Complex Analysis with Applications
Princeton University MAT330
HW6, Due April 7th 2023

April 9, 2023

1 Taylor and Laurent series

1. State the domain of convergence for each of the following power series: (a) $\sum_{n=1}^{\infty}(-1)^n(z-z_0)^n$, and (b) $\sum_{n=1}^{\infty}\left(2n^3+6\right)^2 z^n$.
2. Find a Taylor expansion of the given functions about the indicated point z_0 and provide the domain of convergence for: (a) $z \mapsto \cos(z)$ about $z_0 = \frac{\pi}{2}$ and, (b) $z \mapsto \frac{1}{z+1}$ about $z_0 = 1$.
3. Find the radius of convergence of the associated Taylor series without actually calculating the series for: (a) $\frac{\sin(z)}{z^2+1}$ about $z_0 = 0$, (b) $\frac{z}{\mathrm{e}^z+1}$ about $z_0 = 0$ and (c) $\frac{z+1}{z^2-5z+4}$ about $z_0 = 2$.
4. By rewriting $\frac{1}{z-z_0}$ as either $\frac{1}{z_0}\frac{1}{\frac{z}{z_0}-1}$ or as $\frac{1}{z}\frac{1}{1-\frac{z_0}{z}}$ and expanding the second factor of each version in negative or positive powers of z, find two versions of a Laurent series of $z \mapsto \frac{1}{z-z_0}$ and determine their domain of convergence.

2 Residue calculation, contour integration, and singularities

5. Calculate the residues of the following functions at their singularities: (a) $z \mapsto \frac{1}{z+z^2}$, (b) $z \mapsto \frac{z-\sin(z)}{z}$, (c) $z \mapsto \frac{\exp(-z)}{(z-1)^2}$, (d) $z \mapsto \frac{z+1}{z^2-2z}$ and (e) $z \mapsto \frac{\mathrm{e}^{\mathrm{i}z}}{z^4+4}$.
6. Show that for any $R \in (0,\infty)$,

$$\oint_{\partial B_R(0)} \exp\left(z+\frac{1}{z}\right)\mathrm{d}z = 2\pi\mathrm{i}\sum_{n=0}^{\infty}\frac{1}{n!\,(n+1)!}\,.$$

7. Show that $z \mapsto \frac{1-\cos(z)}{z^2}$ has a removable singularity at the origin and determine the analytic function which is its continuation.

3 The argument principle, Rouché and the maximum modulus principle

8. In HW4Q11 we proved the maximum principle using the Cauchy integral formula. Now we will prove it using the open mapping theorem:
   (a) Define $f : \Omega \to \mathbb{C}$ to be open iff it maps open sets into open sets (cf. with a continuous function, which is defined so that the preimage of open sets is open).
   (b) (*The open mapping theorem*) Show that if $f : \Omega \to \mathbb{C}$ is holomorphic and non-constant on $\Omega \in \mathrm{Open}(\mathbb{C})$ and connected, then f is an open mapping.
   (c) Show that the composition of two open maps is open.
   (d) Show that the absolute value function $|\cdot| : \mathbb{C} \to [0,\infty)$ is an open mapping.
   (e) Show that a subset of $\mathbb{R}$ cannot be open (open in $\mathbb{R}$) if it contains its maximum.
   (f) Prove the maximum modulus principle (phrased in Corollary 7.52) now by assuming $|f|$ has a maximum within Ω, using that $|f|$ is open (why?) and reaching a contradiction.

9. Find some $\varepsilon > 0$ such that
$$\begin{bmatrix} 1 & t \\ t & 1 \end{bmatrix}$$
is invertible whenever $t \in B_\varepsilon(0) \subseteq \mathbb{C}$.

10. Let an $n \times n$ matrix B with complex entries by defined via its elements as
$$B_{ij} := \begin{cases} 1 & i = j \\ t & i \neq j \end{cases} \qquad (i, j = 1, \dots, n) \ .$$
Find some $\varepsilon > 0$ so that if $t \in B_\varepsilon(0) \subseteq \mathbb{C}$ then B is invertible.
You may find it useful to work with the Hilbert-Schmidt norm instead of the operator norm, so that you get a bound on matrix elements (and hence on t) directly. To do so, you would have to use equation (7.7) in the lecture notes to relate the condition in Lemma 7.47 to a bound on the Hilbert-Schmidt norm of the perturbation.
What happens to your bound as $n \to \infty$?

11. [extra] For $f, g : \mathbb{C} \to \mathbb{C}$ meromorphic D some disc such that ∂D contains no zeros of f, g, fg and $f \circ g$, show that
$$\operatorname{index}_D(f \circ g) \quad = \quad \operatorname{index}_D(f) \cdot \operatorname{index}_D(g)$$
and
$$\operatorname{index}_D(fg) \quad = \quad \operatorname{index}_D(f) + \operatorname{index}_D(g) \ .$$