Complex Analysis with Applications
Princeton University MAT330
HW4, Due March 3rd 2023

February 25, 2023

1 Basic functions

Read the chapter (in particular the examples and exercises) on the complex logarithm and complex power in Brown and Churchill.

2 Contour integrals

1. Define $f:\mathbb{C}\to\mathbb{C}$ via $f(z)=x^2-\mathrm{i}y^2$ and calculate the contour integral $\int_\Gamma f$ given that
   (a) $\Gamma=\left\{\, x+\mathrm{i}y\in\mathbb{C} \;\middle|\; y=2x^2\wedge x\in[1,2]\,\right\}$ (carry out the calculation for both orientations).
   (b) Γ is the straight line from $(1,8)$ to $(2,8)$.
   (c) Γ is the straight line from $(1,2)$ to $(2,8)$.
2. Define the curve $\gamma:[0,2\pi]\to\mathbb{C}$ given by
   $$\gamma(t) \quad = \quad (\mathrm{i}+t)-\mathrm{i}e^{-\mathrm{i}t} \qquad (t\in[0,2\pi])\ .$$
   (a) Draw a plot of it.
   (b) Is it simple? Is it closed? Is it smooth or piecewise smooth?
   (c) Calculate $\int_\gamma f$ where $f:\mathbb{C}\to\mathbb{C}$ is given by $f(z)=z^2+1$ for all $z\in\mathbb{C}$.
3. Evaluate
   $$\oint\left(\frac{1}{z}-1\right)\mathrm{d}z$$
   around
   (a) The circle $\{\, z\in\mathbb{C} \mid |z-2|=1\,\}$.
   (b) The square with vertices at $1\pm\mathrm{i}$, $-1\pm\mathrm{i}$.
4. Let γ be any simple closed curve bounding a region having area A.
   (a) Prove that
   $$A \quad = \quad \frac{1}{2\mathrm{i}}\oint_\Gamma \overline{z}\mathrm{d}z\,.$$
   (b) Find the area bounded by the ellipse define via
   $$\begin{cases} x=a\cos(\theta) \\ y=b\sin(\theta)\end{cases} \qquad \theta\in[0,2\pi]$$
   for some $a,b>0$.

5. Let $\varphi : [\alpha, \beta] \to [a, b]$ be a given smooth function with $\varphi' > 0$. Find a counter example (i.e., concrete γ, f) to the statement that

$$\int_{[a,b]} f \circ \gamma = \int_{[\alpha,\beta]} f \circ \gamma \circ \varphi \,.$$

6. Let $\gamma : [0, 2\pi] \to \mathbb{C}$ be given by $\gamma(t) = z_0 + R\mathrm{e}^{\mathrm{i}t}$. Calculate

$$\oint_\Gamma (z - z_0)^k \,\mathrm{d}z \qquad (k \in \mathbb{Z}) \,.$$

Why does this not contradict Cauchy's integral theorem when $k = -1$?

7. Evaluate

$$\frac{1}{2\pi\mathrm{i}} \oint_\Gamma \frac{z^2}{z-1} \mathrm{d}z$$

for Γ the circle $|z| = \frac{1}{2}$ as well as Γ the circle $|z| = 5$, each with positive orientation.

8. Evaluate

$$\frac{1}{2\pi\mathrm{i}} \oint_\Gamma \frac{\mathrm{e}^{2z}}{z} \mathrm{d}z$$

for Γ the circle $|z| = 4$.

9. Let $F, G : \mathbb{R}^2 \to \mathbb{R}$ be a pair of harmonic conjugate functions. Show that the closed loop integral of the vector field

$$V = \begin{bmatrix} G \\ F \end{bmatrix}$$

is zero around any closed contour.

3 Cauchy's integral formula

10. Starting from Cauchy's integral formula

$$f(z_0) = \frac{1}{2\pi\mathrm{i}} \oint \frac{f(z)}{z - z_0} \mathrm{d}z$$

(with the conditions for it to be true...) show that if f is holomorphic in the interior of $B_R(z_0)$ then

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f\left(z_0 + R\mathrm{e}^{\mathrm{i}\theta}\right) \mathrm{d}\theta \,.$$

Show that this actually holds separately for the real and imaginary parts of f. Make a conclusion about a similar formula for a harmonic function $F : \mathbb{R}^2 \to \mathbb{R}$.

11. Prove *the maximum modulus principle*, stating that if $f : \Omega \to \mathbb{C}$ is holomorphic and non-constant on some subset $\Omega \subseteq \mathbb{C}$ then $|f| : \Omega \to [0, \infty)$ has no maximum points in Ω, i.e., all maximum points of $|f|$ in the closure of Ω lie on $\partial\Omega$.

12. Evaluate

$$\frac{1}{2\pi} \int_0^{2\pi} \cos\left(\frac{\pi}{3} + 2\mathrm{e}^{\mathrm{i}\theta}\right) \mathrm{d}\theta \,.$$

13. Evaluate

$$\frac{1}{2\pi\mathrm{i}} \oint \frac{\mathrm{e}^{zt}}{z^2 + 1} \mathrm{d}z$$

on the circle $|z| = 3$.

4 Cauchy's inequality

14. Let f be a holomorphic function satisfying

$$|f(z)| \leq 1 + |z|^{1.5} \qquad (z \in \mathbb{C}) \,.$$

Prove that f is a polynomial and give an upper bound on its degree.

15. Using Liouville's theorem, prove that any polynomial of degree $n \geq 1$ has at least one root.