APR 3 2023

MAT 330 — HW6 Sample Solutions

Q1 Domain of conv. of a power series via the Cauchy–Hadamard formula:

$$R := \frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}}$$

(a) $a_n = (-1)^n \longrightarrow R = 1$

(b) $a_n = (2n^3+6)^2 \longrightarrow |a_n|^{1/n} = (2n^3+6)^{2/n}$

$$= \exp\Big(\underbrace{\tfrac{2}{n}}_{\to 0} \underbrace{\log(2n^3+6)}_{\to\infty}\Big)$$

But poly beats log, so $|a_n|^{1/n} \longrightarrow e^0 = 1$.

Q2 (a) Taylor series for cos about $z_0 = \frac{\pi}{2}$:

Since $\cos(z) = \frac{1}{2}(e^{iz} + e^{-iz})$ and exp is entire, so is cos, so its Taylor series converges w/ $R = \infty$.

To find the coeff. we use

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

$$\cos^{(n)}(z) = \begin{cases} \cos(z) & n=0 \\ -\sin(z) & n=1 \\ -\cos(z) & n=2 \\ \sin(z) & n=3 \\ \cos(z) & n=4 \end{cases}$$

$$= \begin{cases} \pm \cos(z) & n = 0,2 \mod 4 \\ \mp \sin(z) & n = 1,3 \mod 4 \end{cases}$$

So $\cos^{(n)}(\frac{\pi}{2}) = \begin{cases} 0 & n = 0,2 \mod 4 \\ \mp 1 & n = 1,3 \mod 4 \end{cases}$

Hence $\cos(z) = -\sum_{n=0}^{\infty} \frac{1}{(4n+1)!}(z-\frac{\pi}{2})^{4n+1} +$

$$+ \sum_{n=0}^{\infty} \frac{1}{(4n+3)!}(z-\frac{\pi}{2})^{4n+3} .$$

(b) $f(z) := \frac{1}{z+1}$ to be expanded about $z_0 = 1$.

We seek $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(z-1)^n$.

We may take a shortcut via

the geometric series;

$$\frac{1}{1+z} = \frac{1}{z-1+2} = \frac{1}{2}\,\frac{1}{1-\left(\frac{1-z}{2}\right)}$$

$$= \frac{1}{2}\sum_{j=0}^{\infty}\left(\frac{1-z}{2}\right)^{j}$$

$$= \sum_{j=0}^{\infty} 2^{-j-1}(1-z)^{j}$$

$$= \sum_{j=0}^{\infty} \underbrace{2^{-j-1}(-1)^{j}}_{\text{expansion coeff.}}(z-1)^{j}$$

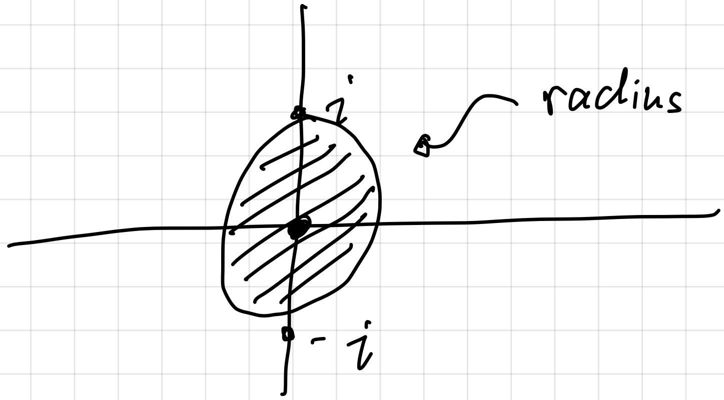
and we know disc of conv. is when

$$\left|\frac{1-z}{2}\right| < 1 \iff \boxed{|1-z| < 2}.$$

$\boxed{Q3}$ Find disc of conv. w/o calc. the series for: (distance to nearest singularity)

(a) $z \mapsto \dfrac{\sin(z)}{z^2+1}$: Since sin is entire, the only constraint will come from the denominator.

Hence when $z^2+1=0 \iff z=\pm i$

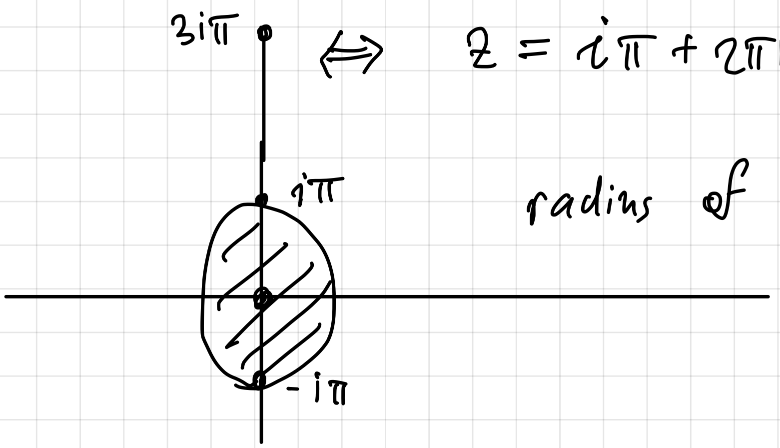


radius of conv. $= 1$

(b) $z \mapsto \dfrac{z}{e^z + 1}$ about $z_0 = 0$.

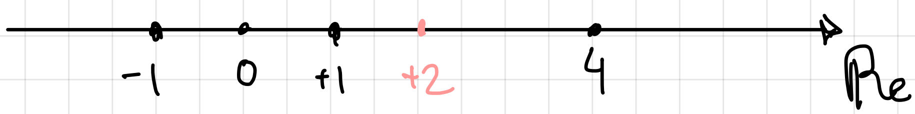
$e^z + 1 = 0 \iff e^z = -1$

$\iff z = i\pi + 2\pi i n \quad \exists\, n \in \mathbb{Z}$



radius of conv. $= \pi$

(c) $z \mapsto \dfrac{z+1}{z^2 - 5z + 4} = \dfrac{z+1}{(z-1)(z-4)}$



$\Rightarrow$ radius of 1 about $+2$.

$\boxed{Q4}$ Write

HWQ4

$$(z-z_0)^{-1} = z_0^{-1}\left(\frac{z}{z_0}-1\right)^{-1} = -z_0^{-1}\sum_{n=0}^{\infty}\left(\frac{z}{z_0}\right)^n$$

when $\left|\frac{z}{z_0}\right|<1 \iff |z|<|z_0|$.

Conversely,

$$(z-z_0)^{-1} = z^{-1}\left(1-\frac{z_0}{z}\right)^{-1} = z^{-1}\sum_{n=0}^{\infty}\left(\frac{z_0}{z}\right)^n$$

when $\left|\frac{z_0}{z}\right|<1 \iff |z|>|z_0|$.

$\boxed{Q5}$ Calc. residues for:

(a) $z \mapsto \frac{1}{z+z^2} = \frac{1}{z(1+z)}$

res @ 0: 1

res @ -1: -1

(b) $z \mapsto \frac{z-\sin(z)}{z}$ is NOT singular at $z=0$

since there $\sin(z) \approx z - \frac{1}{6}z^3 + \dots$

So there is a removable sing. there.

(c) $z \mapsto \frac{e^{-z}}{(z-1)^2}$

res @ 1 : $\lim\limits_{z\to 1} \partial_z e^{-z} = \lim\limits_{z\to 1} (-1)e^{-z}$

$$= -e^{-1}$$

(d) $z \mapsto \dfrac{z+1}{z^2-2z} = \dfrac{z+1}{z(z-2)}$

res @ 0 : $-\dfrac{1}{2}$

res @ 2 : $\dfrac{3}{2}$

(e) $z \mapsto \dfrac{e^{iz}}{z^4+4}$ $\qquad \exp\left(\dfrac{2\pi i n}{4}\right)$

$$z^4+4 = (z-\sqrt{2})(z-\sqrt{2}\underbrace{e^{i\frac{\pi}{2}}}_{i})(z-\sqrt{2}\underbrace{e^{i\pi}}_{-1})(z-\underbrace{e^{i\frac{3\pi}{2}}}_{-i})$$

res @ $\sqrt{2}$: $\dfrac{e^{i\sqrt{2}}}{(\sqrt{2}-\sqrt{2}i)(\sqrt{2}+\sqrt{2})(\sqrt{2}+i\sqrt{2})}$

$$= \ldots$$

res @ $\sqrt{2}i$ $= \ldots$

$\ldots$

Q6 **Claim:** $\oint_{\partial B_R(0)} \exp\left(z+\frac{1}{z}\right)dz = 2\pi i \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!}$

Proof: Parametrize $\gamma(t) = Re^{it}$ $t \in [0, 2\pi)$.

So

$$\oint_{\partial B_R(0)} \exp\left(z+\frac{1}{z}\right)dz = \int_{t=0}^{2\pi} \exp\left(Re^{it} + \frac{1}{Re^{it}}\right) Re^{it}\, i\, dt$$

$$Re^{it} + \frac{1}{Re^{it}} = R(1+R^{-2})\cos(t) + iR(1-R^{-2})\sin(t)$$

Since the singularity is at $z=0$ we may pick any $R \in (0,\infty)$ to do the calc. So pick $R=1$, whence the integral becomes

$$i \int_{t=0}^{2\pi} \exp(2\cos(t) + it)\, dt$$

Now expand $e^{2\cos(t)} = \sum_{n=0}^{\infty} \frac{2^n}{n!} \cos(t)^n$ and exchange series w/ integral [series converges uniformly] to get $i \sum_{n=0}^{\infty} \frac{2^n}{n!} \int_{t=0}^{2\pi} [\cos(t)]^n e^{it}\, dt$

$$2^n \int_{t=0}^{2\pi} [\cos(t)]^n e^{it}\, dt = \int_{t=0}^{2\pi} (e^{it} + e^{-it})^n e^{it}\, dt$$

$$= \sum_{k=0}^{n} \binom{n}{k} \int_{t=0}^{2\pi} \underbrace{e^{ikt}\, e^{-i(n-k)t}\, e^{it}}_{e^{i(1+2k-n)}}\, dt$$

$$= \sum_{k=0}^{n} \binom{n}{k}\, 2\pi\, \underbrace{\delta_{2k-n+1,0}}_{k=\frac{n-1}{2}}$$

$$= \begin{cases} 2\pi \binom{n}{\frac{n-1}{2}} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

So the result for the int. is:

$$2\pi i \sum_{n=0}^{\infty} \frac{1}{n!} \chi_{2\mathbb{Z}+1}(n) \binom{n}{\frac{n-1}{2}} =$$

$$= 2\pi i \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \binom{2n+1}{\frac{(2n+1)-1}{2}} = 2\pi i \sum_{n=0}^{\infty} \frac{1}{\cancel{(2n+1)!}} \frac{\cancel{(2n+1)!}}{n!\,(2n+1-n)!}$$

$$= 2\pi i \sum_{n=0}^{\infty} \frac{1}{n!\,(n+1)!}\,. \qquad \blacksquare$$

$\boxed{Q7}$ **Claim:** $z \overset{f}{\longmapsto} \dfrac{1-\cos(z)}{z^2}$ has a removable sing. @ $z=0$.

Proof: Define $\tilde{f} := \begin{cases} f(z) & z \neq 0 \\ 1/2 & z=0 \end{cases}$.

Claim: $\tilde{f}$ is holo. @ $z=0$.

Proof: $\lim\limits_{z\to 0} \dfrac{\tilde{f}(z)-\tilde{f}(0)}{z} =$

$$= \lim_{z\to 0} \frac{f(z)-1/2}{z}$$

$$= \lim_{z\to 0} \frac{\frac{1-\cos(z)}{z^2} - 1/2}{z}$$

$$= \lim_{z\to 0} \frac{1-\frac{1}{2}z^2-\cos(z)}{z^3}$$

$\cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}$

$$\overset{\curvearrowright}{=} \lim_{z\to 0} \sum_{n=2}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n-3}$$

exchange lim and power series by uniformity of cos Tay. Series $\overset{\curvearrowright}{=} 0$.

$\boxed{\checkmark}$

$\boxed{Q8}$ See Section 7.8 in Lecture notes.

$\boxed{Q9}$ The matrix $\begin{bmatrix} 1 & t \\ t & 1 \end{bmatrix}$ has det $1-t^2$ and is hence inv. whenever $t \neq \pm 1$.

So take $\varepsilon = 1$.

$\boxed{Q10}$ Now since n may be large an explicit calc. may be difficult.

Instead: $A := \mathrm{diag}(1, \dots, 1) \in \mathrm{Mat}_{n \times n}(\mathbb{C})$.

A has eig. val. $+1$ w/ mul. n.

Define $D := B_1(1)$. So $\mathrm{gap}_D(A) = +1$.

Need a bound on $\|B - A\|$ so that

$$\|B - A\| < \frac{1}{2n \cdot n!} \, \mathrm{gap}_D(A) = \frac{1}{2n \cdot n!}$$

and then apply Lemma 7.47.

Use $\|B - A\| \leq \|B - A\|_{HS}$

$$\|B - A\|_{HS}^2 \equiv \sum_{i,j=1}^{n} |B_{ij} - A_{ij}|^2 = \sum_{i \neq j} |t|^2 = |t|^2 (n^2 - n)$$

$$= |t|^2 n(n-1) \overset{!}{<} \frac{1}{2n \cdot n!}$$

$$\Rightarrow |t|^2 < \frac{1}{2n^2(n-1)\, n!}$$

$$\boxed{|t| < (2n^2(n-1)\, n!)^{-1/2}}$$

This becomes meaningless as $n \to \infty$.

Q11 ① Let $f, g : \Omega \to \mathbb{C}$ be meromorphic on $\Omega \in \mathrm{Open}(\mathbb{C})$ s.t. $D \subseteq \Omega$ is a disc: f, g both have no zeros / poles on ∂D.

Claim: $\mathrm{index}_D(fg) = \mathrm{index}_D(f) + \mathrm{index}_D(g)$.

Proof: $$\frac{(fg)'}{fg} = \frac{f'g + fg'}{fg} = \frac{f'}{f} + \frac{g'}{g}$$ ■

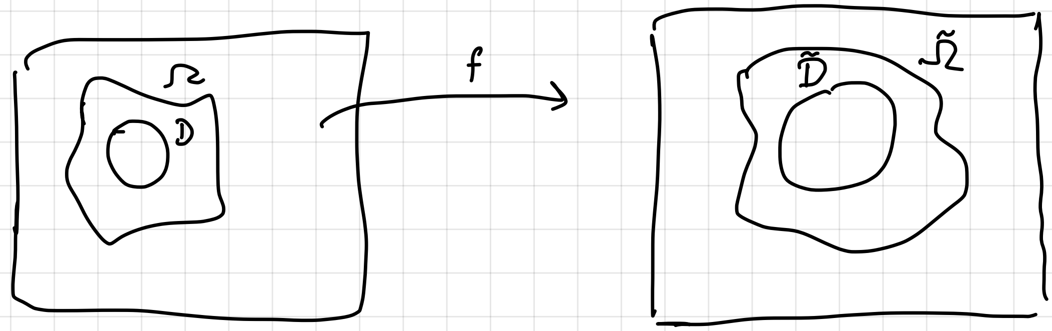
② Let $f : \Omega \to \mathbb{C}$, $g : \tilde{\Omega} \to \mathbb{C}$ w/ $\Omega, \tilde{\Omega}$ conn.

Let $D \subseteq \Omega$ and $\tilde{D} \subseteq \tilde{\Omega}$ be two discs:

∂D has no poles/zeros of f

$\partial \tilde{D}$ has no poles/zeros of g.

Furthermore, we have $g \circ f : \Omega \to \mathbb{C}$



Assume $g \circ f : \Omega \to \mathbb{C}$ has no zeros or poles on ∂D. E.g., w/ $\tilde{D} := f(D)$.

Claim: $\operatorname{index}_D(g \circ f) = \operatorname{index}_D(f)\, \operatorname{index}_{\tilde{D}}(g)$.

Proof: Sketch: ① Show index_D is homotopy stable (takes some work...).

② Deform $f \rightsquigarrow z \mapsto z^n$ $\exists n \in \mathbb{Z}$.
$g \rightsquigarrow z \mapsto z^m$ $\exists m \in \mathbb{Z}$.

$\Rightarrow$ $g \circ f \rightsquigarrow z \mapsto z^{nm}$.