APR 17 2023

MAT 330 $\mathbb{C}$-Anal. — HW8 Sample Sol-ns

$\boxed{Q1}$ (a) Let $a > 1$.

$$I(a) := \int_{\theta=0}^{2\pi} [a + \cos(\theta)]^{-2} d\theta .$$

Define $\lambda := e^{i\theta} \Rightarrow \cos(\theta) = \frac{1}{2}(\lambda + \frac{1}{\lambda})$

$$d\lambda = e^{i\theta} i d\theta$$

$$d\theta = \frac{1}{i\lambda} d\lambda$$

$$[a + \cos(\theta)]^{-2} d\theta = [a + \tfrac{1}{2}(\lambda + \tfrac{1}{\lambda})]^{-2} \frac{1}{i\lambda} d\lambda$$

$$= \left[\frac{1}{2\lambda}(2a\lambda + \lambda^2 + 1)\right]^{-2} \frac{1}{i\lambda} d\lambda$$

$$= (\lambda^2 + 2a\lambda + 1)^{-2} (-4i\lambda) \, d\lambda$$

and the intl. is over $\lambda \in \partial B_1(0)$.

$$f(\lambda) := \frac{-4i\lambda}{(\lambda^2 + 2a\lambda + 1)^2}$$

$$\Rightarrow \quad I(\lambda) = \oint_{\partial B_1(0)} f .$$

f is meromorphic and has two

2^{nd} order poles at $-a \pm \sqrt{a^2-1}$.

Clearly, since $a>1$, $-a-\sqrt{a^2-1} < -1$

and that pole is outside of $B_1(0)$.

OTOH, $\left|-a+\sqrt{a^2-1}\right| < 1$.

Indeed,
$$\begin{aligned}
-a+\sqrt{a^2-1} &> -1 \\
\sqrt{a^2-1} &> a-1 \\
a^2-1 &> (a-1)^2 \\
(a-1)(a+1) &> (a-1)^2 \\
a+1 &> a-1. \quad \checkmark \text{ (always)}
\end{aligned}$$

Also,
$$\begin{aligned}
-a+\sqrt{a^2-1} &< 1 \\
\sqrt{a^2-1} &< a+1 \\
a-1 &< a+1 \quad \checkmark \text{ (always)}.
\end{aligned}$$

For simplicity, $\lambda_\pm := -a \pm \sqrt{a^2-1}$.

$$\begin{aligned}
\operatorname{res}_{\lambda_+}(f) &\underset{2^{nd}\text{ order}}{=} \lim_{z\to\lambda_+} \partial_z (z-\lambda_+)^2 f(z) \\
&= \lim_{z\to\lambda_+} \partial_z \frac{-4iz}{(z-\lambda_-)^2} \\
&= \lim_{z\to\lambda_+} \frac{4i(z+\lambda_-)}{(z-\lambda_-)^3} \\
&= \frac{4i(\lambda_+ + \lambda_-)}{(\lambda_+ - \lambda_-)^3}.
\end{aligned}$$

We find:

$$I(a) = \oint_{\partial B_1(0)} f = 2\pi i \, \mathrm{res}_{\lambda_+}(f)$$

$$= 2\pi i \frac{4i(\lambda_+ + \lambda_-)}{(\lambda_+ - \lambda_-)^3}.$$

$$\lambda_+ + \lambda_- = -2a$$

$$\lambda_+ - \lambda_- = 2\sqrt{a^2-1}$$

$$\Rightarrow I(a) = 2\pi i \frac{4i(-2a)}{8(a^2-1)^{3/2}} = \frac{2\pi a}{(a^2-1)^{3/2}}.$$

Cf.: ① HW5Q15
② HW6Q6

(b) Let $a, b \in \mathbb{R}$: $a > |b|$.

$$I(a,b) := \int_{\theta=0}^{2\pi} [a + b\cos(\theta)]^{-1} \, d\theta.$$

Following a similar scheme we have

$$\lambda := e^{i\theta} \longrightarrow d\theta = \frac{1}{i\lambda} d\lambda$$

$$\Rightarrow I(a,b) = \oint_{\partial B_1(0)} \underbrace{[a + \tfrac{1}{2} b(\lambda + \tfrac{1}{\lambda})]^{-1} \frac{1}{i\lambda}}_{-2i(b\lambda^2 + b + 2a\lambda)^{-1}} d\lambda$$

Case 1: $b = 0$, whence $I(a,0) = \frac{2\pi}{a}$.

Case 2: $b \neq 0$, whence

$$f(z) := \frac{-2i/b}{z^2 + 2cz + 1} \qquad c := \frac{a}{b}$$

is meromorphic, has two poles at

$$\lambda_\pm \equiv -c \pm \sqrt{c^2 - 1}$$

and $a > |b| \iff \frac{a}{|b|} > 1 \iff |c| > 1$.

If $b > 0$ the situation is as before
$(\lambda_+ \in B_1(0), \lambda_- \notin B_1(0).)$
If $b < 0$, $c < 0$ and then

$$\lambda_+ > 1 \Rightarrow \lambda_+ \notin B_1(0)$$
$$|\lambda_-| < 1 \Rightarrow \lambda_- \in B_1(0).$$

We find

$$I(a,b) = \oint_{\partial B_1(0)} f = \begin{cases} \xrightarrow{b>0} 2\pi i \operatorname{res}_{\lambda_+}(f) \\ \xrightarrow{b<0} 2\pi i \operatorname{res}_{\lambda_-}(f) \end{cases}$$

Since these poles are simple we find:

$$\operatorname{res}_{\lambda_+}(f) = \lim_{z \to \lambda_+} (z - \lambda_+) f(z)$$
$$= \frac{-2i/b}{\lambda_+ - \lambda_-}$$

and $\operatorname{res}_{\lambda_-}(f) = \dfrac{-2i/b}{\lambda_- - \lambda_+}$

Thus together,

$$I(a,b) = \frac{4\pi/|b|}{\lambda_+ - \lambda_-} = \frac{4\pi/|b|}{2\sqrt{c^2-1}} = \frac{2\pi}{\sqrt{a^2-b^2}}.$$

(c) Let $a>0$.

$$I(a) := \int_{x=0}^{\infty} \frac{\log(x)}{x^2+a^2}\, dx .$$

First of all, clearly the integral abs. converges:

at $x=0$, it behaves like $\underbrace{\int_{x=0}^{\varepsilon} \frac{\log(x)}{\varepsilon^2+a^2}\, dx}_{\approx \frac{\varepsilon(\log(\varepsilon)-1)}{\varepsilon^2+a}} \xrightarrow{\varepsilon\to 0} 0$

at $x=\infty$, it behaves like

$$\frac{\log(x)}{x^2} \lesssim \frac{1}{x^{1.5}}$$

which is integrable at ∞.

For the actual explicit integration:

<u>First route:</u> Seek change of var $y=f(x)$

s.t. $\quad dy = f'(x)\, dx \overset{!}{=} \log(x)\, dx$

so $\quad f'(x) = \log(x)$

or $\quad f(x) = [\log(x) - 1]\, x$

Invert this relation?

$$x = f^{-1}(y) = \text{NOT GOOD}.$$

<u>Second route:</u> Consider the function

$$f(z) := \frac{\widetilde{\text{Log}}(z)}{z^2 + a^2}$$

which has a branch cut at our choice and two poles at $z = \pm i a$.

E.g. if we pick $\widetilde{\text{Log}}$ to have a cut on negative imaginary axis and take values $[-\frac{\pi}{2}, \frac{3\pi}{2})$ then for $x > 0$,

$$\widetilde{\text{Log}}(-x) = \log(x) + i\pi$$

and so, $\displaystyle\int_{x=-\infty}^{0} \frac{\widetilde{\text{Log}}(x)}{x^2+a^2}\, dx \underset{y := -x}{=} \int_{y=\infty}^{0} \frac{\widetilde{\text{Log}}(-y)}{y^2+a^2}\, (-dy)$

$$= \int_{y=0}^{\infty} \frac{\log(y) + i\pi}{y^2 + a^2}\, dy$$

$$= I(a) + i\pi \underbrace{\int_{y=0}^{\infty} \frac{1}{y^2+a^2}\, dy}_{\text{converges to smth.}}$$

$$\Rightarrow \frac{1}{2} \mathrm{Re}\left\{ \int_{x=-\infty}^{\infty} \frac{\widetilde{\mathrm{Log}}(x)}{x^2+a^2}\, dx \right\} = I(a) .$$

$$f(z) := \frac{\widetilde{\mathrm{Log}}(z)}{z^2+a^2}$$ meromorphic on $(B_R(0) \setminus B_\varepsilon(0)) \cap \mathbb{H}$.

Integrate then the indented upper-semicircle:

$$\oint_{\frown} f = 2\pi i \,\mathrm{res}_{ia}(f)$$

$$\overset{\text{simple pole}}{=} \cancel{2}\pi \cancel{i} \frac{\widetilde{\mathrm{Log}}(ia)}{\cancel{2}\cancel{i}a} = \frac{\pi}{a}\left(\log(a) + i\frac{\pi}{2} \right)$$

OTOH, on the big semi-circle we have:

$$\left| \int_{\frown} f \right| = \left| \int_{\theta=0}^{\pi} \frac{\widetilde{\mathrm{Log}}(Re^{i\theta})}{R^2 e^{2i\theta} + a^2} Re^{i\theta}\, i d\theta \right|$$

$$\leqslant \frac{R}{R^2 - a^2} \int_{\theta=0}^{\pi} |\log(R) + i\theta| \, d\theta$$

$$\leqslant \frac{\pi \log(R) R}{R^2 - a^2} + \frac{\frac{1}{2}\pi^2 R}{R^2 - a^2} \xrightarrow{R \to \infty} 0.$$

and on the small semi-circle

$$\left| \int_{\frown} f \right| = \left| \int_{\theta=0}^{\pi} \frac{\widetilde{\operatorname{Log}}(\varepsilon e^{i\theta})}{\varepsilon^2 e^{2i\theta} + a^2} \varepsilon e^{i\theta} \, i \, d\theta \right|$$

$$\leqslant \frac{\pi \log(\varepsilon) \varepsilon}{a^2 - \varepsilon^2} + \frac{\frac{1}{2}\pi^2 \varepsilon}{a^2 - \varepsilon^2} \xrightarrow{\varepsilon \to 0} 0.$$

(Note small semi-circle was necessary since $\widetilde{\operatorname{Log}}$ is NOT analytic at zero, where it has a branch point!)

$$\Rightarrow \quad I(a) = \frac{1}{2} \operatorname{Re} \left\{ \int_{x=-\infty}^{\infty} f(x) \, dx \right\}$$

$$= \frac{1}{2} \operatorname{Re} \left\{ \frac{\pi}{a} \left[\log(a) + i \frac{\pi}{2} \right) \right\}$$

$$= \frac{\pi \log(a)}{2a}.$$

(d) Let $a \in \overline{B_1(0)}$.

$$I(a) := \int_{\theta=0}^{2\pi} \log(|1-ae^{i\theta}|)\, d\theta$$

$$= \mathrm{Re}\left\{ \int_{\theta=0}^{2\pi} \widetilde{\mathrm{Log}}(1-ae^{i\theta})\, d\theta \right\}$$

$\gamma(\theta) := 1-ae^{i\theta}$

$\gamma'(\theta) = -ae^{i\theta} i$

$= i(-1+1-ae^{i\theta})$

$= i(-1+\gamma(\theta))$

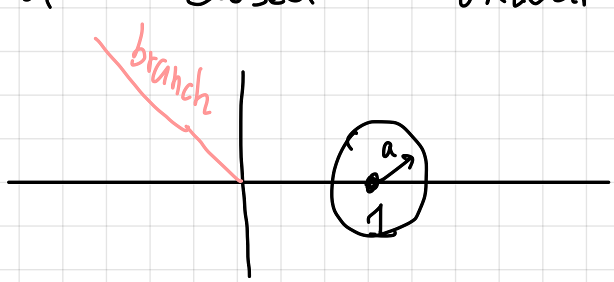
$$= \mathrm{Re}\left\{ \int_{\theta=0}^{2\pi} \widetilde{\mathrm{Log}}(\gamma(\theta)) \frac{1}{\gamma'(\theta)} \gamma'(\theta)\, d\theta \right\}$$

$$= \mathrm{Re}\left\{ \int_{\theta=0}^{2\pi} \widetilde{\mathrm{Log}}(\gamma(\theta)) \frac{1}{i(\gamma(\theta)-1)} \gamma'(\theta)\, d\theta \right\}$$

$$= \mathrm{Re}\left\{ \oint_{B_{|a|}(1)} \widetilde{\mathrm{Log}}(z) \frac{1}{i(z-1)}\, dz \right\}$$

If $|a|<1$, and we take $\widetilde{\mathrm{Log}}$ to have a branch cut away from right half space,

we get the integral of a meromorphic f^n on a closed contour

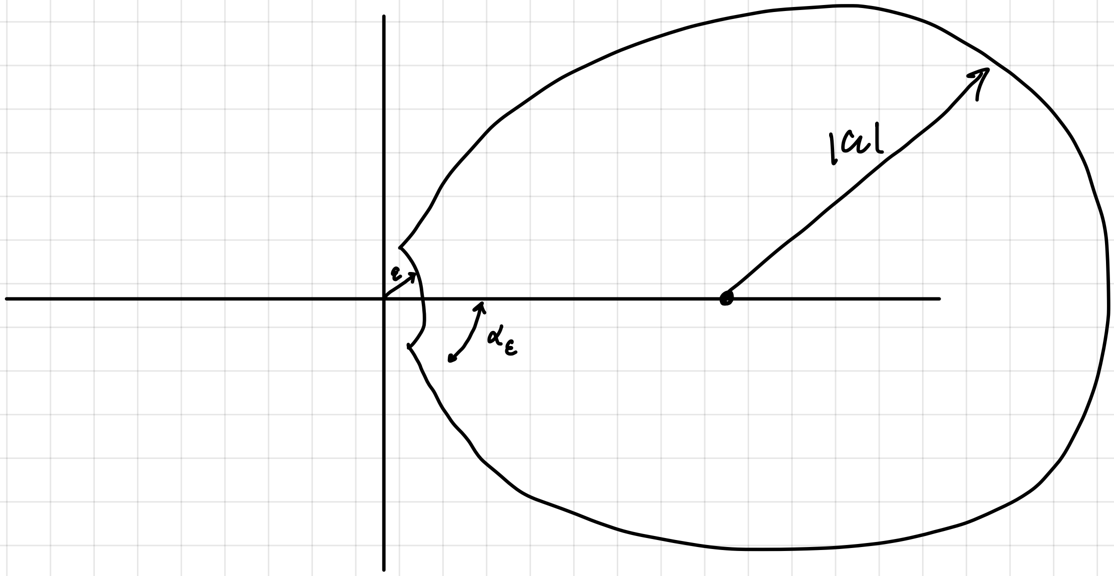


$$\Longrightarrow \ I(a) = 2\pi i \, \mathrm{res}_1\left(z \mapsto \frac{\widetilde{\mathrm{Log}}(z)}{i(z-1)}\right) = 2\pi \widetilde{\mathrm{Log}}(1) = 0.$$

(Note we could have also used HW4Q10 as a shortcut).

Next, if $|a|=1$, the contour passes through a branch pt. for $\widetilde{\mathrm{Log}}$.

Hence deform the contour around zero:



On small ε-arc we find:

$\alpha_\varepsilon \equiv$ the angle determined by ε

$$\left| \int_{\theta=-\alpha_\varepsilon}^{\alpha_\varepsilon} \frac{\widetilde{\mathrm{Log}}(\varepsilon e^{i\theta})}{i(\varepsilon e^{i\theta}-1)} \varepsilon e^{i\theta} \, i \, d\theta \right|$$

$$\leq \frac{\varepsilon}{1-\varepsilon} \int_{\theta=-\alpha_\varepsilon}^{\alpha_\varepsilon} \left(|\log(\varepsilon)| + |\theta| \right) d\theta \xrightarrow{\varepsilon \to 0} 0. \ \checkmark$$

$\Rightarrow \quad I(a) = 2\pi i \operatorname{res}_1(f) = 0$ for $|a|=1$ too.

$\boxed{Q2}$ $\quad S(u) := \sum_{n\in\mathbb{Z}} (n+u)^{-2} \qquad (u\in\mathbb{C}\setminus\mathbb{Z})$

Let $\quad f_u(z) := \dfrac{\pi\, \mathrm{ctg}(\pi z)}{(u+z)^2}$.

This f^u is meromorphic w/ simple poles on $\mathbb{Z}$ w/ res. equal to 1 there.

Indeed, in Example 7.39 we saw

$$\sum_{n\in\mathbb{Z}} g(n) = \lim_{R\to\infty} \frac{1}{2\pi i} \oint_{\Gamma_R} g(z)\, \pi\, \mathrm{ctg}(\pi z)\, dz$$

where Γ_R is any CCW closed contour containing $[-R,R]$ and within which g is analytic. We have a pole of order 2 at $z=-u$ which we would need to account for:

$$\operatorname{res}_{-u}(f) = \lim_{z\to u} \partial_z (z+u)^2 f(z)$$

$$= \lim_{z\to u} \partial_z \, \pi \operatorname{ctg}(\pi z)$$

$$= \lim_{z\to u} \frac{-\pi^2}{\sin(\pi z)^2} = -\frac{\pi^2}{\sin(\pi u)^2}.$$

Now on a big circle we have

$$\left| \int_{\theta=0}^{2\pi} f(Re^{i\theta}) \, R e^{i\theta} \, i \, d\theta \right| =$$

$$= \left| \int_{\theta=0}^{2\pi} \frac{\pi \operatorname{ctg}(\pi R e^{i\theta})}{(Re^{i\theta}+u)^2} \, R e^{i\theta} \, i \, d\theta \right|$$

$|Re^{i\theta}+u| \geq R - |u|$

$\Rightarrow |(Re^{i\theta}+u)^2| \geq (R-|u|)^2$

$$\leq \frac{\pi R}{(R-|u|)^2} \int_{\theta=0}^{2\pi} |\operatorname{ctg}(\pi R e^{i\theta})| \, d\theta$$

Claim: $\sup_{N\in\mathbb{N}} \int_{\theta=0}^{2\pi} |\operatorname{ctg}(\pi(N+\frac{1}{2}) e^{i\theta})| \, d\theta < \infty$.

Proof: Divide $[0, 2\pi]$ into two regions:

$$\mathrm{I}: \quad \{\theta \in [0, 2\pi] \mid |\sin(\theta)| < \varepsilon\}$$

$$\mathrm{II}: \quad \{\theta \in [0, 2\pi] \mid |\sin(\theta)| \geqslant \varepsilon\}.$$

We have

$$ctg(z) \equiv \frac{\cos(z)}{\sin(z)} = \frac{e^{iz} + e^{-iz}}{i(e^{iz} - e^{-iz})} = \frac{e^{2iz}+1}{i(e^{2iz}-1)}$$

and so

$$ctg(\pi R e^{i\theta}) = \frac{e^{2i\pi R e^{i\theta}} + 1}{i(e^{2i\pi R e^{i\theta}} - 1)} =$$

$$= \frac{e^{2i\pi R\cos(\theta)}\, e^{-2\pi R \sin(\theta)} + 1}{i\left(e^{2i\pi R\cos(\theta)}\, e^{-2\pi R\sin(\theta)} - 1\right)}$$

On II,

$$|ctg(\pi R e^{i\theta})| \leqslant \frac{e^{-2\pi R\sin(\theta)} + 1}{|1 - e^{-2\pi R\sin(\theta)}|}$$

$|x+y| \leqslant |x|+|y|$

$|x+y| \geqslant \big||x|-|y|\big|$

if $\sin(\theta) \geqslant \varepsilon$, $\quad |ctg(\pi R e^{i\theta})| \leqslant \dfrac{1 + e^{-2\pi R \varepsilon}}{1 - e^{-2\pi R\varepsilon}}$

if $\sin(\theta) \leqslant -\varepsilon$, $|\mathrm{ctg}(\pi R e^{i\theta})| \leqslant \dfrac{1+e^{2\pi R\sin(\theta)}}{1-e^{2\pi R\sin(\theta)}}$

$$\leqslant \frac{1+e^{-2\pi R\varepsilon}}{1-e^{-2\pi R\varepsilon}}$$

$$\Rightarrow \int_{\theta\in\{\theta\in[0,2\pi] \,|\, |\sin(\theta)|\geqslant\varepsilon\}} |\mathrm{ctg}(\pi R e^{i\theta})|\, d\theta \leqslant$$

$$\leqslant 2\,\frac{1+e^{-2\pi R\varepsilon}}{1-e^{-2\pi R\varepsilon}}\,(\pi-2\varepsilon)$$

On I, we use $R=N+\frac{1}{2}$, since then for $\theta=0,\pi$, $e^{2\pi i R\cos(\theta)} \approx -1$ and the denominator avoids zero.

More precisely,

$$\left|e^{2i\pi R\cos(\theta)}\, e^{-2\pi R\sin(\theta)} - 1\right|^2 =$$

$$= 1+e^{-4\pi R\sin(\theta)} - 2e^{-2\pi R\sin(\theta)}\cos\left(2\pi R\cos(\theta)\right)$$

$$= \left(1-e^{-2\pi R\sin(\theta)}\right)^2 + 2e^{-2\pi R\sin(\theta)}\left[1-\cos(2\pi R\cos(\theta))\right]$$

$$\geqslant 4 e^{-2\pi R \sin(\theta)} \left[\sin(\pi R \cos(\theta))\right]^2$$

Now, $e^{-2\pi R \sin(\theta)} \geqslant e^{-2\pi R \varepsilon}$

$$\sin(\pi R \cos(\theta)) = \sin(\pi(N+\tfrac{1}{2})\cos(\theta))$$

MVT: $\left|\sin(\pi(N+\tfrac{1}{2})\cos(\theta)) - 1\right| \leqslant \pi|\theta|^2 R \leqslant \pi \varepsilon^2 R$

$$\Rightarrow \left|e^{2i\pi R\cos(\theta)} e^{-2\pi R \sin(\theta)} - 1\right| \geqslant 2 e^{-\pi R \varepsilon}(1-\pi\varepsilon^2 R).$$

Hence

$$\int_{\theta \in \{\theta \in [0,2\pi] \,)\, |\sin(\theta)| < \varepsilon\}} |\operatorname{ctg}(\pi R e^{i\theta})| \, d\theta \leqslant$$

$$\leqslant 2\varepsilon \frac{1 + e^{2\pi R \varepsilon}}{2 e^{-\pi R \varepsilon}(1-\pi\varepsilon^2 R)}.$$

If we pick e.g. $\varepsilon := \frac{1}{R^3}$ both terms will be bdd. $\forall$ R large. ∎

$$\Rightarrow \lim_{R\to\infty} \frac{\pi R}{(R-|w|)^2} \int_{\theta=0}^{2\pi} |\operatorname{ctg}(\pi R e^{i\theta})| \, d\theta = 0.$$

$$\Longrightarrow 0 = \lim_{R\to\infty} \oint_{\partial B_R(0)} f = \lim_{R\to\infty} 2\pi i \sum_{\substack{z \in \text{poles} \\ \text{in } B_R(0)}} \operatorname{res}_z(f)$$

$$\Longrightarrow -\operatorname{res}_{-u}(f) = \sum_{n\in\mathbb{Z}} \operatorname{res}_n(f)$$

$$\boxed{\frac{\pi^2}{\sin(\pi u)^2} = \sum_{n\in\mathbb{Z}} (u+n)^{-2}} .$$

$\boxed{Q3}$

Claim: $f: X \to Y$ is injective $\Leftrightarrow$ $\exists\ g: Y \to X$: $g\circ f = \mathbb{1}_X$

Proof: $\boxed{\Rightarrow}$ If $X = \emptyset$ there's nothing to prove.

For any $y \in Y$, $f^{-1}(\{y\}) \equiv \{x \in X \mid f(x) = y\}$ is at most a one element set.

Indeed, if $x, \tilde{x} \in f^{-1}(\{y\})$,

$$f(x) = y = f(\tilde{x}) \overset{\text{inj.}}{\Longrightarrow} x = \tilde{x}.$$

Let $x_y \in X$ be the unique el. in $f^{-1}(\{y\})$ if it $\exists$.

Define $g: Y \to X$ via

$$y \longmapsto \begin{cases} x_y & x_y \text{ exists.} \\ x_0 & f^{-1}(\{y\}) = \emptyset. \end{cases}$$

By the above, g is well-def.

Also, $(g \circ f)(x) = x$!

$\boxed{\Leftarrow}$ Let $x, \tilde{x} \in X$: $f(x) = f(\tilde{x})$.

Apply g on both sides to get $x = \tilde{x}$. So f is injective.

$\blacksquare$

<u>Claim</u>: f is surjective $\Leftrightarrow \exists\, g: Y \to X$: $f \circ g = \mathbb{1}_Y$.

<u>Proof</u>: $\boxed{\Rightarrow}$ Since $f(X) = Y$, $f^{-1}(\{y\}) \neq \emptyset$ for any $y \in Y$. So define

$$g: Y \to X$$

by picking for any $y \in Y$ an arbitrary $x \in f^{-1}(\{y\})$ (fixed one.). This g is by construction a right inverse to f.

$\boxed{\Leftarrow}$ Let $y \in Y$. Then $f(g(y)) = y$.

So any point $y \in Y$ has an origin $g(y) \in X$ which covers it.

$\blacksquare$

$\boxed{Q4}$ See Example 9.7.

$\boxed{Q5}$ Want $\mathcal{U}, V \in Open(\mathbb{C})$;

$$\sin : \mathcal{U} \to V$$

would be a conformal equiv.

sin is entire so we just need to verify a bijection.

$$\sin(z) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$$

Pick $\mathcal{U} := \{ z \in \mathbb{C} \mid \Re\{z\} \in (0, \tfrac{\pi}{2}) \wedge \Im\{z\} > 0 \}$

$V := \{ z \in \mathbb{C} \mid \Re\{z\}, \Im\{z\} > 0 \} \equiv Q_1$

1st quadrant

Write $\sin(z) \equiv \frac{1}{2i}(e^{iz} - e^{-iz})$

$$\overset{\lambda := e^{iz}}{=} \frac{1}{2i}\left(\lambda - \frac{1}{\lambda}\right)$$

$$= -\frac{1}{2}\left(i\lambda - i\frac{1}{\lambda}\right) = -\frac{1}{2}\left(i\lambda + \frac{1}{i\lambda}\right).$$

Claim: $f: B_1(0) \cap \mathbb{H} \to \mathbb{H}$ (Jukowski map
$z \mapsto -\frac{1}{2}(z+\frac{1}{z})$ from HW6Q6)

is a conformal equivalence.

Proof: **Note:**
$$\frac{\left(\frac{z-1}{z+1}\right)^2+1}{\left(\frac{z-1}{z+1}\right)^2-1} = \frac{(z-1)^2+(z+1)^2}{(z-1)^2-(z+1)^2} = \frac{2z^2+2}{-4z}$$
$$= -\frac{1}{2}\left(z+\frac{1}{z}\right) = f(z).$$

Define $g: \mathbb{H} \cap B_1(0) \to Q_2$ ← 2nd quadrant
$$z \mapsto \frac{z-1}{z+1}$$

$S: Q_2 \to Q_3 \cup Q_4$ ← lower half plane
$$z \mapsto z^2$$

$h: Q_3 \cup Q_4 \to \mathbb{H}$
$$z \mapsto \frac{z+1}{z-1}$$

$$f = h \circ S \circ g$$

So will show each of the three maps is a conformal equiv.

Claim: $g: B_1(0) \cap \mathbb{H} \to Q_2$ is a conf. equiv.

Proof: We've seen g in Example 9.4 that $\mathbb{H} \cong B_1(0)$ via
$$z \mapsto \frac{i-z}{i+z}$$

with inverse $w \mapsto i\frac{1-w}{1+w} = -i\underbrace{\frac{w-1}{w+1}}_{g(w)}$

rotation by $90°$

restriction of injective map is injective

Hence by that same proof when restricting the inverse from $B_1(0)$ to $\mathbb{H} \cap B_1(0)$ we obtain a map whose range is Q_2 and is still inj.

We've seen there e.g. $\mathrm{Im}\{-i\frac{w-1}{w+1}\} > 0$

$$\Leftrightarrow \mathrm{Im}\{-i\, g(w)\} > 0$$

$$\Leftrightarrow -\mathrm{Re}\{g(w)\} > 0$$

$$\Leftrightarrow \mathrm{Re}\{g(w)\} < 0$$

$\Rightarrow \mathrm{im}(g) \subseteq$ left half plane.

Moreover,

$$\mathrm{Im}\{g(w)\} = \mathrm{Im}\left\{\frac{w-1}{w+1}\right\} = \mathrm{Im}\left\{\frac{(w-1)(\overline{w+1})}{|w+1|^2}\right\}$$

$$= |w+1|^{-2}\,\mathrm{Im}\{|w|^2 - 1 + 2i\,\mathrm{Im}\{w\}\}$$

$$= |w+1|^{-2}\, 2\,\mathrm{Im}\{w\} > 0$$

$\Rightarrow \mathrm{im}(g) \subseteq Q_2$. ∎

Claim: $S: Q_2 \to Q_3 \cup Q_4$ is a conf. equiv.

Proof: Properly defining $\widetilde{\operatorname{Arg}}$ yields the correct inverse. $\blacksquare$

Claim: $h: Q_3 \cup Q_4 \longrightarrow \mathbb{H}$ is a conf. equiv.

Proof:
$$\operatorname{Im}\left\{\frac{z+1}{z-1}\right\} = \operatorname{Im}\left\{\frac{(z+1)(\bar{z}-1)}{|z-1|^2}\right\}$$
$$= |z-1|^{-2}\operatorname{Im}\left\{|z|^2 - 1 - 2i\operatorname{Im}\{z\}\right\}$$
$$= |z-1|^{-2}(-2)\underbrace{\operatorname{Im}\{z\}}_{<0} > 0 \checkmark$$

$\Rightarrow$ well-def.

inverse is $\quad w \mapsto \dfrac{w+1}{w-1}$. $\blacksquare$

$\blacksquare$

Claim: $z \mapsto e^{iz}$ is a conf. equiv. from $\mathcal{U} \to B_1(0) \cap Q_1$

Proof: $e^{i(x+iy)} = e^{ix}e^{-y}$

$x \in (0, \frac{\pi}{2}) \Rightarrow Q_1$

$y \in (0, \infty) \Rightarrow e^{-y} \leq 1 \Rightarrow B_1(0)$

Since sin is the composition of maps
$$\mathcal{U} \xrightarrow{z \mapsto e^{iz}} B_1(0) \cap Q_1 \xrightarrow{z \mapsto iz} B_1(0) \cap Q_2 \xrightarrow{z \mapsto -\frac{1}{2}(z + \frac{1}{z})} Q_1$$
we find the result.

Q6

Solve the Dirichlet problem on the set

$$S := \left\{ z \in \mathbb{C} \,\middle|\, \mathbb{Re}\{z\} \in \left(0, \frac{\pi}{2}\right) \wedge \mathbb{Im}\{z\} > 0 \right\}$$

with boundary conditions

$$\begin{aligned} f : \partial S &\to \mathbb{R} \\ z &\mapsto \begin{cases} 1 & \mathbb{Re}\{z\} = 0 \\ 0 & \text{else} \end{cases} . \end{aligned}$$

That is, find the unknown function $u : S \to \mathbb{R}$ such that

$$\begin{cases} -\Delta u = 0 \\ u|_{\partial S} = f \, . \end{cases}$$

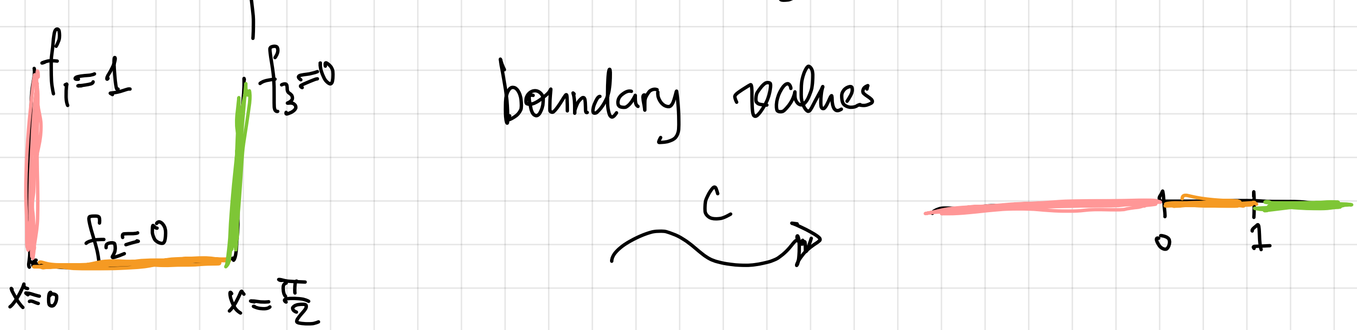
By Lemma 9.16 we are looking for a conformal equiv. $c : S \to B_1(0)$ or $c : S \to \mathbb{H}$

Since we know how to solve the Dirichlet problem on $B_1(0)$ and on $\mathbb{H}$.

Consider $c(z) := \sin(z)^2$.

$$\left.\begin{array}{l} \sin : S \to Q_1 \\ \text{and } \cdot^2 : Q_1 \to \mathbb{H} \end{array}\right\} \text{two conformal equiv.}$$

Let $f : \partial S \to \mathbb{R}$ be given.



$x=0$ gets mapped to $[\sin(iy)]^2 = -\sinh(y)^2$

< 0

$y=0$ gets mapped to $[\sin(x)]^2 \in [0,1]$

$x=\frac{\pi}{2}$ gets mapped to $[\sin(\frac{\pi}{2}+iy)]^2 = \cosh(y)^2 \geq 1$

Hence according to Lemma 9.16 and (9.2),

$$u(z) = (v \circ c)(z)$$

where $v : \mathbb{H} \to \mathbb{R}$ is the sol-n of

$-\Delta v = 0$ w/ B.c. $v|_{\partial\mathbb{H}} = f \circ c^{-1}$.

$$\Rightarrow \quad v(z) = \int_{t=-\infty}^{\infty} dt \, (f \circ c^{-1})(t) \underbrace{\frac{1}{\pi} \frac{y}{(x-t)^2+y^2}}_{\text{Poisson kernel}}$$

$c^{-1} : \partial\mathbb{H} \to \partial S$ is as depicted in the picture above.

Since $f \neq 0$ only on pink region, where it equals 1, we get

$$v(z) = \int_{t=-\infty}^{0} dt \, \frac{1}{\pi} \frac{y}{(x-t)^2+y^2}$$

$$\Longrightarrow \quad u(z) = (v \circ c)(z)$$

$$\mathrm{Re}\left\{ \left[\sin(x+iy) \right]^2 \right\} = \frac{1}{2} - \frac{1}{2} \cos(2x) \cosh(2y)$$

$$\mathrm{Im}\left\{ \left[\sin(x+iy) \right]^2 \right\} = \frac{1}{2} \sin(2x) \sinh(2y)$$

$$\Longrightarrow$$

$$\boxed{u(x+iy) = \int_{t=-\infty}^{0} dt \; \frac{1}{\pi} \; \frac{\frac{1}{2}\sin(2x)\sinh(2y)}{\left(\frac{1}{2} - \frac{1}{2}\cos(2x)\cosh(2y) - t\right)^2 + \frac{1}{4}\sin(2x)^2 \sinh(2y)^2}} \, .$$

$\boxed{\text{Q7}}$ is [extra] credit and its sol-n will appear later.

$\boxed{Q8}$ $\Omega \in Open(\mathbb{C})$ and bounded.

$f, g : \Omega \to \mathbb{C}$ analytic and extend cont. to $\partial\Omega$.

Assume ① $|f(z)| \leq |g(z)|$ $(z \in \partial\Omega)$

② $g \neq 0$ on Ω

Claim: $|f(z)| \leq |g(z)|$ $(z \in \Omega)$.

Proof: Since $g \neq 0$ we may divide by g to get $h := f/g$ also analytic and bdd.

If h is const. we're finished.

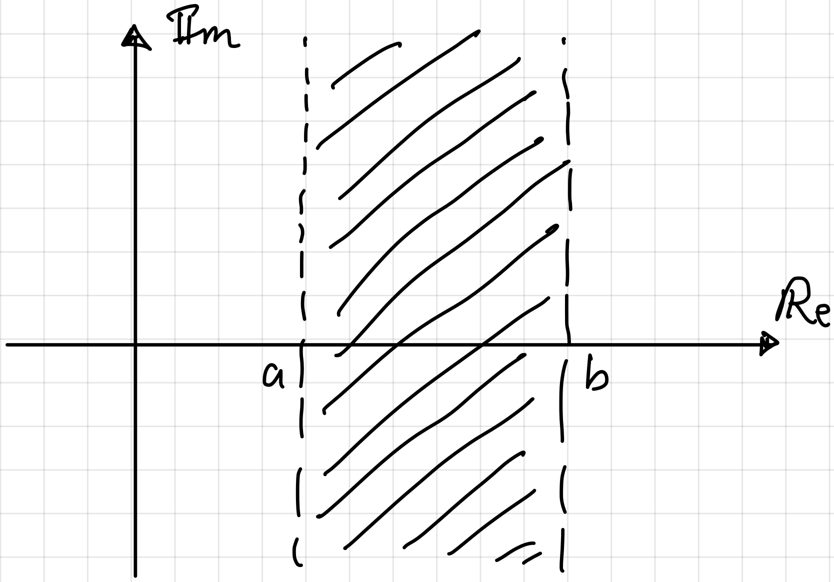
Else, know $|h| \leq 1$ on $\partial\Omega$.

Want $|h| \leq 1$ on Ω.

But $\sup_{z \in \Omega} |h(z)| > 1$, since $\overline{\Omega}$ is compact and $|h| : \overline{\Omega} \to [0, \infty)$ is cont., $|h|$ attains its max on $\overline{\Omega}$.

But apparently this max is within Ω, which is in contradiction to the max mod. principle. $\boxtimes$

$\boxed{Q9}$



Let $f: S_{(a,b)} \to \mathbb{C}$ analytic and bdd. s.t. it extends cont. to $S_{[a,b]}$.

Claim: For any fixed $y \in \mathbb{R}$,

$$|f(\cdot + iy)| : (a,b) \longrightarrow [0,\infty)$$

is log-convex. (Hadamard 3-line lemma)

Proof: WTS $\forall$ $x_1, x_2 \in (a,b)$, $t \in [0,1]$

$$\log(|f(tx_1 + (1-t)x_2 + iy)|) \leq t\log(|f(x_1+iy)|) + (1-t)\log(|f(x_2+iy)|)$$

Taking an exp of this eq-n we find:

$$|f(tx_1 + (1-t)x_2 + iy)| \leq |f(x_1+iy)|^t \, |f(x_2+iy)|^{1-t}$$

Define

$$F : S_{(a,b)} \to \mathbb{C} \quad \text{via}$$

$$F(z) := f(z)\, |f(a+iy)|^{\frac{z-b}{b-a}}\, |f(b+iy)|^{\frac{z-a}{a-b}}$$

Claim: $|F(z)| \leq 1 \quad \forall \quad z \in \partial S_{(a,b)}$

Proof: If $z = a+iy$

$$\frac{z-b}{b-a} = \frac{a+iy-b}{b-a} = -1 + i\frac{y}{b-a}$$

$$\frac{z-a}{a-b} = \frac{iy}{a-b}$$

$$|F(a+iy)| = |f(a+iy)|^{i\frac{y}{b-a}}\, |f(b+iy)|^{i\frac{y}{a-b}}$$

$$a^{i\alpha} \equiv \exp(\log(a^{i\alpha}))$$
$$= \exp(i\alpha \log(a))$$

$$\Rightarrow |a^{i\alpha}| = 1. \Rightarrow |F(a+iy)| \leq 1 \checkmark$$

If $z = b+iy$,

$$\frac{z-b}{b-a} = i\frac{y}{b-a}$$

$$\frac{z-a}{a-b} = \frac{b+iy-a}{a-b} = -1 + i\frac{y}{a-b}$$

$$\Rightarrow \quad |F(b+iy)| = |f(a+iy)|^{i\frac{y}{b-a}} |f(b+iy)|^{i\frac{y}{a-b}} \quad ▣$$

Now we'd like to say $|F(z)| \leqslant 1$ also in the interior.

But F is analytic on open domain, and on the boundary $|F| \leqslant 1$.

So by max mod. princ., $|F| \leqslant 1$ also in interior.

$$\Rightarrow \quad \forall \quad x \in [a,b], \; y \in \mathbb{R}$$

$$|F(x+iy)| \leqslant 1$$

$$\Leftrightarrow \left| f(z) \, |f(a+iy)|^{\frac{z-b}{b-a}} |f(b+iy)|^{\frac{z-a}{a-b}} \right| \leqslant 1$$

For $p > 0$, $\quad |p^z| \equiv |\exp(z \log(p))|$

$$= \exp(\mathrm{Re}\{z\} \log(p))$$

$$= p^{\mathrm{Re}\{z\}}.$$

$$\Rightarrow \left| \, |f(a+iy)|^{\frac{z-b}{b-a}} |f(b+iy)|^{\frac{z-a}{a-b}} \right| =$$

$$= |f(a+iy)|^{\frac{x-b}{b-a}} \quad |f(b+iy)|^{\frac{x-a}{a-b}}$$

$\Rightarrow$ $|F(z)| \leq 1$ implies

$$|f(z)| \leq |f(a+iy)|^{-\frac{x-b}{b-a}} \; |f(b+iy)|^{-\frac{x-a}{a-b}}$$

$$-\frac{x-b}{b-a} = \frac{b-x}{b-a} = \frac{b-a+a-x}{b-a} = 1 + \frac{a-x}{b-a}$$

$$= 1 - \frac{x-a}{b-a}$$

and $-\frac{x-a}{a-b} = \frac{x-a}{b-a}$

$\Rightarrow$

$$\boxed{|f(x+iy)| \leq |f(a+iy)|^{1-\frac{x-a}{b-a}} \; |f(b+iy)|^{\frac{x-a}{b-a}}}\;.$$

This box shows two things:

① Since a, b were arbitrary, could replace them by $x_1 < x_2$ to get the log-convexity we set out to prove.

② The more specific claim in the problem: if $|f(a+iy)| \leq A$, $|f(b+iy)| \leq B$

$|f(x+iy)| \leq A^{1-\frac{x-a}{b-a}} B^{\frac{x-a}{b-a}}$ follows immediately.

$\blacksquare$