APR 10 2023

MAT 330 — HW7 Sample Sol-ns

$\boxed{Q1}$ (a) $I = \displaystyle\int_{x\in\mathbb{R}} \frac{e^{ix}}{x+2i}$

$f(z) := \dfrac{e^{iz}}{z+2i}$ has pole @ $z=-2i$.

If you try to create a closed contour downwards to capture the pole you see that the arc integral does not converge.

Instead if we manage to close contour upwards we'll show $I=0$.
This indeed can be done via

$$\left| \int_{\substack{\text{upper}\\ \text{semi-circle of}\\ \text{radius } R}} f(z)\,dz \right| =$$

$$= \left| \int_{\theta=0}^{\pi} f(Re^{i\theta})\, Re^{i\theta}\, i\, d\theta \right|$$

$$= R\left| \int_{\theta=0}^{\pi} \frac{e^{iRe^{i\theta}+i\theta}}{Re^{i\theta}+2i}\, i\, d\theta \right|$$

$$\leqslant R \int_{\theta=0}^{\pi} \frac{e^{-R\sin(\theta)}}{|Re^{i\theta}+2i|}\, d\theta$$

$|Re^{i\theta}+2i| \geqslant |Re^{i\theta}| - |2i| = R-2$

$$\leqslant \frac{R}{R-2} \underbrace{\int_{\theta=0}^{\pi} e^{-R\sin(\theta)}\, d\theta}_{\sim \frac{1}{R} \text{ via Big-arc}}$$

Lemma A.2

$$\xrightarrow{R\to\infty} 0$$

Hence via the Cauchy int. thm.,

$$\int_{x\in[-R,R]} \frac{e^{ix}}{x+2i}\, dx = \int_{\substack{\text{upper}\\ \text{semi-circle of}\\ \text{radius } R}} f(z)\, dz \xrightarrow{R\to\infty} 0.$$

(b) $$\int_{x=0}^{\infty} \frac{1}{1+x^2}\, dx = \frac{1}{2}\int_{x=-\infty}^{\infty} \frac{1}{1+x^2}\, dx \overset{\text{Example 7.31}}{=} \frac{\pi}{2}.$$

(c) $$\int_{x=0}^{\infty} \frac{\cos(x)}{1+x^4}\, dx = \frac{1}{2}\int_{x=-\infty}^{\infty} \frac{\cos(x)}{1+x^4}\, dx$$

$$= \frac{1}{2}\,\mathrm{Re}\left\{ \int_{x=-\infty}^{\infty} \frac{e^{ix}}{1+x^4}\, dx \right\}$$

$$f(z) := \frac{\exp(iz)}{1+z^4} \quad \text{is meromorphic.}$$

Close contour upwards:

$$\left| \int_{\underbrace{\frown}_{R}} f(z)\,dz \right| = \left| \int_{\theta=0}^{\pi} f(Re^{i\theta}) Re^{i\theta}\, i\, d\theta \right|$$

as before

$$\leq \frac{R}{R^4-1} \underbrace{\int_{\theta=0}^{\pi} e^{-R\sin(\theta)}\, d\theta}_{\lesssim \frac{1}{R}} \xrightarrow{R\to\infty} 0 .$$

Now, f has 2 poles in upper half plane: $\quad 1+z^4=0 \Leftrightarrow z = \sqrt[4]{-1} = \sqrt[4]{e^{i\pi}}$

$$= \left\{ \exp\left(i \frac{\pi+2\pi n}{4}\right) \middle| n\in\mathbb{Z} \right\}$$

$$= \{ \underbrace{e^{i\frac{\pi}{4}}, e^{i\frac{3\pi}{4}}}_{\text{upper half plane}}, \underbrace{e^{i\frac{5\pi}{4}}, e^{i\frac{7\pi}{4}}}_{\text{lower half plane}} \}$$

We thus find: $\displaystyle\int_{x=-\infty}^{\infty} \frac{e^{ix}}{1+x^4}\,dx = 2\pi i \left(\mathrm{res}_{e^{i\frac{\pi}{4}}}(f) + \mathrm{res}_{e^{i\frac{3\pi}{4}}}(f) \right)$

Let us calculate the residues at the two poles. They are simple poles, so:

$$\mathrm{res}_{e^{i\frac{\pi}{4}}}(f) = \lim_{z \to e^{i\frac{\pi}{4}}} \left(z - e^{i\frac{\pi}{4}}\right) f(z)$$

$$= \frac{e^{ie^{i\frac{\pi}{4}}}}{\left(e^{i\frac{\pi}{4}} - e^{i\frac{3\pi}{4}}\right)\left(e^{i\frac{\pi}{4}} - e^{i\frac{5\pi}{4}}\right)\left(e^{i\frac{\pi}{4}} - e^{i\frac{7\pi}{4}}\right)}$$

and similarly at $e^{i\frac{3\pi}{4}}$.

$$\Rightarrow \int_{x=-\infty}^{\infty} \frac{e^{ix}}{1+x^4}\,dx = \frac{1}{2\pi i}\left[\frac{e^{ie^{i\frac{\pi}{4}}}}{\left(e^{i\frac{\pi}{4}} - e^{i\frac{3\pi}{4}}\right)\left(e^{i\frac{\pi}{4}} - e^{i\frac{5\pi}{4}}\right)\left(e^{i\frac{\pi}{4}} - e^{i\frac{7\pi}{4}}\right)} + \right.$$

$$\left. + \frac{e^{ie^{i\frac{3\pi}{4}}}}{\left(e^{i\frac{3\pi}{4}} - e^{i\frac{\pi}{4}}\right)\left(e^{i\frac{3\pi}{4}} - e^{i\frac{5\pi}{4}}\right)\left(e^{i\frac{3\pi}{4}} - e^{i\frac{7\pi}{4}}\right)}\right] =$$

$$= \dots = \frac{\pi}{\sqrt{2}\exp(1/\sqrt{2})}\left[\cos\left(\frac{1}{\sqrt{2}}\right) + \sin\left(\frac{1}{\sqrt{2}}\right)\right]$$

$$\Rightarrow \int_{x=0}^{\infty} \frac{\cos(x)}{1+x^4}\,dx = \frac{\pi}{2\sqrt{2}\exp(1/\sqrt{2})}\left[\cos\left(\frac{1}{\sqrt{2}}\right) + \sin\left(\frac{1}{\sqrt{2}}\right)\right].$$

(d) $\displaystyle\int_{x\in\mathbb{R}} \frac{x}{(x^2+1)(x^2+2x+2)}\,dx = ?$

$$f(z) := \frac{z}{(z^2+1)(z^2+2z+2)}$$ has four poles at:

$$z = \pm i \quad \text{and at} \quad z = -1 \pm i$$

We may close contour up <u>or</u> down since in either case the integrand behaves like $\frac{R^2}{R^4}$.

Anyway closing it up yields

$$\int_{x\in\mathbb{R}} \frac{x}{(x^2+1)(x^2+2x+2)}\,dx = \frac{1}{2\pi i}\left[\operatorname{res}_i(f) + \operatorname{res}_{-1+i}(f)\right]$$

These poles are simple so we find:

$$\operatorname{res}_i(f) = \frac{i}{(i+i)(i^2+2i+2)}$$

$$\operatorname{res}_{-1+i}(f) = \frac{i}{((-1+i)^2+1)((-1+i)-(-1-i))}$$

so that

$$\int_{x\in\mathbb{R}} \frac{x}{(x^2+1)(x^2+2x+2)}\,dx = -\frac{\pi}{5}.$$

(e) Let $a>0$.

$$\int_{x=0}^{\infty} \frac{\cos(ax)}{1+x^2}\,dx =$$

$$= \frac{1}{2}\,\mathrm{Re}\left\{ \underbrace{\int_{x=-\infty}^{\infty} \frac{e^{iax}}{1+x^2}\,dx}_{\equiv\, \mathcal{F}\left(x\mapsto \frac{1}{1+x^2}\right)\left(-\frac{a}{2\pi}\right)} \right\}$$

In Example 8.7 we saw

$$\mathcal{F}\left(x\mapsto \frac{1}{1+x^2}\right)(\xi) = \pi\exp(-2\pi|\xi|)$$

so we find

$$\int_{x=0}^{\infty} \frac{\cos(ax)}{1+x^2}\,dx = \frac{\pi}{2}\exp(-|a|) \qquad (a\in\mathbb{R}).$$

(f) $a>0$

$$\int_{x\in\mathbb{R}} \frac{x\sin(ax)}{4+x^4}\,dx =$$

$$= \mathrm{Im}\left\{ \underbrace{\int_{x\in\mathbb{R}} \frac{x e^{iax}}{4+x^4}\, dx}_{\mathcal{F}\left(x \mapsto \frac{x}{4+x^4}\right)\left(-\frac{a}{2\pi}\right)} \right\}$$

$$f(z) := \frac{z e^{iaz}}{4+z^4}$$

Close contour up where integral behaves like

$$\frac{R^2}{R^4-4}\,\frac{1}{R} \xrightarrow{R\to\infty} 0 \quad \text{(using } a>0\text{)}.$$

So we may pick only the residues in the upper half plane.

$$z^4=-4 \longrightarrow z\in\sqrt{2}\left\{e^{i\frac{\pi}{4}}, e^{i\frac{3\pi}{4}}, e^{i\frac{5\pi}{4}}, e^{i\frac{7\pi}{4}}\right\}$$

as above.

We find

$$\int_{x\in\mathbb{R}} \frac{x e^{iax}}{4+x^4}\, dx = 2\pi i\left[\mathrm{res}_{\sqrt{2}e^{i\frac{\pi}{4}}}(f) + \mathrm{res}_{\sqrt{2}e^{i\frac{3\pi}{4}}}(f)\right]$$

$$\mathrm{res}_{\sqrt{2}e^{i\frac{\pi}{4}}}(f) = \frac{\sqrt{2}e^{i\frac{\pi}{4}}\, e^{ia\sqrt{2}e^{i\frac{\pi}{4}}}}{\left(\sqrt{2}e^{i\frac{\pi}{4}}-\sqrt{2}e^{i\frac{3\pi}{4}}\right)\left(\sqrt{2}e^{i\frac{\pi}{4}}-\sqrt{2}e^{i\frac{5\pi}{4}}\right)(\ldots 7)}$$

and similarly for the other residue.

Collecting and simplifying we find:

$$\int_{x\in\mathbb{R}} \frac{x e^{iax}}{4+x^4}\, dx = \frac{i\pi}{2} \exp(-a)\sin(a)$$

$$\Rightarrow \boxed{\int_{x\in\mathbb{R}} \frac{x\sin(ax)}{4+x^4}\, dx = \frac{\pi}{2} e^{-a} \sin(a)} .$$

(g) Let $a\in\mathbb{C}$.

$$I := \int_{x=0}^{\infty} \frac{x^{-a}}{1+x}\, dx = ?$$

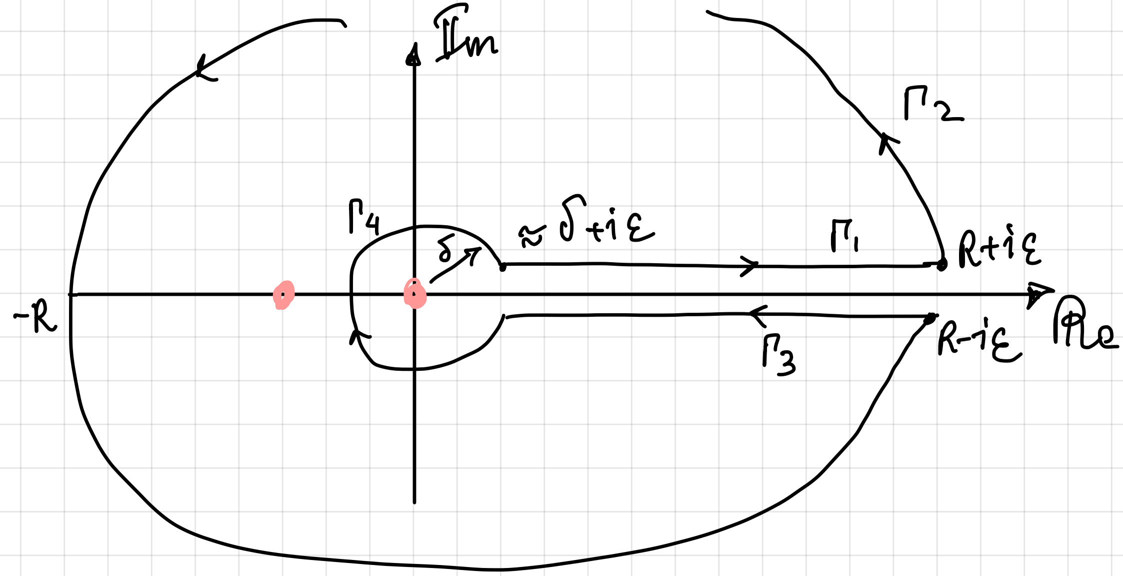
Pick $\widetilde{\mathrm{Log}}$ analytic on $\underbrace{\mathbb{C}\setminus\{z\in\mathbb{C} \mid \mathrm{Re}\{z\}\geq 0\}}_{\text{different than usually...}}$.

to avoid pole at $z=-1$. Hence $\widetilde{\mathrm{Arg}}\in[0,2\pi)$.

Then $f(z) := \dfrac{z^{-a}}{1+z} = \dfrac{\exp(-a\,\widetilde{\mathrm{Log}}(z))}{1+z}$

is analytic on $\mathbb{C}\setminus\{z\in\mathbb{C} \mid \mathrm{Re}\{z\}\geq 0\}\cup\{-1\}$.

Define the following contour:



This closed contour has 4 legs.

$$I_{\Gamma_1} = \int_{x=\delta}^{R} f(x+i\varepsilon)\,dx$$

$$= \int_{x=\delta}^{R} \frac{e^{-a\widetilde{\mathrm{Log}}(x+i\varepsilon)}}{1+x+i\varepsilon}\,dx$$

$$\widetilde{\mathrm{Log}}(x+i\varepsilon) \approx \log(x) + i\,0$$

$$\widetilde{\mathrm{Log}}(x-i\varepsilon) \approx \log(x) + 2\pi i$$

So $\quad I_{\Gamma_1} \xrightarrow[\varepsilon\to 0]{\substack{R\to\infty \\ \delta\to 0}} \displaystyle\int_{x=0}^{\infty} \frac{e^{-a\log(x)}}{1+x}\,dx = I.$

$$I_{\Gamma_3} \xrightarrow[\varepsilon\to 0]{\substack{R\to\infty \\ \delta\to 0}} e^{-2\pi i a}\, I.$$

$$I_{\Gamma_2} = \int_{\theta=0}^{2\pi} \frac{e^{-a\widetilde{\mathrm{Log}}(Re^{i\theta})}}{1+Re^{i\theta}}\, Re^{i\theta}\, i\,d\theta$$

$$= \int_{\theta=0}^{2\pi} \frac{R^{-a+1} e^{-ia\theta}}{1+Re^{i\theta}} e^{i\theta} i d\theta$$

$$|I_{\Gamma_2}| \leqslant 2\pi \frac{|R^{-a+1}|}{R-1} = 2\pi \frac{R^{-\mathrm{Re}\{a\}+1}}{R-1}$$

If $\mathrm{Re}\{a\} > 0$, $|I_{\Gamma_2}| \xrightarrow{R\to\infty} 0$.

Similarly,

$$I_{\Gamma_4} = \int_{\theta=0}^{2\pi} \frac{\delta^{-a+1} e^{-ia\theta}}{1+\delta e^{i\theta}} e^{i\theta} i d\theta$$

If $\mathrm{Re}\{a\} < 1$, $|I_{\Gamma_4}| \xrightarrow{\delta\to 0} 0$ (using the dominated convergence thm.).

The residue @ $z=-1$ yields:

$$\mathrm{res}_{-1}(f) = (-1)^{-a} = \exp(-a \,\mathrm{Log}(-1))$$

$$= \exp(-a i\pi)$$

Thus the residue thm yields

$$2\pi i e^{-i\pi a} = I - e^{-2\pi i a} I$$

$$\Rightarrow \boxed{I = \frac{\pi}{\sin(\pi a)}} \quad (a \in \{z \in \mathbb{C} \mid \mathrm{Re}\{z\} \in (0,1)\}).$$

Claim: I may be analytically continued to $\tilde{I} : \mathbb{C} \setminus \mathbb{Z} \to \mathbb{C}$ using the same formula.

Proof: Clearly $\tilde{I}$ is meromorphic on the open set $\mathbb{C} \setminus \mathbb{Z}$ and

$$\tilde{I}\big|_{\{z \in \mathbb{C} \mid \operatorname{Re}\{z\} \in (0,1)\}} = I.$$

By uniqueness of analytic contin. we conclude. $\blacksquare$

$$\Rightarrow \boxed{I = \frac{\pi}{\sin(\pi a)}} \quad (a \in \mathbb{C} \setminus \mathbb{Z}).$$

(h) $$\int_{x=0}^{\infty} \frac{1}{\sqrt{x}\,(1+x^2)}\,dx \overset{y := \sqrt{x} \quad dy = \frac{1}{2\sqrt{x}}dx}{=} 2\int_{y=0}^{\infty} \frac{1}{1+y^4}\,dy$$

$$= 1 \cdot \int_{y=-\infty}^{\infty} \frac{1}{1+y^4}\,dy$$

was 4 erroneouly before

$$= 1 \cdot 2\pi i \sum_{\substack{z \in \text{ poles in} \\ \text{upper half} \\ \text{plane}}} \operatorname{res}_z \left(z \mapsto \frac{1}{1+y^4} \right)$$

$$= 4\pi i \left[\frac{1}{(e^{i\frac{\pi}{4}} - e^{i\frac{3\pi}{4}})(e^{i\frac{\pi}{4}} - e^{i\frac{5\pi}{4}})(e^{i\frac{\pi}{4}} - e^{i\frac{7\pi}{4}})} + \frac{1}{(e^{i\frac{3\pi}{4}} - e^{i\frac{\pi}{4}})(e^{i\frac{3\pi}{4}} - e^{i\frac{5\pi}{4}})(e^{i\frac{3\pi}{4}} - e^{i\frac{7\pi}{4}})} \right]$$

$$= 4\pi i \frac{-i}{2\sqrt{2}} = \frac{\pi}{\sqrt{2}} .$$

Note it is also possible to integrate this w/ a similar contour as in question g) to avoid the change of var.

(i) Let $n \in \mathbb{N}$.

$$I := \int_{x \in \mathbb{R}} (1+x^2)^{-n-1} \, dx = ?$$

$$f(z) := (1+z^2)^{-n-1} = (z-i)^{-n-1} (z+i)^{-n-1}$$

Close contour w/ upper semicircle since integrand $\sim R^{-2(n+1)+1} \longrightarrow 0$.

Thus we have

$$I = 2\pi i \; \mathrm{res}_i (f)$$

$$= 2\pi i \lim_{z\to i} \frac{1}{n!} \partial_z^n (z-i)^{n+1} f(z)$$

pole of order $n+1$

$$= 2\pi i \lim_{z\to i} \frac{1}{n!} \partial_z^n (z+i)^{-n-1}$$

$$= 2\pi i \lim_{z\to i} \frac{1}{n!} (-n-1)(-n-2)\cdots(-2n)(z+i)^{-2n-1}$$

$$= 2\pi i \frac{(2i)^{-2n-1} (-1)^n (n+1)\cdots(2n)}{n!}$$

$$= \pi (2i)^{-2n} (-1)^n \frac{(2n)!}{(n!)^2}$$

$i^2=-1$

$$= \boxed{\frac{\pi}{2^{2n}} \frac{(2n)!}{(n!)^2}}.$$

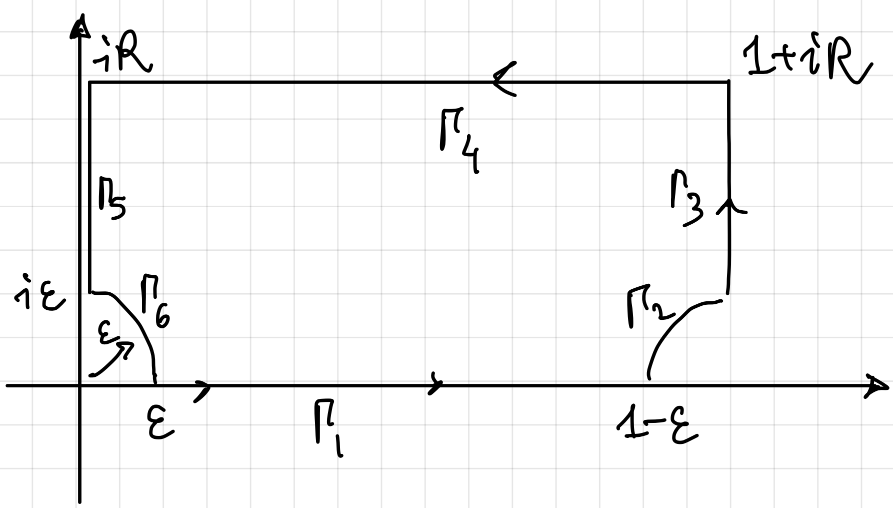
(j) $I := \int_0^1 \log(\sin(\pi x))\, dx$

$$f(z) := \mathrm{Log}(\sin(\pi z)) + i\pi z + \mathrm{Log}(-2i)$$

$$\int_0^1 f(x)dx = \int_0^1 \log(\sin(\pi x))dx + i\pi \tfrac{1}{2} + \underbrace{\mathrm{Log}(-2i)}_{\log(2) - i\frac{\pi}{2}}$$

$$= \int_0^1 \log(\sin(\pi x))dx + \log(2).$$

Define now



Note $f(z) = \mathrm{Log}(1 - e^{2\pi i z}) = \mathrm{Log}\left(e^{\pi i z}\, 2i\, \frac{e^{-\pi i z} - e^{\pi i z}}{2i}\right)$

is holomorphic within interior of contour, and

$$\oint_{\Gamma} f(z)\,dz = 0$$

$$\Rightarrow \int_{\Gamma_1} f = -\sum_{j=2}^{6} \int_{\Gamma_j} f$$

$\int_{\Gamma_2} f - \int_{\Gamma_5} f = 0$ due to symmetry:

$$\int_{\Gamma_2} f = \int_{y=\varepsilon}^{R} \mathrm{Log}\left(1-e^{2\pi i(iy)}\right) i\,dy$$

$$= i\int_{y=\varepsilon}^{R} \mathrm{Log}\left(1-e^{-2\pi y}\right) dy$$

$$\int_{\Gamma_5} f = \int_{y=\varepsilon}^{R} \mathrm{Log}\left(1-e^{2\pi i(1+iy)}\right) i\,dy$$

$$= i\int_{y=\varepsilon}^{R} \mathrm{Log}\left(1-e^{-2\pi y}\right) dy$$

Small quarter circle int. vanish:

$$\int_{\Gamma_2} f = \int_{\theta=\pi}^{\pi/2} \mathrm{Log}\left(1-e^{2\pi i \varepsilon e^{i\theta}}\right) \varepsilon e^{i\theta} i\,d\theta$$

$\xrightarrow{\varepsilon\to 0} 0$ essentially since $\mathrm{Log}(\varepsilon)\varepsilon \xrightarrow{\varepsilon\to 0} 0$.

Same w/ other quarter circle.

For Γ_4:

$$\int_{\Gamma_4} f = \int_{x=\varepsilon}^{1-\varepsilon} \underbrace{\mathrm{Log}(1-e^{2\pi i(x+iR)})}_{\mathrm{Log}(1-e^{-2\pi R}e^{2\pi i x})}\,dx$$

$$\underbrace{\mathrm{Log}(\underbrace{1-\underbrace{e^{-2\pi R}e^{2\pi i x}}_{\to 0}}_{\to 1})}_{\to 0}$$

We find:

$$\boxed{I = -\log(2)}.$$

Note This integral may also be solved via real analysis and symmetries.

Q2

Claim: If $a > e$, $az^n = e^z$ has n roots within $B_1(0)$.

Proof: Define $\begin{cases} f(z) := az^n \\ g(z) := az^n - e^z \end{cases}$

By fund. thm. of alg. know f has n roots in $\mathbb{C}$. Actually the roots are all at $z=0$!

Note $|g(z) - f(z)| = |e^z|$

$$\leq e^{|z|} = e \text{ on } z \in \partial B_1(0).$$

OTOH, $|f(z)| = a$ on $z \in \partial B_1(0)$.

So Rouché applies as soon as $e < a$. $\blacksquare$

Q3

Claim: If $f : \mathbb{C} \to \mathbb{C}$ is entire : $\exists\, k \in \mathbb{N}$, $A, B \in [0, \infty)$:

$$\sup_{|z|=R} |f(z)| \leq AR^k + B \qquad (R > 0)$$

Then f is a poly. of deg. $\leq k$.

Proof: If f is const then we're done.

Otherwise, by max. mod. principle,

$$\sup_{z \in B_R(0)} |f(z)| \leqslant \sup_{|z|=R} |f(z)|$$

$$\Longrightarrow \sup_{z \in B_R(0)} |f(z)| \leqslant AR^k + B$$

By Cauchy's ineq., Thm. 6.30,

$$|f^{(n)}(0)| \leqslant \frac{n!}{R^n} \sup_{z \in B_R(0)} |f(z)|$$

$$\leqslant \frac{n!}{R^n} (AR^k + B)$$

for all $R > 0$.

Taking $R \to \infty$ yields

$$f^{(n)}(0) = 0 \quad \forall \quad n > k.$$

The same is true also for $\tilde{f}(z) := f(z - z_0)$ for any $z_0 \in \mathbb{C}$.

$$\Rightarrow f^{(n)}(z) = 0 \quad \forall \quad z \in \mathbb{C},\ n > k.$$

$$\Leftrightarrow f \text{ is a poly of deg} \leqslant k.$$

$\blacksquare$

$\boxed{Q4}$ **Claim:** If $f: \mathbb{C} \to \mathbb{C}$ is entire: $f_R: \mathbb{C} \to \mathbb{R}$ is bdd. Then f is const.

Proof: Let $g := \exp \circ f$. g is entire.

Then $|g| = \exp \circ f_R$.

If f_R is bdd., g is bdd.

By Liouville's thm. Thm. 6.32,

if g is entire & bdd. it is const.

$\Leftrightarrow f$ is const. (Otherwise we could say f takes only the values in $2\pi i \mathbb{Z}$ and is hence NOT const., but to be continuous it can't jump!)

$\blacksquare$

Alt. proof: Let $R > 0$. By max. mod. prin. $|g|$ cannot attain a max on $B_R(0)$ $\forall$ $R > 0$, if g is non-const!

$\Rightarrow$ Contradiction w/ g bdd.

So g is const.

$\blacksquare$

Q5 Let $\tilde{\mathcal{F}}_c : L^2(\mathbb{S}^1) \to \ell^2(\mathbb{Z})$

$$\hat{\psi}(n) := \frac{1}{c}\int_{\theta=0}^{2\pi} e^{-in\theta}\psi(\theta)\,d\theta \qquad (n\in\mathbb{Z})$$

<u>Claim</u>: If $c = (2\pi)^{-1/2}$ then $\tilde{\mathcal{F}}_c$ is unitary.

<u>Proof</u>: See proof of Thm. 8.5. $\blacksquare$

Q6 See Thm. 8.5.

Q7 See Thm. 8.6.

Q8 See Lemma 8.4.

Q9 To show a map is conformal, suffice to show it is analytic and injective on its domain $\mathcal{U}$ and then make sure its image is its codomain V ($\equiv$ surjective)

⊛ For $z \mapsto a+z$ clearly this map is entire and we may take $\mathcal{U}=\mathcal{V}=\mathbb{C}$.

⊛ For $z \mapsto bz$, will have injectivity only if $b \neq 0$, and analyticity always. Then we may take $\mathcal{U}=\mathcal{V}=\mathbb{C}$.

⊛ For $z \mapsto \frac{1}{z}$, we must take $\mathcal{U} = \mathbb{C} \setminus \{0\}$, whence it is injective: $\frac{1}{z} = \frac{1}{\tilde{z}} \Leftrightarrow z = \tilde{z}$ $\forall\, z, \tilde{z} \in \mathcal{U}$.

Surjectivity follows w/ $\mathcal{V} = \mathbb{C} \setminus \{0\}$:

Let $w \in \mathcal{V}$. Then $\frac{1}{(1/w)} = w$. ✓

Note $\mathcal{U}=\mathcal{V} \in \text{Open}(\mathbb{C})$ as required.

$\boxed{Q10}$ ⊛ $\exp: \mathbb{C} \to \mathbb{C}$ is entire but not injective.

We may define

$$\mathcal{U} := \{ z \in \mathbb{C} \mid \text{Im}\{z\} \in (0, 2\pi) \} \in \text{Open}(\mathbb{C}).$$

$$\mathcal{V} := \mathbb{C} \setminus [0, \infty) \in \text{Open}(\mathbb{C})$$

Claim: $\exp: \mathcal{U} \to \mathcal{V}$ is bijective.

Proof: $\exp(z) = \exp(\tilde{z}) \Leftrightarrow z = \tilde{z} + 2\pi i n$
$\exists\, n \in \mathbb{Z}$

But since $z, \tilde{z} \in \mathcal{U}$, $n = 0$ and $z = \tilde{z}$.

$\Rightarrow$ exp is injective.

Next, let $w \in V$. Then define

$$z := \log(|w|) + i\,\widetilde{\mathrm{Arg}}(w) \equiv \widetilde{\mathrm{Log}}(w)$$

argument w/ branch on positive real axis taking values in $[0, 2\pi)$.

Then $z \in \mathcal{U}$ since $w \in \mathbb{C} \setminus [0, \infty)$, so $\widetilde{\mathrm{Arg}}(w) \in (0, 2\pi)$.

$\Longrightarrow$ $\exp(z) = w$ and so exp is surjective. $\blacksquare$

(*) This shows that $\widetilde{\mathrm{Log}} : V \to \mathcal{U}$ is also a conformal equivalence.

(*) For $z \mapsto z^{\alpha + i\beta} \equiv \exp(\alpha\,\widetilde{\mathrm{Log}}(z) + i\beta\,\widetilde{\mathrm{Log}}(z))$

where Log is holomorphic on its domain

$\mathcal{U}$ to be specified below.

$$f(z) \equiv \exp(\alpha \widetilde{\mathrm{Log}}(z) + i\beta \widetilde{\mathrm{Log}}(z))$$

$$= \exp(\alpha \log(|z|) + i\alpha \widetilde{\mathrm{Arg}}(z) + i\beta \log(|z|) - \beta \widetilde{\mathrm{Arg}}(z))$$

$$= |z|^\alpha e^{-\beta \widetilde{\mathrm{Arg}}(z)} \exp(i(\alpha \widetilde{\mathrm{Arg}}(z) + \beta \log(|z|)))$$

Define

$$\mathcal{U} := \{ z \in \mathbb{C} \mid 0 < \alpha \widetilde{\mathrm{Arg}}(z) + \beta \log(|z|) < 2\pi \}$$

which is an open s/set of $\mathbb{C}$.

Define $V := \mathbb{C} \setminus [0, \infty)$.

Can we pick a branch cut for $\widetilde{\mathrm{Arg}}$ which is outside of $\mathcal{U}$?

$$0 < \alpha\theta + \beta \log(r) < 2\pi$$

So need $-\beta \log(r) < \alpha\theta < 2\pi - \beta \log(r)$

For every r, this defines a range of θ which are allowed. So in principle we may define $\widetilde{\mathrm{Arg}}$ to have an r-dependent branch cut to make sure the cut would <u>not</u> intersect $\mathcal{U}$ so that f is holomorphic on $\mathcal{U}$. Furthermore, as we always land in $[0, 2\pi)$ in the argument we never wrap around $\leftrightarrow$ injectivity.

The codomain is thus $V := \mathbb{C} \setminus [0, 2\pi)$ and we get conformal equivalence.