Complex Analysis with Applications
Princeton University MAT330
HW5v3, Due Apr 7th 2023 (but not Apr 9th!)

April 1, 2023

Note: the present HW is graded up to 100+10=100 points with 10 points bonus for legibility and coherence as is our tradition. New this week, you have the opportunity of obtaining bonus points counting for your overall grade: the exercises marked [extra] are not part of the HW's 100 points count: a fully correct solution of each [extra] exercise counts as one point increase to your final course grade.

Second note: this homework has been edited to make it more accessible. If you struggled with any question in the previous version (or haven't even tried it yet) please consult the question presented in this version. The added text is mostly in *italics. Importantly, questions 5 and 12 are now [extra] credit. Only for questions 5 and 12, if you've already worked on them and wish to receive partial credit, please hand in your work (even if it's partial) and I'll make sure you get those extra two points in your course grade.*

1 Sequences, series and power series

In the following three examples, we will examine how things break down badly if interchanging the order of limits, or interchanging limits and integration, without justification.

1. Let a function $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) \quad := \quad \begin{cases} \frac{x}{x+y} & x \neq -y \\ 0 & x = -y \end{cases} .$$

   Calculate the limits:

   (a) $\lim_{y\to\infty} \lim_{x\to\infty} f(x,y)$.

   (b) $\lim_{x\to\infty} \lim_{y\to\infty} f(x,y)$.

   (c) $\lim_{t\to\infty} f(t,t)$.

2. Let a sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ be given by

$$f_n(x) \quad = \quad \frac{x^2}{(1+x^2)^n} \qquad (x \in \mathbb{R}, n \in \mathbb{N}_{\geq 0}) .$$

   Define the partial sum sequence $g_N : \mathbb{R} \to \mathbb{R}$ via

$$g_N(x) \quad := \quad \sum_{n=0}^{N} f_n(x) \qquad (x \in \mathbb{R}, N \in \mathbb{N}) .$$

   Calculate

$$\lim_{N\to\infty} g_N(x)$$

   for any $x \in \mathbb{R}$ and determine whether the convergence is uniform (in x) or not. *Note: you do not have to prove the statement about uniform convergence but please at least heuristically justify it.* Finally, determine whether g_N is a continuous function for finite N and whether the limit function g_∞ is continuous.

3. Define a sequence of functions $f_n : [0,1] \to \mathbb{R}$ via

$$f_n(x) \quad := \quad n^2 x \left(1-x^2\right)^n \qquad (x \in [0,1]\,, n \in \mathbb{N})\ .$$

   (a) Calculate the limit function $\lim_{n\to\infty} f_n$.

   (b) Is the convergence uniform? *You do not have to justify you answer.*

   (c) Calculate the limit number

$$\lim_{n\to\infty} \int_0^1 f_n\ .$$

   (d) Now considering the modified sequence of functions $\tilde{f}_n := \frac{1}{n} f_n$, calculate $\lim_{n\to\infty} \tilde{f}_n$ and $\lim_{n\to\infty} \int_0^1 \tilde{f}_n$.

4. (*Summation by parts formula*) Let $\{ a_n \}_{n=1}^N, \{ b_n \}_{n=1}^N \subseteq \mathbb{C}$ be two finite sequences. Define $B_k := \sum_{n=1}^k b_n$ be the partial sums, with $B_0 \equiv 0$. Show that

$$\sum_{n=M}^{N} a_n b_n \quad = \quad a_N B_N - a_M B_{M-1} - \sum_{n=M}^{N-1} \left(a_{n+1} - a_n\right) B_n\ .$$

5. [extra] (*Abel's theorem on "Abel summation"*) For some sequence $\{ a_n \}_n \subseteq \mathbb{C}$, assume that $\sum_{n=1}^\infty a_n$ converges. Show that

$$\lim_{\varepsilon\to 0^+} \sum_{n=1}^\infty (1-\varepsilon)^n a_n = \sum_{n=1}^\infty a_n\ .$$

   Here are some hints on how to approach this:

   (a) In the first step, assume that $\sum_{n=1}^\infty a_n = 0$ so your goal is to show $\lim_{\varepsilon\to 0^+} \sum_{n=1}^\infty (1-\varepsilon)^n a_n = 0$.

      i. Apply the previous summation by parts formula on (there) a_n (here) $(1-\varepsilon)^n$ and (there) b_n (here) a_n to obtain an identity for $\sum_{n=1}^N (1-\varepsilon)^n a_n$ (at some *finite* partial sum N).

      ii. Since $A_N := \sum_{n=1}^N a_n \to 0$ by assumption, for any $\delta > 0$, there's some $N_\delta \in \mathbb{N}$ such that $|A_n| < \delta$ if $n \geq N_\delta$. So pick some arbitrary δ and divide the sum in the RHS of the identity you obtained in the previous step to $\sum_{n=1}^{N-1} \cdots = \sum_{n=1}^{N_\delta} \cdots + \sum_{n=N_\delta+1}^{N-1} \cdots$.

      iii. Bound each of these terms (in absolute value from above) separately. For the first term, use the fact that $(1-\varepsilon)^n \leq 1$ and for the second term use the fact that within it, $|A_n| \leq \delta$, and afterwards you may calculate the resulting geometric sum of the second term.

      iv. Now take the limits, in the following order: first $N \to \infty$, then $\varepsilon \to 0^+$, and finally $\delta \to 0$. In doing so, note that δ and N_δ are independent of both N and ε by construction.

      v. Conclude that $\lim_{\varepsilon\to 0^+} \left|\sum_{n=1}^\infty (1-\varepsilon)^n a_n\right| \leq 0$ and hence the claim.

   (b) To obtain the general case, define $a_0 := -\sum_{n=1}^\infty a_n$ and add that to both sides of the putative equation to get the equivalent claim

$$\lim_{\varepsilon\to 0^+} \sum_{n=0}^\infty (1-\varepsilon)^n a_n = \sum_{n=0}^\infty a_n\ .$$

      i. Define $b_n = a_{n-1}$, and after factoring out $(1-\varepsilon)^{-1}$, apply the claim to $\lim_{\varepsilon\to 0^+} \sum_{n=1}^\infty (1-\varepsilon)^n b_n$ to get the result, using the fact that $\lim_{\varepsilon\to 0^+} (1-\varepsilon)^{-1} = 1$ and the product of limits is the limit of products.

6. [*extra*] Find the radius of convergence of the following power series $\sum_{n=1}^\infty a_n z^n$. You may employ the so-called *Cauchy-Hadamard formula* for the radius R of absolute convergence of a power series

$$R \quad = \quad \frac{1}{\limsup_{n\to\infty} |a_n|^{\frac{1}{n}}}$$

   (cf. the root test of series). Find R for the following choices:

   (a) $a_n = (\log(n))^2$.

   (b) $a_n = n!$.

   (c) $a_n = \frac{1}{n^2}$.

(d) $a_n = \frac{n^2}{4^n+3n}$.

(e) $a_n = \frac{(n!)^3}{(3n)!}$. You may use the upper and lower bounds on the factorial

$$\frac{n^n}{e^{n-1}} \leq \quad n! \quad \leq \frac{n^{n+1}}{e^{n-1}} \,.$$

(f) $a_n = \frac{f_n(\alpha)f_n(\beta)}{n!f_n(\gamma)}$ for some $\alpha, \beta \in \mathbb{C}$, $\gamma \in \mathbb{C} \setminus \{ n \in \mathbb{Z} \mid n \leq 0 \}$ and

$$f_n(\xi) \quad := \quad \prod_{j=0}^{n-1} (\xi + j) \qquad (\xi \in \mathbb{C}) \,.$$

(g) $a_{2n+1} = 0$ and $2_{2n} = \frac{(-1)^n}{n!(n+r)!} \frac{1}{2^{2n}}$ for some $r \in \mathbb{N}$.

7. Let f be a power series centered at the origin. Prove that f has a power series expansion around any point in its disc of convergence. *If you are curious calculate the coefficients of the new series (but you don't have to do so to obtain credit).*

8. Find a power series expansion of $(1-z)^{-m}$ about $z_0 = 0$ for some $m \in \mathbb{N}$.

9. Calculate the Taylor series coefficients *at* $x = 0$ of the $\mathbb{R} \to \mathbb{R}$ function

$$f(x) \quad := \quad \begin{cases} 0 & x \leq 0 \\ e^{-\frac{1}{x^2}} & x > 0 \end{cases} \,.$$

What does this mean about analyticity about $x_0 = 0$?

10. Find $s \in \mathbb{R}$ for which the following series are convergent, and for which they are absolutely convergent:

(a) $\sum_{n=1}^{\infty} \frac{1}{n^s}$.

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^s}$.

You may use the integral test for convergence of a series and the alternating series test.

11. Show that the following two series are convergent to a finite number and are equal:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \quad = \quad \sum_{n=1}^{\infty} \frac{1}{(2n-1)\,2n} \,.$$

You may find the identity

$$\frac{1}{a} - \frac{1}{b} = \frac{1}{a}(b-a)\frac{1}{b}$$

useful.

12. [extra] (*Riemann's theorem*) For a given $\alpha \in \mathbb{R}$, find a re-arrangement $f : \mathbb{N} \to \mathbb{N}$ (i.e., a bijection) so that

$$\sum_{n=1}^{\infty} \frac{(-1)^{f(n)+1}}{f(n)} \quad = \quad \alpha \,.$$

2 Calculation of contour integrals

In the following integral calculation, to obtain full credit please justify every step of your calculation to reasonable extent.

13. For some $a \in (0,1)$, calculate

$$\int_{x=-\infty}^{\infty} \frac{e^{ax}}{1+e^x} dx \,.$$

14. How many distinct values does

$$\oint_{\partial B_R(0)} \frac{\cos(\pi z)}{z(z-5)^2} dz$$

take as R ranges over $(0,\infty) \setminus \{ 5 \}$? Calculate them.

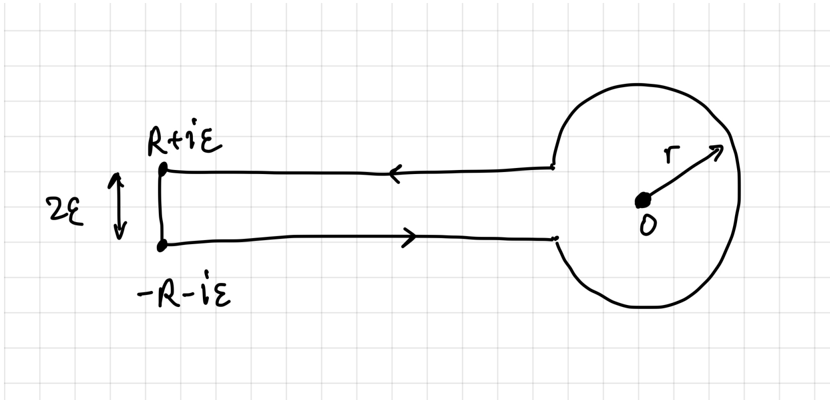

Figure 1: Keyhole contour.

15. Let Γ be a simple closed CCW contour. Show that for all $t \in \mathbb{C}$,

$$\oint_\Gamma \frac{z^3 + tz}{(z-w)^3} \mathrm{d}z \quad = \quad \begin{cases} 6\pi \mathrm{i} w & w \in \text{interior}(\Gamma) \\ 0 & w \notin \text{interior}(\Gamma) \end{cases} .$$

16. [*extra*] For some $z \in \mathbb{C} \setminus [0,4]$, calculate

$$\int_{k=0}^{2\pi} \frac{1}{2 - 2\cos(k) - z} \mathrm{d}k$$

and (second, unrelated part) for some $\xi \in \mathbb{R}$, calculate

$$\int_{x=-\infty}^{\infty} \frac{\mathrm{e}^{-2\pi \mathrm{i} x \xi}}{\cosh(\pi x)} \mathrm{d}x \, .$$

3 Miracles

17. Prove that if $f : \Omega \to \mathbb{C}$ is non-constant analytic on some open connected Ω then f_R cannot have a maximum on interior (Ω).

18. [*extra*] Define $\Gamma : \Omega \to \mathbb{C}$ on $\Omega := \{ z \in \mathbb{C} \mid \mathbb{Re}\{z\} > 0 \}$ via

$$\Gamma(z) \quad = \quad \int_0^\infty t^{z-1} \mathrm{e}^{-t} \mathrm{d}t \, .$$

Show that Γ is analytic on Ω. Define now $\tilde{\Gamma} : \mathbb{C} \to \mathbb{C}$ via

$$\tilde{\Gamma}(z) \quad := \quad \frac{1}{2\mathrm{i}\sin(\pi z)} \oint_C t^{z-1} \mathrm{e}^t \mathrm{d}t$$

where C is a key-hole contour about the negative real axis (see Figure 1). Show that $\tilde{\Gamma}$ is *the* analytic extension of Γ, and that it possesses simple poles on $\{ n \in \mathbb{Z} \mid n \leq 0 \} \subseteq \mathbb{C}$.