Complex Analysis with Applications
Princeton University MAT330
HW8, Due April 21st 2023

April 21, 2023

1 Contour integrals

1. Calculate the following integrals:
   (a) For any $a > 1$,
   $$\int_0^{2\pi} \frac{1}{(a+\cos(\theta))^2} \mathrm{d}\theta\,.$$
   (b) for any $a, b \in \mathbb{R}$ such that $a > |b|$,
   $$\int_0^{2\pi} \frac{1}{a+b\cos(\theta)} \mathrm{d}\theta\,.$$
   (c) for any $a > 0$,
   $$\int_0^{\infty} \frac{\log(x)}{x^2+a^2} \mathrm{d}x\,.$$
   (d) for any $a \in \mathbb{C}$ with $|a| \leq 1$,
   $$\int_0^{2\pi} \log\left(\left|1-a\mathrm{e}^{\mathrm{i}\theta}\right|\right) \mathrm{d}\theta\,.$$
   (first show it for $|a| < 1$)
2. Calculate the series
   $$\sum_{n\in\mathbb{Z}} \frac{1}{(u+n)^2}$$
   for $u \in \mathbb{C} \setminus \mathbb{Z}$ by integrating $z \mapsto \frac{\pi\cot(\pi z)}{(u+z)^2}$ over $\partial B_{N+\frac{1}{2}}(0)$ for some $N \in \mathbb{N}$ with $N \geq |u|$ as $N \to \infty$.

2 Set theory maintenance

Let X, Y be two sets. A function $f : X \to Y$ is called injective (one-to-one) iff
$$f(a) = f(b) \implies a = b \qquad (a, b \in X)\,.$$
f is called surjective (onto) iff
$$y \in Y \implies \exists x_y \in X : f(x_y) = y\,.$$
The identity mapping on X, denoted by $\mathbb{1}_X : X \to X$ is the function that maps
$$x \mapsto x \qquad (x \in X)\,.$$
A function $f : X \to Y$ is said to have a left-inverse iff there exists some $g : Y \to X$ such that
$$g \circ f = \mathbb{1}_X\,.$$
A function $f : X \to Y$ is said to have a right-inverse iff there exists some $g : Y \to X$ such that
$$f \circ g = \mathbb{1}_Y\,.$$
A function is bijective iff it is both injective and surjective.

3. Prove that $f : X \to Y$ is injective iff it has a left-inverse; prove that $f : X \to Y$ is surjective iff it has a right-inverse.

3 Conformal maps

4. Provide an example of a function $f : U \to V$ (for some open $U, V \subseteq \mathbb{C}$) which is holomorphic and for which $f' \neq 0$, but which is *not* a conformal equivalence.

5. Find $U, V \subseteq \mathbb{C}$ open such $\sin : U \to V$ is a conformal equivalence. Prove that it is so.

6. Solve the Dirichlet problem on the set

$$S := \left\{ z \in \mathbb{C} \,\middle|\, \mathbb{Re}\{z\} \in \left(0, \frac{\pi}{2}\right) \wedge \mathbb{Im}\{z\} > 0 \right\}$$

with boundary conditions

$$\begin{aligned} f : \partial S &\to \mathbb{R} \\ z &\mapsto \begin{cases} 1 & \mathbb{Re}\{z\} = 0 \\ 0 & \text{else} \end{cases} . \end{aligned}$$

That is, find the unknown function $u : S \to \mathbb{R}$ such that

$$\begin{cases} -\Delta u = 0 \\ u|_{\partial S} = f . \end{cases}$$

7. [extra] Find the flow $V : \mathbb{R}^2 \to \mathbb{R}^2$, draw its vector field, and describe the obstacle, for flows associated to the complex velocity potential $f : \mathbb{C} \to \mathbb{C}$ given by:

   (a) $f(z) = wz$ for some $w \in \mathbb{C}$.

   (b) $f(z) = wz^n$ for some $w \in \mathbb{C}$ and $n \in \mathbb{N}_{\geq 2}$.

   (c) $f(z) = w\sqrt{z}$ for some $w \in \mathbb{C}$.

   (d) $f(z) = \frac{w}{2\pi \mathrm{i}} \log(z)$ for some $w > 0$.

4 The maximum modulus principle

8. Let $\Omega \subseteq \mathbb{C}$ be open and bounded and $f, g : \Omega \to \mathbb{C}$ analytic and which extend to $\partial\Omega$ continuously and furthermore satisfy

$$|f(z)| \leq |g(z)| \qquad (z \in \partial\Omega)$$

as well as

$$g(z) \neq 0 \qquad (z \in \Omega) .$$

Show that

$$|f(z)| \leq |g(z)| \qquad (z \in \Omega) .$$

9. A function $f : X \to \mathbb{R}$ is called *convex* iff

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2) \qquad (t \in [0,1], x_1, x_2 \in X) .$$

A function $f : X \to \mathbb{R}$ is called *log-convex* iff $\log \circ f : X \to \mathbb{R}$ is convex. Define the closed strip

$$S_{[a,b]} := \{ z \in \mathbb{C} \mid \mathbb{Re}\{z\} \in [a,b] \} .$$

Let $f : \text{interior}\left(S_{[a,b]}\right) \to \mathbb{C}$ be analytic and bounded, such that it extends continuously to $\partial S_{[a,b]}$. Show that for fixed $y \in \mathbb{R}$, $|f(\cdot + \mathrm{i}y)| : (a,b) \to [0,\infty)$ is *log-convex.* I.e., show that if there exist $A, B \in (0,\infty)$ such that for all $y \in \mathbb{R}$,

$$\begin{aligned} |f(a + \mathrm{i}y)| &\leq A \\ |f(b + \mathrm{i}y)| &\leq B \end{aligned}$$

then

$$|f(z)| \leq A^{1-\frac{\mathbb{Re}\{z\}-a}{b-a}} B^{\frac{\mathbb{Re}\{z\}-a}{b-a}} \qquad \left(z \in S_{[a,b]}\right) .$$

Find an example where this is an equality.