1 Contour integrals

1. Calculate the following integrals:
   
   (a) For any $a > 1$,
   
   \[ \int_0^{2\pi} \frac{1}{(a + \cos \theta)^2} \, d\theta. \]
   
   (b) for any $a, b \in \mathbb{R}$ such that $a > |b|$,
   
   \[ \int_0^{2\pi} \frac{1}{a + b \cos \theta} \, d\theta. \]
   
   (c) for any $a > 0$,
   
   \[ \int_0^{\infty} \frac{\log(x)}{x^2 + a^2} \, dx. \]
   
   (d) for any $a \in \mathbb{C}$ with $|a| \leq 1$,
   
   \[ \int_0^{2\pi} \log(|1 - ae^{i\theta}|) \, d\theta. \]
   
   (first show it for $|a| < 1$)
   
2. Calculate the series
   
   \[ \sum_{n \in \mathbb{Z}} \frac{1}{(u + n)^2} \]
   
   for $u \in \mathbb{C} \setminus \mathbb{Z}$ by integrating $z \mapsto \frac{\pi \cot(\pi z)}{(u + z)^2}$ over $\partial B_{N + \frac{1}{2}}(0)$ for some $N \in \mathbb{N}$ with $N \geq |u|$ as $N \to \infty$.

2 Set theory maintenance

Let $X, Y$ be two sets. A function $f : X \to Y$ is called injective (one-to-one) iff

\[ f(a) = f(b) \implies a = b \quad (a, b \in X). \]

$f$ is called surjective (onto) iff

\[ g \in Y \implies \exists x_g \in X : f(x_g) = y. \]

The identity mapping on $X$, denoted by $1_X : X \to X$ is the function that maps

\[ x \mapsto x \quad (x \in X). \]

A function $f : X \to Y$ is said to have a left-inverse iff there exists some $g : Y \to X$ such that

\[ g \circ f = 1_X. \]

A function $f : X \to Y$ is said to have a right-inverse iff there exists some $g : Y \to X$ such that

\[ f \circ g = 1_Y. \]

A function is bijective iff it is both injective and surjective.

3. Prove that $f : X \to Y$ is injective iff it has a left-inverse; prove that $f : X \to Y$ is surjective iff it has a right-inverse.
3 Conformal maps

4. Provide an example of a function \( f : U \to V \) (for some open \( U, V \subseteq \mathbb{C} \)) which is holomorphic and for which \( f' \neq 0 \), but which is not a conformal equivalence.

5. Find \( U, V \subseteq \mathbb{C} \) open such that \( f : U \to V \) is a conformal equivalence. Prove that it is so.

6. Solve the Dirichlet problem on the set 
\[
S := \{ z \in \mathbb{C} \mid \text{Re} \{ z \} \in \left(0, \frac{\pi}{2}\right) \land \text{Im} \{ z \} > 0 \}
\]
with boundary conditions 
\[
f : \partial S \to \mathbb{R} \quad z \mapsto \begin{cases} 1 & \text{Re} \{ z \} = 0 \\ 0 & \text{else} \end{cases}.
\]
That is, find the unknown function \( u : S \to \mathbb{R} \) such that 
\[
\begin{cases} -\Delta u = 0 \\ u|_{\partial S} = f. \end{cases}
\]

7. [extra] Find the flow \( V : \mathbb{R}^2 \to \mathbb{R}^2 \), draw its vector field, and describe the obstacle, for flows associated to the complex velocity potential \( f : \mathbb{C} \to \mathbb{C} \) given by:
   (a) \( f(z) = wz \) for some \( w \in \mathbb{C} \).
   (b) \( f(z) = wz^n \) for some \( w \in \mathbb{C} \) and \( n \in \mathbb{N}_{\geq 2} \).
   (c) \( f(z) = w\sqrt{z} \) for some \( w \in \mathbb{C} \).
   (d) \( f(z) = \frac{w}{2\pi i} \log(z) \) for some \( w > 0 \).

4 The maximum modulus principle

8. Let \( \Omega \subseteq \mathbb{C} \) be open and bounded and \( f, g : \Omega \to \mathbb{C} \) analytic and which extend to \( \partial \Omega \) continuously and furthermore satisfy 
\[
|f(z)| \leq |g(z)| \quad (z \in \partial \Omega)
\]
as well as 
\[
g(z) \neq 0 \quad (z \in \Omega).
\]
Show that 
\[
|f(z)| \leq |g(z)| \quad (z \in \Omega).
\]

9. A function \( f : X \to \mathbb{R} \) is called convex iff 
\[
f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2) \quad (t \in [0,1], x_1, x_2 \in X).
\]
A function \( f : X \to \mathbb{R} \) is called log-convex iff \( \log f : X \to \mathbb{R} \) is convex. Define the closed strip 
\[
S_{[a,b]} := \{ z \in \mathbb{C} \mid \text{Re} \{ z \} \in [a, b] \}.
\]
Let \( f : \text{interior} \left(S_{[a,b]}\right) \to \mathbb{C} \) be analytic and bounded, such that it extends continuously to \( \partial S_{[a,b]} \). Show that for fixed \( y \in \mathbb{R} \), \( |f(\cdot + iy)| : (a, b) \to [0, \infty) \) is log-convex. I.e., show that if there exist \( A, B \in (0, \infty) \) such that for all \( y \in \mathbb{R} \),
\[
|f(a+iy)| \leq A \\
|f(b+iy)| \leq B
\]
then 
\[
|f(z)| \leq A|\text{Re}(z) - a|^{\frac{a}{b-a}}B|\text{Re}(z) - a|^{\frac{b}{b-a}} \quad (z \in S_{[a,b]}).
\]
Find an example where this is an equality.