Complex Analysis with Applications
Princeton University MAT330
HW7, Due April 14th 2023

April 15, 2023

1 Contour integrals

1. Calculate the following integrals:
   (a) $\int_{x\in\mathbb{R}} \frac{e^{ix}}{x+2i}dx$.
   (b) $\int_{x=0}^{\infty} \frac{1}{1+x^2}dx$.
   (c) $\int_{x=0}^{\infty} \frac{\cos(x)}{1+x^4}dx$.
   (d) $\int_{x\in\mathbb{R}} \frac{x}{(x^2+1)(x^2+2x+2)}dx$.
   (e) For $a>0$, $\int_{x=0}^{\infty} \frac{\cos(ax)}{1+x^2}dx$
   (f) For $a>0$, $\int_{x\in\mathbb{R}} \frac{x\sin(ax)}{4+x^4}dx$.
   (g) For $a\in\mathbb{C}$, $\int_{x=0}^{\infty} \frac{x^{-a}}{1+x}dx$.
   (h) $\int_{x=0}^{\infty} \frac{1}{\sqrt{x}(1+x^2)}dx$.
   (i) For $n\in\mathbb{N}$, $\int_{x\in\mathbb{R}} \frac{1}{(1+x^2)^{n+1}}dx$.
   (j) $\int_0^1 \log(\sin(\pi x))\,dx$.

2 Rouché and the maximum modulus principle

2. Show that for $a>e$, the equation $az^n = e^z$ has n roots inside $|z|=1$.
3. Prove using the maximum modulus principle or any other way that if f is as an entire function such that for some $k\in\mathbb{N}$, $A,B\in(0,\infty)$

$$\sup_{|z|=R} |f(z)| \quad \leq \quad AR^k + B \qquad (R>0)$$

   then f is a polynomial of degree at most k.
4. Using the maximum modulus principle or however you like, show that if $f:\mathbb{C}\to\mathbb{C}$ is entire such that $f_R \equiv \mathbb{Re}\{f\}$ is bounded then f is constant. This question is very similar to HW5Q17.

3 Fourier series

In this part, recall that: $\mathbb{S}^1 \equiv \{\, z\in\mathbb{C} \mid |z|=1 \,\}$, the "L-2 spaces" are

$$L^2\left(\mathbb{S}^1\to\mathbb{C}\right) \quad \equiv \quad \left\{ \psi:\mathbb{S}^1\to\mathbb{C} \;\middle|\; \int_{\theta=0}^{2\pi} |\psi(\theta)|^2 d\theta < \infty \right\}$$

with inner product

$$\langle\psi,\varphi\rangle_{L^2} \quad \equiv \quad \int_{\theta=0}^{2\pi} \overline{\psi(\theta)}\varphi(\theta)\,d\theta$$

and

$$\ell^2(\mathbb{Z}\to\mathbb{C}) \quad\equiv\quad \left\{\hat{\psi}:\mathbb{Z}\to\mathbb{C}\,\middle|\,\sum_{n\in\mathbb{Z}}\left|\widehat{\psi}(n)\right|^2<\infty\right\}$$

with inner product

$$\left\langle\hat{\psi},\hat{\varphi}\right\rangle_{\ell^2} \quad\equiv\quad \sum_{n\in\mathbb{Z}}\overline{\hat{\psi}(n)}\hat{\varphi}(n)\ .$$

5. Find a definition of the Fourier series $\mathscr{F}:L^2\left(\mathbb{S}^1\right)\to\ell^2(\mathbb{Z})$ which, unlike the one presented in the lecture notes, *is* unitary.

6. Prove Parseval's theorem: $\langle\psi,\varphi\rangle_{L^2}=C\langle\mathscr{F}\psi,\mathscr{F}\varphi\rangle_{\ell^2}$. Find C for $\mathscr{F}$ defined in the lecture notes and the one defined in the previous question.

7. Prove the uncertainty principle: For any ψ that is sufficiently regular (i.e. that $\hat{\psi}$ decays at infinity sufficiently fast),

$$1\leq\mathbb{E}_{\psi}\left[\theta^2\right]\left(\sqrt{\mathbb{E}_{\hat{\psi}}\left[n^2\right]}+\sqrt{\mathbb{E}_{\hat{\psi}}\left[(n+1)^2\right]}\right)^2\ .$$

8. [extra] Prove that if $\psi:\mathbb{S}^1\to\mathbb{C}$ is analytic (i.e. it extends to a holomorphic function $\tilde{\psi}:A_\varepsilon\to\mathbb{C}$ where A_ε is an annulus of width $\varepsilon>0$ about $\mathbb{S}^1\subseteq\mathbb{C}$ (but may not extend to a holomorphic function in a whole disc that contains $\mathbb{S}^1$, and this is hence a more general situation than the one presented in the lecture notes) then $\hat{\psi}:\mathbb{Z}\to\mathbb{C}$ exhibits exponential decay, in the sense that

$$\left|\hat{\psi}(n)\right| \quad\leq\quad C\mathrm{e}^{-\mu|n|}$$

for some constants C,μ. Find μ in terms of ε.

4 Conformal maps

9. Show that $z\mapsto z+a$ (shift), $z\mapsto bz$ (scaling and or rotation) and $z\mapsto\frac{1}{z}$ are conformal, and find their domain and codomain of conformal equivalence.

10. Show that exp, log, and $z\mapsto z^a$ for $a\in\mathbb{C}$ are conformal equivalences, specifying the appropriate domains and co-domains.