MAR 30 2023

MAT330 — $\mathbb{C}$ Analysis — HW5 Sample Sol-ns

$\boxed{Q1}$

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

$$(x,y) \longmapsto \begin{cases} \dfrac{x}{x+y} & x \neq -y \\ 0 & x = -y \end{cases}$$

(a) $$\lim_{y\to\infty} f(x,y) = \lim_{y\to\infty} \frac{x}{x+y} = 0.$$

$$\Longrightarrow \lim_{x\to\infty} \lim_{y\to\infty} f(x,y) = \lim_{x\to\infty} 0 = 0.$$

(b) $$\lim_{x\to\infty} f(x,y) = \lim_{x\to\infty} \frac{x}{x+y} = 1$$

$$\lim_{y\to\infty} \lim_{x\to\infty} f(x,y) = \lim_{y\to\infty} 1 = 1.$$

(c) $$\lim_{t\to\infty} f(t,t) = \lim_{t\to\infty} \frac{t}{2t} = $$

$$= \lim_{t\to\infty} \frac{1}{2} = \frac{1}{2}.$$

Conclusion: exchanging limits must be justified!

$\boxed{Q2}$

$$f_n : \mathbb{R} \to \mathbb{R} \qquad (n \in \mathbb{N}_{\geq 0})$$
$$x \mapsto \frac{x^2}{(1+x^2)^n}$$

$$g_N : \mathbb{R} \to \mathbb{R} \qquad (N \in \mathbb{N}_{\geq 0})$$
$$x \mapsto \sum_{n=0}^{N} f_n(x)$$

$$\lim_{N\to\infty} g_N(x) = \lim_{N\to\infty} \sum_{n=0}^{N} \frac{x^2}{(1+x^2)^n}.$$

Now, $$\sum_{n=0}^{N} (1+x^2)^{-n} = \frac{1+x^2 - (1+x^2)^{-N}}{x^2}$$

so $g_N(x) = 1+x^2 - (1+x^2)^{-N}$.

Note $g_N(0) = 0$, so $\lim_{N\to\infty} g_N(0) = \lim_{N\to\infty} 0 = 0$.

If $x \neq 0$, $1+x^2 > 1$ so $(1+x^2)^{-N} \xrightarrow{N\to\infty} 0$ so $\lim_{N\to\infty} g_N(x) = 1+x^2$.

We find $$g(x) \equiv \lim_{N\to\infty} g_N(x) = \begin{cases} 0 & x=0 \\ 1+x^2 & x\neq 0. \end{cases}$$

<u>Claim</u>: $\{g_N(x)\}_N$ does not converge uniformly in x.

<u>Proof</u>: Assume otherwise. Then $\forall\, \varepsilon>0\ \exists\, N_\varepsilon \in \mathbb{N}$: if $N \geqslant N_\varepsilon$ then

$$\underbrace{|g_N(x) - g(x)| < \varepsilon}_{(1+x^2)^{-N} < \varepsilon} \qquad \forall\, x. \quad (*)$$

Take log of both sides

$$-N \log(1+x^2) < \log(\varepsilon)$$

assume $\varepsilon \leq 1$ WLOG

But $\log \leqslant 0$ on $(0,1]$, so $\log(\varepsilon) \leq 0$ and $\log(1+x^2) \geqslant 0$. Hence

$$N \log(1+x^2) > -\log(\varepsilon) = \log\left(\frac{1}{\varepsilon}\right)$$

$$N > \frac{\log(1/\varepsilon)}{\log(1+x^2)}.$$

Take x: $2N_\varepsilon = \dfrac{\log(1/\varepsilon)}{\log(1+x^2)}$

$$\log(1+x^2) = \frac{\log(1/\varepsilon)}{2N_\varepsilon}$$

$$x = \sqrt{\exp\left(\frac{\log(1/\varepsilon)}{2N_\varepsilon}\right) - 1}.$$

This x, for any $\varepsilon \leq 1$, will violate $(*)$. Indeed, then

$$\left(1+\exp\left(\frac{\log(1/\varepsilon)}{2N_\varepsilon}\right)-1\right)^{-N} =$$

$$= \exp\left(\tfrac{1}{2}\log(\varepsilon)\right) = \sqrt{\varepsilon} > \varepsilon .$$

for $\varepsilon \lesssim 1$.

So $\{g_N(x)\}_N$ does NOT converge uniformly. ☒

For any fixed N,

$\mathbb{R} \ni x \longmapsto g_N(x) = 1+x^2 - (1+x^2)^{-N}$ is continuous.

The limit f^n, $g_\infty(x) = \begin{cases} 0 & x=0 \\ 1+x^2 & x\neq 0 \end{cases}$

is NOT continuous. It has a jump discont. at $x=0$.

<u>Conclusion</u>: The pointwise limit g_∞ of a sequence of continuous f^n's $\{g_N\}_N$ is NOT continuous. This is bcs. the limit is pointwise

but not uniform.

Beware that $\exists$ theorem that says:

"The uniform limit of cont. f^n's is cont." (See Rudin Ch. 7).

Q3

$$f_n : [0,1] \to \mathbb{R} \qquad (n \in \mathbb{N})$$
$$x \mapsto n^2 x (1-x^2)^n$$

(a) $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} n^2 x (1-x^2)^n = 0.$

if $x \in (0,1)$, exp $n \mapsto (1-x^2)^n =$
$= \exp(n \underbrace{\log(1-x^2)}_{\leq 0})$

beats poly. $n \mapsto n^2$.

if $x=0,1$ we have the limit of the zero sequence.

(b) Claim: $\{f_n\}_n \to 0$ but the convergence is merely pointwise, not uniform.

<u>Proof</u>: If, $\forall\ \varepsilon > 0$, there were some $N_\varepsilon \in \mathbb{N}$ s.t. any $n \geq N_\varepsilon$ would imply

$$|f_n(x) - 0| < \varepsilon$$

$$\Leftrightarrow$$

$$|n^2 x (1-x^2)^n| < \varepsilon$$

$$\Leftrightarrow$$

$$n^2 x (1-x^2)^n < \varepsilon$$

Take log of both sides:

$$2\log(n) + \log(x) + n\log(1-x^2) < \log(\varepsilon).$$

Since $\varepsilon \leq 1$ (WLOG), $\log(\varepsilon) \leq 0$, and similarly, $\log(1-x^2) \leq 0$. So mul. both sides by (-1) to get

$$-2\log(n) - \log(x) + n\log\left(\frac{1}{1-x^2}\right) > \log\left(\frac{1}{\varepsilon}\right)$$

As $x \to 0$, $\frac{1}{1-x^2} \to 1$, so $\log\left(\frac{1}{1-x^2}\right) \to 0$, and it becomes harder and harder to satisfy the ineq. for fixed ε.

More formally, since

$$\log(n) \leq \frac{n^s}{s} \qquad \forall\ s > 0,\ n \geq 0$$
$$\log(n) \leq 2\sqrt{n} \qquad \forall\ n \geq 0$$

So if $n \geq \frac{64}{\alpha^2}$ $\exists\, \alpha > 0$,

$$2\log(n) \leq 4\sqrt{n} \leq \frac{1}{2}\alpha n$$

Use this with $\alpha = \log\left(\frac{1}{1-x^2}\right)$.

Hence if $n \geq \frac{64}{\log\left(\frac{1}{1-x^2}\right)^2}$,

$$2\log(n) \leq \frac{1}{2} \log\left(\frac{1}{1-x^2}\right) n$$

whence

$$-2\log(n) - \log(x) + n\log\left(\frac{1}{1-x^2}\right) \geq$$

$$\geq \frac{1}{2}\log\left(\frac{1}{1-x^2}\right) n - \log(x) \overset{!}{>} \log\left(\frac{1}{\varepsilon}\right)$$

$$\Longrightarrow \qquad n \overset{!}{>} \frac{\log(x) + \log\left(\frac{1}{\varepsilon}\right)}{\frac{1}{2}\log\left(\frac{1}{1-x^2}\right)}$$

We see that as $x \to 0$, this becomes harder and harder to satisfy. No one $N_\varepsilon \in \mathbb{N}$ may work for one x. ∎

Note: I wrote the above for the sake of completeness. To obtain full credit it was certainly enough to argue less explicitly, just indicating the constraint becomes harder to satisfy as $x \to 0$.

(c) $$\int_0^1 f_n = \int_0^1 n^2 x(1-x^2)^n \, dx$$

$$= n^2 \int_{y=1}^{0} \left(-\tfrac{1}{2}\right) y^n \, dy$$

$y := 1-x^2$

$dy = -2x\,dx$

$$= \frac{n^2}{2} \int_{y=0}^{1} y^n \, dy$$

$$= \tfrac{1}{2} n \, y^{n+1} \Big|_0^1 = \tfrac{1}{2} n \xrightarrow{n\to\infty} \infty .$$

Conversely, $\int_0^1 \lim_{n\to\infty} f_n = \int_0^1 0 = 0$.

<u>Conclusion</u>: If $f_n \to f$ pointwise but not uniformly it may certainly happen that $\lim \int f_n \neq \int \lim f_n$

(d) If we replace f_n w/ $\tilde{f}_n := \frac{1}{n} f_n$,

we find

$$\lim_{n\to\infty} \tilde{f}_n = 0 \qquad \text{still}$$

but now

$$\lim_{n\to\infty} \int_0^1 \tilde{f}_n = \frac{1}{2} \neq \int_0^1 \lim_{n\to\infty} \tilde{f}_n = 0.$$

<u>Conclusion</u>: Even if all limits are finite this problem may occur, of not being able to exchange int. and limits.

$\boxed{Q4}$ Let $N \in \mathbb{N}_{\geq 1}$, $\{a_n\}_{n=1}^N, \{b_n\}_{n=1}^N \subseteq \mathbb{C}$,

$$B_k := \sum_{n=1}^{k} b_n \quad (k \geq 1); \quad B_0 := 0$$

<u>Claim</u>: For any $M \in \{1, \ldots, N-1\}$,

$$\sum_{n=M}^{N} a_n b_n = a_N B_N - a_M B_{M-1} - \sum_{n=M}^{N-1} (a_{n+1} - a_n) B_n$$

<u>Proof</u>: Note $B_k - B_{k-1} = b_k$ so

$$\sum_{n=M}^{N} a_n b_n = \sum_{n=M}^{N} a_n (B_n - B_{n-1})$$

open bracket

$$= \sum_{n=M}^{N} a_n B_n - \sum_{n=M}^{N} a_n B_{n-1}$$

Separate 1st & last term from both sums resp.

$$= a_N B_N + \sum_{n=M}^{N-1} a_n B_n - a_M B_{M-1} - \sum_{n=M+1}^{N} a_n B_{n-1}$$

Shift sum var. in 2nd sum

$$= a_N B_N - a_M B_{M-1} + \sum_{n=M}^{N-1} a_n B_n - \sum_{n=M}^{N-1} a_{n+1} B_n$$

Combine two sums

$$= a_N B_N - a_M B_{M-1} - \sum_{n=M}^{N-1} (a_{n+1} - a_n) B_n .$$

$\blacksquare$

Q5

Claim: Let $\{a_n\}_{n=1}^{\infty} \subseteq \mathbb{C}$: $\sum_{n=1}^{\infty} a_n < \infty$.

Then $\lim_{\varepsilon \to 0^+} \sum_{n=1}^{\infty} (1-\varepsilon)^n a_n = \sum_{n=1}^{\infty} a_n$.

Proof: Step 1: Assume $\sum_{n=1}^{\infty} a_n = 0$.

Then we're trying to prove

$\lim_{\varepsilon \to 0^+} \sum_{n=1}^{\infty} (1-\varepsilon)^n a_n$ $\exists$ and equals zero.

Apply $\boxed{Q4}$ on $\sum_{n=1}^{N} (1-\varepsilon)^n a_n$ but

w/ $a_n \mapsto (1-\varepsilon)^n$, $b_n \mapsto a_n$ to get

$$\sum_{n=1}^{N} (1-\varepsilon)^n a_n \underset{A_0 \equiv 0}{=} (1-\varepsilon)^N A_N - \sum_{n=1}^{N-1} \left[(1-\varepsilon)^{n+1} - (1-\varepsilon)^n\right] A_n$$

$$= \underbrace{\underbrace{(1-\varepsilon)^N}_{\to 0} \underbrace{A_N}_{\to 0}}_{\to 0} + \varepsilon \sum_{n=1}^{N-1} (1-\varepsilon)^n A_n$$

Let $\delta > 0$. Pick $N_\delta \in \mathbb{N}$: if $n \geq N_\delta$, $|A_n| < \delta$.
(Possible since $A_n \to 0$).
Then

$$\varepsilon \sum_{n=1}^{N-1} (1-\varepsilon)^n A_n = \underbrace{\varepsilon \sum_{n=1}^{N_\delta} (1-\varepsilon)^n A_n}_{=: \mathrm{I}} + \underbrace{\varepsilon \sum_{n=N_\delta+1}^{N-1} (1-\varepsilon)^n A_n}_{=: \mathrm{II}}$$

$$|\mathrm{II}| \leq \varepsilon \delta \sum_{n=N_\delta+1}^{N-1} (1-\varepsilon)^n = \delta \left[(1-\varepsilon)^{N_\delta+1} - (1-\varepsilon)^N\right]$$

$$= \delta (1-\varepsilon)^{N_\delta+1} \left(1 - (1-\varepsilon)^{N-N_\delta-1}\right)$$

$$|\mathrm{I}| \leq \left| \varepsilon \sum_{n=1}^{N_\delta} (1-\varepsilon)^n A_n \right| \leq \varepsilon \sum_{n=1}^{N_\delta} (1-\varepsilon)^n |A_n|$$

$$\leq \varepsilon \sum_{n=1}^{N_\delta} |A_n|$$

We find

$$\left| \sum_{n=1}^{N} (1-\varepsilon)^n a_n \right| \leq (1-\varepsilon)^N |A_N| + \varepsilon \sum_{n=1}^{N_\delta} |A_n| + \delta (1-\varepsilon)^{N_\delta+1} \left(1-(1-\varepsilon)^{N-N_\delta-1}\right)$$

Take now the limit $N \to \infty$ to get:

$$\lim_{N\to\infty} \left| \sum_{n=1}^{N} (1-\varepsilon)^n a_n \right| \leq \varepsilon \sum_{n=1}^{N_\delta} |A_n| + \delta (1-\varepsilon)^{N_\delta+1}$$

(Note δ, N_δ are indep. of N!)

Take now the limit $\varepsilon \to 0$ to get:

$$\lim_{\varepsilon\to 0} \lim_{N\to\infty} \left| \sum_{n=1}^{N} (1-\varepsilon)^n a_n \right| \leq \delta$$

(Note δ, N_δ are indep. of ε!)

Since $\delta > 0$ was arbitrary, we conclude

$$\lim_{\varepsilon\to 0^+} \lim_{N\to\infty} \sum_{n=1}^{N} (1-\varepsilon)^n a_n = 0.$$

<u>Step 2</u>: If $\sum_{n=1}^{\infty} a_n \neq 0$, define

$$a_0 := -\sum_{n=1}^{\infty} a_n$$

W.T.S. $\lim_{\varepsilon\to 0^+} \sum_{n=1}^{\infty} (1-\varepsilon)^n a_n = \sum_{n=1}^{\infty} a_n$

$\Leftrightarrow a_0 + \lim_{\varepsilon\to 0^+} \sum_{n=1}^{\infty} (1-\varepsilon)^n a_n = a_0 + \sum_{n=1}^{\infty} a_n$

$\Leftrightarrow \lim_{\varepsilon\to 0^+} \sum_{n=0}^{\infty} (1-\varepsilon)^n a_n = \sum_{n=0}^{\infty} a_n$.

Define $b_n := a_{n-1}$ $\forall\ n \geq 1$

Note $\sum_{n=1}^{\infty} b_n = 0$ by construction.

W.T.S.

$$\lim_{\varepsilon\to 0^+} \sum_{n=0}^{\infty} (1-\varepsilon)^n b_{n+1} = \sum_{n=0}^{\infty} b_{n+1}$$

$$\Leftrightarrow \lim_{\varepsilon\to 0^+} \sum_{n=1}^{\infty} (1-\varepsilon)^{n-1} b_n = \sum_{n=1}^{\infty} b_n$$

$$\Leftrightarrow \lim_{\varepsilon\to 0^+} (1-\varepsilon)^{-1} \sum_{n=1}^{\infty} (1-\varepsilon)^n b_n = \sum_{n=1}^{\infty} b_n$$

Now since we've just shown in step 1 that

$$\lim_{\varepsilon\to 0^+} \sum_{n=1}^{\infty} (1-\varepsilon)^n b_n = \sum_{n=1}^{\infty} b_n$$

and since $\lim_{\varepsilon\to 0^+} (1-\varepsilon)^{-1} = 1$, we're finished.

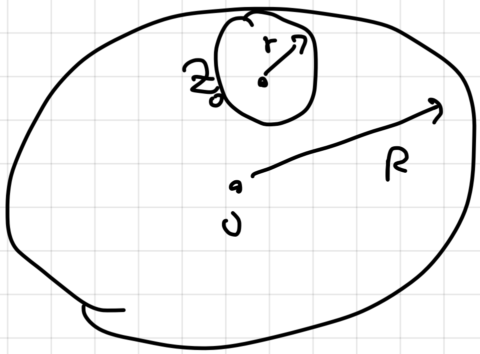
$\blacksquare$

⚠ [extra] Q's will be solved in the end.

$\boxed{\text{Q7}}$ Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ $\exists \{a_n\}_{n=0}^{\infty} \subseteq \mathbb{C}$, w/ some radius $R>0$ of convergence.

$\Leftrightarrow f$ is analytic on $B_R(0)$

So we may choose any other point in $B_R(0)$ to make an expansion, just so long as we make the new radius of conv. smaller:



$r := R - |z_0|$

Though this was not explicitly asked (and not required for full credit), let's calculate the new coeff.:

$$\sum_{n=0}^{\infty} a_n z^n \overset{!}{=} \sum_{n=0}^{\infty} b_n (z-z_0)^n$$

To find b_n, let us calc.

$$\begin{aligned} f(z) &= f(z - z_0 + z_0) \\ &= \sum_{n=0}^{\infty} a_n (z - z_0 + z_0)^n \\ &= \sum_{n=0}^{\infty} a_n \sum_{j=0}^{n} \binom{n}{j} (z-z_0)^j z_0^{n-j} \end{aligned}$$

$$= \sum_{n=0}^{\infty}\sum_{j=0}^{n} a_n \binom{n}{j} (z-z_0)^j z_0^{n-j}$$

Sum on $j \leq n$:

$$\sum_{n=0}^{\infty}\sum_{j=0}^{n} S(n,j) = \sum_{j=0}^{\infty}\sum_{n=j}^{\infty} S(n,j)$$

$$\Rightarrow f(z) = \sum_{j=0}^{\infty}\sum_{n=j}^{\infty} a_n \binom{n}{j} (z-z_0)^j z_0^{n-j}$$

$$= \sum_{j=0}^{\infty} \underbrace{\left(\sum_{n=j}^{\infty} a_n \binom{n}{j} z_0^{n-j} \right)}_{=: b_j} (z-z_0)^j$$

$\boxed{\text{Q8}}$ Want a power series expansion of $(1-z)^{-m}$ $\exists\, m \in \mathbb{N}_{\geq 1}$ @ $z_0 = 0$.

$f(z) := (1-z)^{-m}$ $\quad \forall\ |z| < 1.$

Taylor's thm. says

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$$

$$f^{(n)}(z) \equiv \partial_z^n (1-z)^{-m}$$

$$= (-m)(-m-1)\cdots(-m-n+1)(1-z)^{-m-n}$$

Hence $f^{(n)}(0) = (-1)^n m(m+1)\cdots(m+n-1)$

$$= (-1)^n \frac{(m+n-1)!}{(m-1)!}$$

Hence $f(z) = \sum_{n=0}^{\infty} (-1)^n \frac{(n+m-1)!}{(m-1)!\, n!} z^n$

$$= \sum_{n=0}^{\infty} \underbrace{(-1)^n \binom{n+m-1}{n}}_{\text{desired expansion coeff.}} z^n .$$

$\boxed{Q9}$ $f: \mathbb{R} \to \mathbb{R}$, $x \mapsto \begin{cases} 0 & x \leq 0 \\ \exp(-1/x^2) & x > 0 \end{cases}$

Want Taylor coeff. about zero.

$f^{(n)}(0) = 0$ bcs. $f^{(n)}(x) = \exp(-1/x^2) \cdot \text{poly}_n(x)^{-1}$

for some $\text{poly}_n(x)$ which will always lose against how quickly $\exp(-1/x^2) \to 0$ as $x \to 0$,

$$\Rightarrow \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = 0 \quad \text{for } |x| \text{ small}$$

But we know $f \neq 0$ for $x > 0$ small! ($\exp$ is never zero) so f cannot be analytic at $x=0$, though it is smooth on $\mathbb{R}$ and analytic on $\mathbb{R} \setminus \{0\}$, and has an ess. sing. on $x=0$.

$\boxed{Q10}$ (a) Define $\sum_{n=1}^{\infty} n^{-s}$ $\exists$ $s \in \mathbb{R}$.

Seek phase diagram of s (conv., abs. conv.,)

This is the famous Riemann zeta f^n.

Claim: It converges as long as $s > 1$.

Proof: By the integral test for convergence of a series (see ...), we have

$$\sum_{n=1}^{\infty} n^{-s} \quad \text{converges} \iff$$

$$\int_1^\infty x^{-s}\,dx < \infty$$

since $n \mapsto n^{-s}$ is decreasing.

But
$$\int_1^\infty x^{-s}\,dx = \frac{1}{-s+1} x^{-s+1}\Big|_1^\infty$$
$$= \begin{cases} \frac{1}{1-s} & s>1 \\ \infty & s \leq 1. \end{cases}$$
$\blacksquare$

Since $n^{-s} \geq 0$, the conv. is abs.

Claim: If $s \leq 1$ series diverges.

Proof: Same integral comparison test.

(b) Do the same for $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^s}$.

Note this is minus the famous Dirichlet eta f^n.

By the above clearly if $s>1$, we get abs. conv. and if $s \leq 1$, no abs. conv.

Claim: If $s>0$ we have convergence.
If $s \leq 0$ we get divergence.

Proof: By the alternating series test (see...)

$\sum_{n=1}^{\infty} (-1)^n a_n$ converges if

(1) $n \mapsto |a_n|$ is monotone decr.

(2) $\lim_{n\to\infty} a_n = 0$.

(1) and (2) hold if $s>0$.

Now if $s \leq 0$ we cannot converge since $n \mapsto n^{-s}$ is not $\to 0$ as $n \to \infty$.

$\boxtimes$

$\boxed{Q11}$ The LHS is $\boxed{Q10}$ (b) @ $s=1$ (up to a minus sign).

Also,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

$$= \sum_{n=1}^{\infty} \underbrace{\frac{1}{2n-1} - \frac{1}{2n}}_{\frac{1}{2n-1}\left[2n-(2n-1)\right]\frac{1}{2n} = \frac{1}{2n(2n-1)}}$$

Since the two are equal, the RHS $< \infty$ too.

$\boxed{Q12}$ For any $\alpha \in \mathbb{R}$, want a rearrangement $f: \mathbb{N} \to \mathbb{N}$ (i.e. a permutation, = bijection) s.t. $$\sum_{n=1}^{\infty} \frac{(-1)^{f(n)+1}}{f(n)} = \alpha.$$

(This is the famous Riemann thm.)

Proceed as follows:

By the integral test for conv. of series,

$$\underbrace{\int_1^N \frac{1}{x}\,dx}_{\log(N)} \leq \sum_{n=1}^{N} \frac{1}{n} \leq 1 + \underbrace{\int_1^N \frac{1}{x}\,dx}_{\log(N)}$$

$$\Rightarrow \frac{\sum_{n=1}^{N} \frac{1}{n}}{\log(N)} \xrightarrow{N\to\infty} 1$$

So write $$\sum_{n=1}^{N} \frac{1}{n} = a + \log(N) + \varepsilon_N$$

$\exists$ $a \in (0,1)$ and $\varepsilon_N \in (0,1)$ depends on N but

s.t. $$\frac{\varepsilon_N}{\log(N)} \longrightarrow 0 \quad \text{as} \quad N \to \infty.$$

$$\Rightarrow \sum_{n=1}^{N} \frac{1}{2n} = \frac{1}{2}a + \frac{1}{2}\log(N) + \frac{1}{2}\varepsilon_N$$

$$\sum_{n=1}^{N} \frac{1}{2n-1} = \sum_{n=1}^{2N-1} \frac{1}{n} - \sum_{n=1}^{N-1} \frac{1}{2n}$$

$$= a + \log(2N-1) + \varepsilon_{2N-1}$$

$$- \frac{1}{2}a - \frac{1}{2}\log(N-1) - \frac{1}{2}\varepsilon_{N-1}$$

$$= \frac{1}{2}a + \log\Big(\underbrace{\frac{2N-1}{\sqrt{N-1}}}\Big) + \varepsilon_{2N-1} - \frac{1}{2}\varepsilon_{N-1}$$

$$\frac{2N-2+1}{\sqrt{N-1}} = 2\frac{N-1}{\sqrt{N-1}} + \frac{1}{\sqrt{N-1}}$$

$$= \frac{1}{2}a + \log\Big(2\sqrt{N-1} + \frac{1}{\sqrt{N-1}}\Big) + \varepsilon_{2N-1} - \frac{1}{2}\varepsilon_{N-1}$$

$$=: \frac{1}{2}a + \log(2) + \frac{1}{2}\log(N) + \underbrace{\tilde{\varepsilon}_N}_{\to 0}$$

So let us make the rearrangement:

N odd (positive) terms

M even (negative) terms

N odd (positive) terms

⋮

We get:

$$\sum_{n=1}^{N} \frac{1}{2n-1} - \sum_{n=1}^{M} \frac{1}{2n} + \sum_{n=N+1}^{2N+1} \frac{1}{2n-1} - \sum_{n=M+1}^{2M+1} \frac{1}{2n} + \dots$$

If we take now a partial sum w/ fixed

ratio of these terms, say, $L := KN + KM$ terms,

w/ N, M fixed, but $K \to \infty$, then that

partial L sum equals:

$$\frac{1}{2}a + \log(2) + \frac{1}{2}\log(KN) + \varepsilon_{KN} -$$

$$- \frac{1}{2}a - \frac{1}{2}\log(KM) + \varepsilon_{KM}$$

$$=: \log(2) + \frac{1}{2}\log(N/M) + \tilde{\varepsilon}_K$$

so as $K \to \infty$ we find

$$\log(2\sqrt{N/M})$$

$\Rightarrow$ can get any # of this form.

More generally, let $\{N_K\}_K$, $\{M_K\}_K$ be variable lengths, so that

$$\frac{N_K}{M_K} \xrightarrow{K \to \infty} \beta \in \mathbb{R}$$

Then we get $\log(2\sqrt{\beta}) \overset{!}{=} \alpha$.

Pick $\boxed{\beta = \frac{1}{4} e^{2\alpha}}$.

Q13 See Example 7.32 in the lecture notes.

Q14 $$I(R) := \oint_{\partial B_R(0)} \underbrace{\frac{\cos(\pi z)}{z(z-5)^2}}_{=: f(z)} dz$$

⊛ If $R \in (0,5)$, $\exists$ only one pole (@ $z=0$)

so by residue formula,

$$I(R) = 2\pi i \operatorname{res}_0(f).$$

$$\operatorname{res}_0(f) = \lim_{z\to 0} z \frac{\cos(\pi z)}{z(z-5)^2} = \frac{1}{25}.$$

$$\Longrightarrow \quad I(R) = 2\pi i \frac{1}{25}.$$

⊛ If $R>5$, $\exists$ two poles so we need to add also the other contribution:

$$\operatorname{res}_5(f) = \lim_{z\to 5} \partial_z (z-5)^2 \frac{\cos(\pi z)}{z(z-5)^2} =$$

$$= \lim_{z\to 5} \partial_z \frac{\cos(\pi z)}{z} =$$

$$= \lim_{z\to 5} \left[-\frac{\cos(\pi z)}{z^2} - \frac{\pi \sin(\pi z)}{z} \right]$$

$$= \frac{1}{25}$$

$$\Rightarrow \quad I(R) = 2 \times 2\pi i \frac{1}{25} = \frac{4\pi i}{25}.$$

$\boxed{Q15}$ Let Γ be a simple closed CCW contour.

<u>Claim</u>: $$\oint_\Gamma \frac{z^3 + tz}{(z-w)^3}\, dz = 6\pi i w \begin{cases} 0 & w \notin \operatorname{int}(\Gamma) \\ 1 & w \in \operatorname{int}(\Gamma). \end{cases}$$

for any $t \in \mathbb{C}$.

<u>Proof</u>: Since $f(z) := \dfrac{z^3 + tz}{(z-w)^3}$ is analytic if

$w \notin \operatorname{int}(\Gamma)$ we get the result by Cauchy.

If $w \in \operatorname{int}(\Gamma)$, this is covered by the residue thm.:

$$\operatorname{res}_w(f) = \lim_{z \to w} \frac{1}{2} \partial_z^2 (z-w)^3 f(z)$$

$$= \lim_{z \to w} \frac{1}{2} \partial_z^2 (z^3 + tz)$$

$$= \lim_{z \to w} \frac{6}{2} z = 3w.$$

Hence then $\oint f(z)\,dz = 2\pi i \cdot 3w = 6\pi i w$.

$\blacksquare$

$\boxed{Q17}$

<u>Claim</u>: Let $f : \Omega \to \mathbb{C}$ be an analytic non-const. f^n on $\Omega \in \text{Open}(\mathbb{C}) \cap \text{Conn.}(\mathbb{C})$. Then f_R cannot have a max on Ω.

<u>Proof</u>: Let
$$g : \Omega \to \mathbb{C}$$
$$z \mapsto \exp(f(z)).$$

Since exp is analytic, g is.

Moreover, $|g(z)| = |\exp(f_R(z) + i f_I(z))|$
$$= \exp(f_R(z))$$

Apply max principle (Corollary 7.52 in the notes) on g to get that $z \mapsto \exp(f_R(z))$ cannot have a max on Ω, and since exp is monotone incr. and injective on $\mathbb{R}$, our result.

$\blacksquare$

Extras

Q5 and Q12 have already appeared above.

$\boxed{Q6}$ We calc. the radius of conv. of series via the Cauchy-Hadamard formula:

$$R = \left(\limsup_{n\to\infty} |a_n|^{1/n}\right)^{-1}.$$

Note: ① One may also use the ratio test:

$$z \mapsto \sum_{n=0}^{\infty} a_n z^n \quad \text{converges when} \quad \left(\lim_{n\to\infty} \left|\frac{a_{n+1} z^{n+1}}{a_n z^n}\right|\right) < 1.$$

② Cauchy-Hadamard comes from root test:

$$z \mapsto \sum_{n=0}^{\infty} a_n z^n \quad \text{converges when} \quad \left(\limsup_{n\to\infty} \underbrace{|a_n z^n|^{1/n}}_{|a_n|^{1/n}\,|z|}\right) < 1$$

$$\Leftrightarrow \quad |z| < \left(\limsup_{n\to\infty} |a_n|^{1/n}\right)^{-1}.$$

(a) $a_n = (\log(n))^2$

$$|a_n|^{1/n} = \log(n)^{\frac{2}{n}} = \exp(\log(\log(n)^{\frac{2}{n}}))$$

$$= \exp(\underbrace{\tfrac{2}{n}}_{\to 0} \underbrace{\log(\log(n))}_{\to\infty})$$

But poly. wins over $\log\circ\log$. $\Rightarrow |a_n|^{1/n} \to e^0 = 1$

$\Rightarrow \boxed{R=1}$.

(b) $a_n = n!$

ratio test:

$$\left|\frac{a_{n+1} z^{n+1}}{a_n z^n}\right| = \frac{(n+1)!}{n!}|z| = (n+1)|z| \longrightarrow \infty$$

$$\Longrightarrow \boxed{R=0}.$$

(c) $a_n = n^{-2}$

$$|a_n|^{1/n} = (n^{-2})^{1/n} = n^{-2/n} = \exp(-\tfrac{2}{n}\log(n))$$

$$\longrightarrow e^0 = 1. \quad \Longrightarrow \boxed{R=1}.$$

(d) $a_n = \dfrac{n^2}{4^n+3n}$

$$|a_n|^{1/n} = \exp\left(\tfrac{1}{n}\log\left(\frac{n^2}{4^n+3n}\right)\right)$$

$$= \exp\Big(\underbrace{\tfrac{2}{n}\log(n)}_{\to 0} - \frac{\log(4^n+3n)}{n}\Big)$$

$$\sim \exp(-\log(4(1+3\cdot\underbrace{4^{-n}n}_{\to 0}))$$

$$\sim \exp(-\log(4)) = \frac{1}{4}$$

$$\Longrightarrow \boxed{R=4}.$$

(e) $a_n = \dfrac{(n!)^3}{(3n)!}$

We have by the Stirling approx.:

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

$$= \sqrt{2\pi n}\, \exp(n \log(\tfrac{n}{e}))$$

$$= \sqrt{2\pi n}\, \exp(n(\log(n) - 1))$$

$$\sim \sqrt{n}\, e^{n \log(n)}$$

Hence $a_n \sim \dfrac{n^{3/2}\, e^{3n \log(n)}}{(3n)^{1/2}\, e^{3n \log(3n)}}$

$$\sim n \exp(3n \log(\tfrac{n}{3n}))$$

$$\sim n \exp(-3 \log(3) n)$$

$$a_n^{1/n} \sim \exp(-3 \log(3))$$

$$\Longrightarrow \boxed{R = \exp(3 \log(3)) = 27}.$$

(f) $a_n = \dfrac{f_n(\alpha) f_n(\beta)}{f_n(\gamma)\, n!}$ $\quad \alpha, \beta \in \mathbb{C}, \quad \gamma \in \mathbb{C} \setminus (-\mathbb{N}_{\geq 0})$

$$f_n(z) := \prod_{j=0}^{n-1} (z + j) = z(z+1)(z+2) \cdots (z+n-1)$$

This corresponds to the hypergeometric series.

$$\left|\frac{a_{n+1} z^{n+1}}{a_n z^n}\right| = |z| \left|\frac{f_{n+1}(\alpha)}{f_n(\alpha)}\right| \left|\frac{f_{n+1}(\beta)}{f_n(\beta)}\right| \left|\frac{f_n(\gamma)}{f_{n+1}(\gamma)}\right| \frac{n!}{(n+1)!}$$

$$\left|\frac{f_{n+1}(\xi)}{f_n(\xi)}\right| = |\xi + n|$$

So we find

$$\left|\frac{a_{n+1} z^{n+1}}{a_n z^n}\right| = |z| \frac{|\alpha+n||\beta+n|}{|\gamma+n|(n+1)} \overset{n\to\infty}{\sim} |z| \frac{n^2}{n^2} = |z|.$$

$$\Longrightarrow \boxed{R = 1}.$$

(g) $a_{2n+1} = 0$ and $a_{2n} = \dfrac{(-1)^n}{n!(n+r)!} \dfrac{1}{2^{2n}}$ $\exists\, r \in \mathbb{N}$.

Using Stirling again we have

$$|a_{2n}| \sim \exp(-n\log(n) - (n+r)\log(n+r) - 2n\log(2))$$

$$|a_{2n}|^{1/2n} \sim \exp\left(-\tfrac{1}{2}\log(n) - \tfrac{n+r}{2n}\log(n+r) - \log(2)\right)$$

$$\longrightarrow e^{-\infty} = 0$$

$$\Longrightarrow \boxed{R = \infty}.$$

$\boxed{\text{Q16}}$ Part 1

$z \in \mathbb{C} \setminus [0,4]$

$$I(z) := \int_{k=0}^{2\pi} [2 - 2\cos(k) - z]^{-1} \, dk .$$

Rewrite as a contour int. via $\lambda := e^{ik}$:

$$d\lambda = e^{ik} i dk = i\lambda dk$$

$$I(z) = \oint_{\lambda \in \partial B_1(0)} \left(2 - \lambda - \tfrac{1}{\lambda} - z\right)^{-1} \frac{1}{i\lambda} d\lambda$$

$$= -i \oint_{\lambda \in \partial B_1(0)} \left((2-z)\lambda - 1 - \lambda^2\right)^{-1} d\lambda$$

Write $\beta := \frac{1}{2}(2-z) \Rightarrow z = 2 - 2\beta$

$$\Rightarrow (2-z)\lambda - 1 - \lambda^2 = 2\beta\lambda - 1 - \lambda^2$$

$$= (\lambda \underbrace{- \beta - \sqrt{\beta^2 - 1}}_{=: -\alpha_+(\beta)})(\lambda \underbrace{- \beta + \sqrt{\beta^2 - 1}}_{=: -\alpha_-(\beta)})$$

$$\Rightarrow I(z) = -i \oint_{\lambda \in \partial B_1(0)} d\lambda \frac{1}{(\lambda - \alpha_+(\beta))(\lambda - \alpha_-(\beta))}$$

Now there are cases to be considered according to the value of $\beta \in \mathbb{C}$:

Claim: If $\beta \in [-1,1]$ then $|\alpha_\pm(\beta)| = 1$.

Proof: $$\alpha_\pm(\beta) \equiv \beta \pm \sqrt{\beta^2-1} = \beta \pm i\sqrt{1-\beta^2} \quad \beta \in [-1,1]$$

$$|\alpha_\pm(\beta)|^2 = \beta^2 + 1 - \beta^2 = 1.$$

Claim: If $\beta \in \mathbb{C} \setminus [-1,1]$ then $\alpha_i(\beta) \in B_1(0)$
$\alpha_j(\beta) \in \overline{B_1(0)}^c$.

where $(i,j) \in \{(+,-), (-,+)\}$.

Proof: Consider $f(\lambda) := 2\beta\lambda - 1 - \lambda^2$
and $g(\lambda) := 2\beta\lambda - \lambda^2$
g has two roots: $\lambda = 0$ and
$\lambda = 2\beta$

So if $|\beta| > \frac{1}{2}$, g has the desired property of having only one root within $B_1(0)$.

Then applying Rouche, via

$$\underbrace{|f(\lambda) - g(\lambda)|}_{=1} \overset{!}{<} \underbrace{|g(\lambda)|}_{\underbrace{|\lambda|}_{=1}\underbrace{|2\beta-\lambda|}_{> 2|\beta|-1}} \quad \forall\ |\lambda| = 1$$

$\Rightarrow$ If $|\beta| > 1$, we establish the desired property for f thanks to Rouché.

If $\beta = x + i\varepsilon \quad \exists\ x \in [-1,1], \varepsilon \in \mathbb{R} : |\varepsilon| \ll 1$,

$$\beta \pm \sqrt{\beta^2 - 1} = x + i\varepsilon \pm \sqrt{x^2 - \varepsilon^2 + 2i\varepsilon x - 1}$$
$$\approx x + i\varepsilon \pm \sqrt{-(1-x^2) + 2i\varepsilon x}$$

$$= x + i\varepsilon \pm i\sqrt{1-x^2-2i\varepsilon x}$$

$$\approx x + i\varepsilon \pm i\sqrt{1-x^2} \pm \frac{x\varepsilon}{\sqrt{1-x^2}}$$

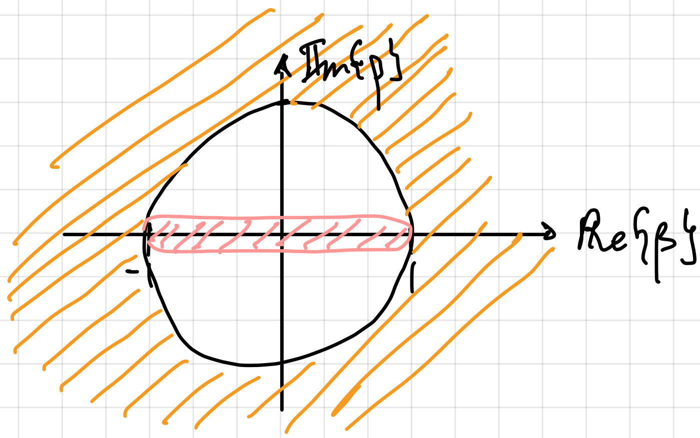
$$|\beta \pm \sqrt{\beta^2-1}|^2 = x^2\left(1 \pm \frac{\varepsilon}{\sqrt{1-x^2}}\right)^2 + \left(\varepsilon \pm \sqrt{1-x^2}\right)^2$$

$$= x^2\left(1 \pm \frac{2\varepsilon}{\sqrt{1-x^2}}\right) + 1 - x^2 \pm 2\varepsilon\sqrt{1-x^2}$$

$$= 1 \pm 2\varepsilon\left(\frac{x^2}{\sqrt{1-x^2}} + \sqrt{1-x^2}\right)$$

$$= 1 \pm 2\varepsilon \frac{1}{\sqrt{1-x^2}} + O(\varepsilon^2)$$

$\Longrightarrow$ The property is established on a sliver around $[-1,1]$.



$$g_t(\lambda) := 2\lambda(x+it) - 1 - \lambda^2$$

$$|g_\varepsilon(\lambda) - g_{t+\varepsilon}(\lambda)| = |2\lambda i t| \underset{|\lambda|=1}{=} 2|t|$$

$$2|t| \overset{!}{<} |g_\varepsilon(\lambda)| =$$

[TODO...]

Since $\alpha_\pm(\beta)$ are in/out of $B_1(0)$,

using the residue formula we find:

$$I(z) = -i \oint_{\lambda \in \partial B_1(0)} d\lambda \, \frac{1}{(\lambda-\alpha_+(\beta))(\lambda-\alpha_-(\beta))} =$$

depending on which one is in/out of $B_1(0)$

$$= -i \, 2\pi i \, \frac{\pm 1}{\alpha_+(\beta)-\alpha_-(\beta)}$$

$$= \frac{\pm 2\pi}{\beta+\sqrt{\beta^2-1}-\beta+\sqrt{\beta^2-1}} = \frac{\pm 2\pi}{2\sqrt{\beta^2-1}}$$

$$= \frac{\pm \pi}{\sqrt{\beta^2-1}} ,$$

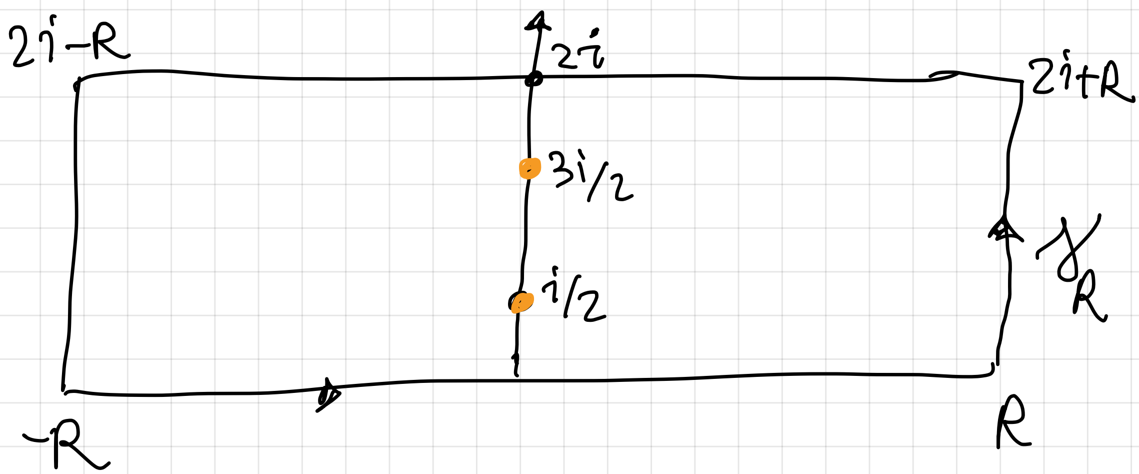
$$\Rightarrow \boxed{I(z) = \frac{\pm \pi}{\sqrt{\frac{1}{4}(2-z)^2-1}}}$$

At least for $|\beta|>1$.

Part 2

$$I(\xi) = \int_{x \in \mathbb{R}} \frac{e^{-2\pi i x \xi}}{\cosh(\pi x)} \, dx$$

$$f(z) := \frac{e^{-2\pi i \xi z}}{\cosh(\pi z)} \qquad \text{meromor.}$$



$$\cosh(\pi z) = 0 \iff e^{\pi z} + e^{-\pi z} = 0$$

$$\iff 1 + e^{2\pi z} = 0$$

$$\iff 2\pi z = i\pi + 2\pi i n \qquad (n \in \mathbb{Z})$$

$$z = \frac{i}{2} + in \qquad (n \in \mathbb{Z})$$

$$z \in i\left\{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots\right\}$$

γ_R includes only the poles at $\frac{i}{2}$ and $\frac{3i}{2}$.

Each pole is simple:

$$(z-\tfrac{il}{2})f(z) = \frac{(z-\frac{il}{2})\exp(-2\pi i\xi z)}{\frac{1}{2}\exp(-\pi z)(1+\exp(2\pi z))}$$

$$= 2e^{-2\pi i\xi z+\pi z}\underbrace{\frac{z-\frac{il}{2}}{e^{2\pi z}-e^{2\pi\frac{il}{2}}}}_{\to\left(\partial_z\big|_{z=\frac{il}{2}} e^{2\pi z}\right)^{-1}}$$

$$= 2e^{-2\pi i\xi\frac{il}{2}+i\pi\frac{l}{2}}\left(2\pi \underbrace{e^{2\pi\frac{il}{2}}}_{-1}\right)^{-1}$$

$$= -\frac{e^{+\pi\xi l+i\pi\frac{l}{2}}}{\pi} = \begin{cases}\nearrow \dfrac{e^{\pi\xi}}{\pi i} & l=1\\[2ex] \searrow -\dfrac{e^{+3\pi\xi}}{\pi i} & l=3\end{cases}$$

Vertical legs:

$$\left|\int_0^2 \frac{e^{-2\pi i\xi(R+ix)}}{\cosh(\pi(R+ix))}\,i\,dx\right| \leq$$

$$\leq \int_0^2 \frac{e^{4\pi|\xi|}}{\underbrace{\frac{1}{2}\left|e^{\pi(R+ix)}-e^{-\pi(R+ix)}\right|}_{\geq e^{\pi R}-e^{-\pi R}}}\,dx \sim 4e^{4\pi|\xi|-\pi R} \to 0.$$

The top leg equals:

$$\int_{-R}^{R} \frac{e^{-2\pi i \xi (x+2i)}}{\cosh(\pi(x+2i))} dx = e^{4\pi\xi} \int_{-R}^{R} \frac{e^{-2\pi i \xi x}}{\underbrace{\cosh(\pi x + 2\pi i)}_{\text{periodic}}} dx$$

$$\xrightarrow{R\to\infty} e^{4\pi\xi} I(\xi).$$

Hence res. formula says

$$(1 - e^{4\pi\xi}) I = 2\pi i \left(\frac{e^{\pi\xi}}{\pi i} - \frac{e^{3\pi\xi}}{\pi i} \right)$$

$$= -2 e^{2\pi\xi} (e^{\pi\xi} - e^{-\pi\xi})$$

$$= -4 e^{2\pi\xi} \sinh(\pi\xi).$$

We find $\quad I(\xi) = \dfrac{-4 e^{2\pi\xi} \sinh(\pi\xi)}{1 - e^{4\pi\xi}}$

$$= \frac{1}{\cosh(\pi\xi)}.$$

//

$\boxed{Q18}$ First define

$$\Gamma_1(z) := \int_0^\infty dt\ e^{-t}\, t^{z-1}\, dt \qquad (z>0)$$

Since the int. conv. near $t=0$ and $t=\infty$, Γ_1 is well-def.

(1) Γ_1 extends to an analytic f^n

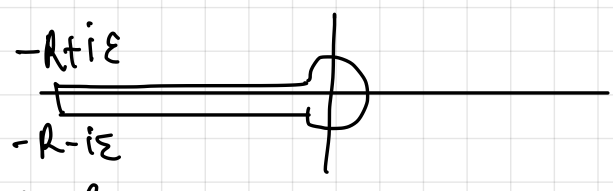
$$\Gamma : \underbrace{\{ z \in \mathbb{C} \mid \mathrm{Re}\{z\} > 0 \}}_{=:\Omega} \longrightarrow \mathbb{C}$$

(Stein & Shakarchi Ch. 6 Prop 1.1)

(2) Define $\tilde{\Gamma} : \mathbb{C} \setminus (-\mathbb{N}_{\geq 0}) \to \mathbb{C}$ via

$$\tilde{\Gamma}(z) := \frac{1}{2i\sin(\pi z)} \oint_G t^{z-1} e^{t}\, dt$$

where G is the so-called "Hankel contour":



radius of circle: r

width of corridor: 2ε

Vertical line:

$$\left| \int_{t=-\varepsilon}^{\varepsilon} (-R+it)^{z-1} e^{-R+it} \, i \, dt \right| \leq$$

$$\leq e^{-R} \int_{-\varepsilon}^{\varepsilon} \underbrace{\left| \exp((z-1) \log(-R+it)) \right|}_{\leq \exp(|z-1| \underbrace{|\log(-R+it)|}_{\leq \sqrt{\log(\sqrt{R^2+t^2})^2 + i}})} dt$$

$$\lesssim \exp(|z-1| \log(R))$$

$$\longrightarrow 0$$

Circle:

$$\left| \int_{\theta=-\pi+\varepsilon}^{\pi-\varepsilon} (re^{i\theta})^{z-1} e^{-re^{i\theta}} re^{i\theta} i \, d\theta \right|$$

$$\leq r |r^{z-1}| \int_{\theta=-\pi+\varepsilon}^{\pi-\varepsilon} \left| e^{i\theta(z-1) - re^{i\theta}} \right| d\theta$$

$$r r^{z-1} = r^{a+ib} \qquad z = a+ib, \ \underline{a > 0}$$

$$|r^{z-1}| = |\exp((a+ib)\log(r))|$$
$$= \exp(a\log(r)) \longrightarrow 0 \quad \text{as} \quad r \to 0$$

Since $a > 0$.

$$\left| e^{i\theta(z-1)} \right| = \left| e^{i\theta(a-1+ib)} \right| = e^{-\theta b} \leq e^{\pi|b|}.$$

The two vertical lines converge to the gamma f^n:

$$\int_{-R}^{-r} (t-i\varepsilon)^{z-1} e^{t-i\varepsilon}\, dt +$$

$$+ \int_{-r}^{-R} (t+i\varepsilon)^{z-1} e^{t+i\varepsilon}\, dt$$

$$(t-i\varepsilon)^{z-1} = \exp((z-1)\log(t-i\varepsilon))$$

$$\log(t-i\varepsilon) = \log(|t-i\varepsilon|) + i\underbrace{\arg(t-i\varepsilon)}_{-\pi}$$

$$\Rightarrow \int_{-R}^{-r} (t-i\varepsilon)^{z-1} e^{t-i\varepsilon}\, dt \xrightarrow{\varepsilon \to 0} \int_{-R}^{-r} dt\, e^{-i\pi(z-1)} |t|^{z-1} e^{t}$$

and similarly,

$$\int_{-r}^{-R} (t+i\varepsilon)^{z-1} e^{t+i\varepsilon} dt \xrightarrow{\varepsilon \to 0} e^{i\pi(z-1)} \int_{-r}^{R} |t|^{z-1} e^{t} dt$$

Summing up we get the $\sin(\pi z)$.

$\Rightarrow$ $\tilde{\Gamma}$ extends Γ since they agree on $\operatorname{Re}\{z\} > 0$.

$\tilde{\Gamma}$ is meromorphic since it is the int. of a meromorphic f^n. The poles are the zeros of $\sin(\pi z)$.

Note when $z \in \mathbb{N}_{\geq 1}$ we get removable singularities:

$$\Gamma(z) = \tilde{\Gamma}(z) \quad \text{for} \quad \operatorname{Re}\{z\} > 0$$

and Γ itself is analytic for $\operatorname{Re}\{z\}$, so $\frac{1}{\sin(\pi z)}$ can only generate poles on $\operatorname{Re}\{z\} < 0$.