x-y \mapsto x$.

Depending on the sign of x we either close the contour up or down w/ a semi-circle. if $x>0$, close it up so that

$$\int_{\xi\in\mathbb{R}} e^{2\pi i \xi x}\left(4\pi^2\xi^2 - z\right)^{-1} d\xi =$$

only one pole in upper half plane as $\mathrm{Im}\{z\} \neq 0$.

$$= 2\pi i \operatorname{res}_{\frac{\sqrt{z}}{2\pi}}\left(\xi \mapsto e^{2\pi i \xi x}\left(4\pi^2\xi^2 - z\right)^{-1}\right)$$

$$= \cancel{2\pi} i \exp\left(\cancel{2\pi} i \frac{\sqrt{z}}{\cancel{2\pi}} x\right) \frac{1}{\cancel{4\pi^2}\left(\frac{\sqrt{z}}{\cancel{2\pi}} + \frac{\sqrt{z}}{\cancel{2\pi}}\right)}$$

$$= \exp\left(i\sqrt{z}\, x\right) \frac{i}{2\sqrt{z}}.$$

Had we done the same exercise w/ $x<0$ we'd have found $\exp(i\sqrt{z}\,|x|)\frac{i}{2\sqrt{z}}$.

We find

$$\boxed{R(z)(x,y) = \frac{i e^{i\sqrt{z}\,|x-y|}}{2\sqrt{z}}}$$

$(z \in \mathbb{C}\setminus\mathbb{R})$
$x,y \in \mathbb{R}$.

Note If $z = E + i\varepsilon$, $E \geq 0$, then:

$$\sqrt{z} \approx \sqrt{E} + i\tilde{\varepsilon}$$

$$\Rightarrow R(E+i\varepsilon)(x,y) \overset{\varepsilon\to 0^+}{\approx} \frac{i e^{i(\sqrt{E}+i\varepsilon)|x-y|}}{2(\sqrt{E}+i\varepsilon)}$$

$$= \frac{i\, e^{i\sqrt{E}\,|x-y|}\, e^{-\varepsilon|x-y|}}{2\sqrt{E}+2i\varepsilon}$$

$$\xrightarrow{\varepsilon\to 0^+} \frac{i\, e^{i\sqrt{E}\,|x-y|}}{2\sqrt{E}}.$$

if $E<0$ $\exists$ exp. decay still!

$\boxed{Q12}$ **Claim:** $L^2([0,1]) \subseteq L^1([0,1])$

Proof: Let $f \in L^2([0,1])$.

$$\Longleftrightarrow \int_{[0,1]} |f|^2 < \infty .$$

$$\int_{[0,1]} |f| = \int_{[0,1]} |f| \cdot 1$$

$$\overset{\text{Cauchy-Schwarz}}{\leq} \underbrace{\left(\int_{[0,1]} |f|^2 \right)^{1/2}}_{= \|f\|_{L^2} < \infty} \underbrace{\left(\int_{[0,1]} 1^2 \right)^{1/2}}_{= 1}$$

$$< \infty .$$

$\blacksquare$

$\boxed{Q13}$ $x \mapsto \frac{e^{-x^2}}{\sqrt{x}}$ is L^1 @ 0 (bcs. $x \mapsto \frac{1}{\sqrt{x}}$ is int. @ 0) and L^1 @ ∞ bcs. $x \mapsto e^{-x^2}$ is int. @ ∞.

Squared, this is no longer integrable!

$\Rightarrow$ NOT L^2.

It is not L^∞ since it is not bounded at 0.

$\boxed{Q14}$ $f(x) := \dfrac{1}{1+x}$

Claim: $f \in L^2([0,\infty))$

Proof:

$$\int_0^\infty \left(\frac{1}{1+x}\right)^2 dx \leq$$

$$\leq \int_0^\infty \frac{1}{x^2+2x+1} dx = -\frac{1}{1+x}\Big|_{x=0}^{\infty} = 1 < \infty.$$

$\blacksquare$

Claim: $f \notin L^1([0,\infty))$

Proof:

$$\int_0^\infty \frac{1}{1+x} dx \geq \int_1^\infty \frac{1}{1+x} dx$$

$$\overset{1+x \leq 2x}{\geq} \int_1^\infty \frac{1}{2x} dx$$

$$= \log(\sqrt{x})\Big|_{x=1}^{\infty} = \infty.$$

$\blacksquare$

$\boxed{\text{Q15}}$ $$\sum_{n\in\mathbb{Z}}' \frac{1}{\pi}\frac{a}{a^2+n^2} \overset{\substack{a>0 \\ \text{Poisson} \\ \text{summ.}}}{=} \sum_{n\in\mathbb{Z}}' \left(\mathcal{F}\left(x\mapsto \frac{1}{\pi}\frac{a}{a^2+x^2}\right)\right)(n)$$

$\boxed{\text{Q1}}$ $$\overset{}{=} \sum_{n\in\mathbb{Z}}' \exp(-2\pi a|n|)$$

$$= 1+2\sum_{n=1}^{\infty}{}' e^{-2\pi a n}$$

geometric series

$$= 1+2\frac{1}{e^{2\pi a}-1}$$

$$= \operatorname{ctgh}(\pi a).$$

$$\boxed{\sum_{n\in\mathbb{Z}}' \frac{1}{\pi}\frac{a}{a^2+n^2} = \operatorname{ctgh}(\pi a)}$$

$\boxed{\text{Q16}}$ $$\sum_{n\in\mathbb{Z}}' (z+n)^{-\ell} = ? \quad \text{w/} \quad \ell\in\mathbb{N}_{\geq 2},\ \operatorname{Im}\{z\}>0$$

Using Poisson summation we need

$$\left(\mathcal{F}\left(x\mapsto (z+x)^{-\ell}\right)\right)(\xi) \equiv$$

$$\equiv \int_{x\in\mathbb{R}} e^{-2\pi i \xi x} (z+x)^{-\ell}\, dx$$

Say $\xi > 0$. Close contour downwards to get zero since $\mathrm{Im}\{z\} > 0$.

Say $\xi < 0$. Close contour upwards to get

$$2\pi i \operatorname{res}_{-z}\left(x \mapsto \frac{e^{-2\pi i \xi x}}{(z+x)^{\ell}} \right)$$

pole of order $\ell \geq 2$

$$\operatorname{res}_{-z}\left(x \mapsto \frac{e^{-2\pi i \xi x}}{(z+x)^{\ell}} \right) =$$

$$= \lim_{x\to -z} \frac{1}{(\ell-1)!} \partial_x^{\ell-1} (z+x)^{\ell} \frac{e^{-2\pi i \xi x}}{(z+x)^{\ell}}$$

$$= \lim_{x\to -z} \frac{1}{(\ell-1)!} (-2\pi i \xi)^{\ell-1} e^{-2\pi i \xi x}$$

$$= \frac{(-2\pi i \xi)^{\ell-1}}{(\ell-1)!} e^{+2\pi i \xi z}$$

$$\Rightarrow \sum_{n\in\mathbb{Z}}' (z+n)^{-\ell} = \sum_{n=1}^{\infty}' 2\pi i \frac{(-2\pi i n)^{\ell-1}}{(\ell-1)!} e^{+2\pi i n z}$$

$$= \frac{(2\pi i)^{\ell}(-1)^{\ell-1}}{(\ell-1)!} \sum_{n=1}^{\infty} n^{\ell-1} e^{2\pi i n z}$$

$$= \frac{(-1)^{\ell-1} 2\pi i}{(\ell-1)!} \sum_{n=1}^{\infty} \partial_z^{\ell-1} e^{2\pi i n z}$$

$$= \frac{(-1)^{\ell-1} 2\pi i}{(\ell-1)!} \partial_z^{\ell-1} \underbrace{\sum_{n=1}^{\infty} e^{2\pi i n z}}_{\text{geometric series}}$$

$$\frac{e^{2\pi i z}}{1-e^{2\pi i z}}$$

Related to poly logarithm & Hurwitz Zeta.

$\boxed{Q17}$

$$f(x) := \sum_{n\in\mathbb{Z}} e^{-\frac{1}{2}\lambda(n-x)^2} \qquad (\lambda>0)$$

$$\overset{\text{Poisson summation}}{=} \sum_{n\in\mathbb{Z}} \sqrt{\frac{2\pi}{\lambda}} \exp\left(-\frac{1}{2}\frac{(2\pi)^2}{\lambda}n^2 - 2\pi i x n\right).$$

$$\Rightarrow |f(x)| \leq \underbrace{\sum_{n\in\mathbb{Z}} \sqrt{\frac{2\pi}{\lambda}} \exp\left(-\frac{1}{2}\frac{(2\pi)^2}{\lambda}n^2\right)}_{\text{independent of } x!}.$$

$$= \sqrt{\frac{2\pi}{\lambda}} \left(1 + 2 \underbrace{\sum_{n=1}^{\infty} e^{-\frac{1}{2}\frac{(2\pi)^2}{\lambda} n^2}}_{\leq e^{-\frac{1}{2}\frac{(2\pi)^2}{\lambda}} + \int_{x=1}^{\infty} e^{-\frac{1}{2}\frac{(2\pi)^2}{\lambda} x^2} dx} \right)$$

integral test for convergence

$$\leq e^{-\frac{1}{2}\frac{(2\pi)^2}{\lambda}} + \underbrace{\int_{x=0}^{\infty} e^{-\frac{1}{2}\frac{(2\pi)^2}{\lambda} x^2} dx}_{= \frac{1}{2}\sqrt{\frac{\lambda}{2\pi}}}$$

$$= \sqrt{\frac{2\pi}{\lambda}} \left(1 + 2\left(e^{-\frac{1}{2}\frac{(2\pi)^2}{\lambda}} + \frac{1}{2}\sqrt{\frac{\lambda}{2\pi}} \right) \right)$$

$$\boxed{\|f\|_{L^\infty} \leq \sqrt{\frac{2\pi}{\lambda}} + 1 + 2\sqrt{\frac{2\pi}{\lambda}}\, e^{-\frac{1}{2}\frac{(2\pi)^2}{\lambda}}}.$$