Q1. Define \( f : \mathbb{C} \rightarrow \mathbb{C} \) by \( f(z) = x^2 - iy^2 \).

(a) \( \Gamma = \{ z \in \mathbb{C} \mid y = 2x^2 \land x \in [1, 2] \} \)

Parameterization of \( \Gamma \):

\[
\gamma : [1, 2] \rightarrow \mathbb{C} \\
\gamma(t) := t + i2t^2 \\
\gamma'(t) = 1 + 4t
\]

\[
\int_{\Gamma} f = \int_{t=1}^{2} f(\gamma(t)) \gamma'(t) \, dt \\
= \int_{t=1}^{2} \left[ \gamma_x(t)^2 - i\gamma_x(t)^2 \right] \gamma'(t) \, dt \\
= \int_{t=1}^{2} \left( t^2 - i4t^4 \right) (1 + i4t) \, dt \\
= \frac{1}{2} (511 - 49i).
\]

Other orientation will have a minus sign.

(b) \( \Gamma \) is the straight line from \((1, 8)\) to \((2, 32)\).
Parametrize $\Gamma$ as

$$\gamma(t) = (1-t)(1+8i) + t(2+8i) \quad t \in [0,1]$$

Then

$$\gamma(0) = (1-t) + 2t + 8i = 1 + 0 + 8i$$

$$\gamma'(t) = 1$$

$$\int f \left( \gamma(t) \right) \gamma'(t) \, dt$$

$$= \int_0^1 \left[ (1+t)^2 - i \cdot 64 \right] \, dt$$

$$= \frac{7}{3} - 64i$$

(c) $\Gamma$ is the straight line from (1, 2) to (2, 8). So

$$\gamma(t) = (1-t)(1+2i) + t(2+8i)$$

$$= (1-t) + 2t + (1-t)2 + t8i$$

$$= 1 + t + (2 + 6t)i$$

$$\gamma'(t) = 1 + 6i$$

So
\[ \int f = \int_{t=0}^{1} f(x(t)) \, x'(t) \, dt = \int_{t=0}^{1} \left[ (1+t)^2 - i \left( 2+6t \right)^2 \right] (1+6i) \, dt = \frac{511}{3} - 14i. \]

Q2

\[ \gamma : [0, \pi] \rightarrow \mathbb{C} \]

\[ t \mapsto (i+t) - i e^{-it} = i + t - i \left[ \cos(t) - i \sin(t) \right] = t - 8i \sin(t) + i \left[ 1 - \cos(t) \right] \]

(a)

(b) It is simple and NOT closed. Indeed, \( \gamma \) is an injective \( \mathbb{R}^n \):

If \( \exists \, t, s \in [0, 2\pi] : \gamma(t) = \gamma(s) \)

\[ \begin{cases} t - 8i \sin(t) = s - \sin(s) \\ 1 - \cos(t) = 1 - \cos(s) \end{cases} \]

\[ \Rightarrow t = s + 2\pi n \ \ \ \text{or} \ \ \ \text{for \ 2^{nd} \ eqn.} \]
Plug into 1st eqn to get

\[ 2\pi n - 8\sin(s + 2\pi n) = 8 - \text{sincs} \]
\[ \Rightarrow 2\pi n = 0 \Rightarrow n = 0 \]
\[ \Rightarrow s = \frac{1}{\pi}. \]

**Claim:** \( \gamma \) is piecewise smooth; it fails to be smooth @ \( t \in 2\pi \mathbb{Z} \).

**Proof:** Consider the point \( t = 0 \):

Calculate the one-sided derivative:

\[
\lim_{\varepsilon \to 0^+} \frac{\gamma(t + \varepsilon) - \gamma(t)}{\varepsilon} = \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \left[ \frac{i}{2} \varepsilon^2 + O(\varepsilon^3) \right] = 0
\]

\[
\Rightarrow \gamma'(0^+) = 0.
\]

But we have defined (Def. 6.7) \( \gamma \) to be smooth iff \( \gamma' \neq 0 \).

Similarly @ \( t = 2\pi \).
To gain a bit more intuition, write the curve implicitly rather than parametrically (i.e. \(y\) as a \(f^n\) of \(x\)).

\[
x = t - \sin(t)
\]

\[
\Rightarrow t = f(x) \text{ for some function } f.
\]

Plug into \(y = 1 - \cos(t)\) to get \(y\) as a \(f^n\) of \(x\):

\[
y(x) = 1 - \cos(f(x))
\]

Calculate the derivative:

\[
y'(x) = \sin(f(x)) f'(x)
\]

To find \(f'(x)\), differentiate \(x = t - \sin(t) \equiv f^{-1}(t)\) w.r.t. \(t\) to get:

\[
1 - \cos(t) = \frac{df}{dt}
\]

Now, since \(f \circ f^{-1} \equiv 1\)

\[
\Rightarrow \frac{df}{dx} = 1
\]

We find:

\[
\frac{1}{(df \circ f^{-1})(t)} = \frac{1}{1 - \cos(t)}
\]
Hence
\[ y'(x) = \frac{\sin(f(x))}{1 - \cos(f(x))}. \]

It is clear from this eqn that \( y' \) explodes when \( t = f(x) \in 2\pi \mathbb{Z} \), since then \( \cos(t) = 1 \).

(This has been a long-winded but elementary explanation to the inverse fn \( c.n.m. \), which is sometimes explained in Leibniz notation as
\[
\frac{dy}{dx} = \frac{dy}{dt} = \frac{y_1}{x_1},
\]
and
\[
\delta_R(t) = t - \sin(t), \quad \delta_R'(t) = 1 - \cos(t) \implies t \in 2\pi \mathbb{Z}.
\]

(c) Define \( f: \mathbb{C} \to \mathbb{C} \) \( \text{via} \) \( f(z) = 1 + z^2 \).

\[
\int_0^{2\pi} (1 + y'(t)^2) y'(t) \, dt = \int_0^{2\pi} (1 + (i + t)^2 - i e^{-it}) \, dt.
\]
Define \( f(z) := \frac{1}{2} - 1 \) on \( \mathbb{C} \setminus \{0\} \).

(a) \( \oint_{\partial D_1(2)} f = ? \)

Parametrize \( \gamma(t) = 2 + e^{it} \quad t \in [0, 2\pi] \),
\[ \gamma'(t) = ie^{it} \]

\[ \Rightarrow \oint_{\partial D_1(2)} f = \int_{0}^{2\pi} f(\gamma(t)) \gamma'(t) \, dt \]
\[ = \int_{0}^{2\pi} \left( \frac{1}{2 + e^{it}} + 1 \right) i e^{it} \, dt = 0. \]

This agrees with Cauchy's Theorem since \( f \) is holomorphic in \( \text{int}(\partial D_1(2)) \).

(b)
Now we expect that the int. need not be zero.

Parametrize the four legs:

\[ \gamma_1(t) = 1 + ti \quad t \in [-1, 1] \]
\[ \gamma_2(t) = i - t \quad t \in [-1, 1] \]
\[ \gamma_3(t) = -1 - ti \quad t \in [-1, 1] \]
\[ \gamma_4(t) = -i + t \quad t \in [-1, 1] \]

\[ I_1 = \int_{t=-1}^{1} \left( \frac{1}{1 + ti} - 1 \right) i \ dt = \frac{\phi}{2} (\pi - 4) \]
\[ I_2 = \int_{t=-1}^{1} \left( \frac{1}{i - t} - 1 \right) (-1) dt = 2 + \frac{i \pi}{2} \]
\[ I_3 = \int_{t=-1}^{1} \left( \frac{1}{-1 - it} - 1 \right) (-i) dt = \frac{i}{2} (4 + \pi) \]
\[ I_4 = \int_{t=-1}^{1} \left( \frac{1}{-i + t} - 1 \right) dt = -2 + \frac{i \pi}{2} \]

\[ I_1 + I_2 + I_3 + I_4 = 2\pi i \]

This could have been foreseen w/ Cauchy's int. formula: \( \frac{1}{2\pi i} \) is holomorphic to it's
int. would be zero and \( z = \frac{1}{2} \) obeys the formula
\[
\frac{1}{2\pi i} \oint \frac{1}{z} \, dz = 1.
\]

**Claim:** If \( \gamma \) is a simple closed curve, then
\[
\text{area} \int_{\gamma} = \frac{1}{2i} \oint \bar{z} \, dz.
\]

**Proof:** Recall Green's Thm.: If \( V: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) is cont. diff. and \( \Omega \subseteq \mathbb{R}^2 \) is simply-conn. then
\[
\oint_{\partial \Omega} V = \iint_{\Omega} \text{curl}(V).
\]

We know (Eqn (6,7,6,8) in lecture notes) that
\[
\oint_{\partial \Omega} f = \oint_{\partial \Omega} U + i \oint_{\partial \Omega} V,
\]
with 
\[
U = \begin{bmatrix} f_R \\ f_I \end{bmatrix}, \quad V = \begin{bmatrix} f_I \\ f_R \end{bmatrix}.
\]

For us \( f(\mathbf{z}) = \bar{z} \), i.e., \( f_R(x,y) = x \) and \( f_I(x,y) = -y \)
and so
\[
U(x,y) = \begin{bmatrix} x \\ y \end{bmatrix}, \quad V(x,y) = \begin{bmatrix} -y \\ x \end{bmatrix}.
\]
Now, \( \text{curl}(U)(x,y) = \partial_x U_2 - \partial_y U_1 = 0 \).  
\( \text{curl}(V)(x,y) = \partial_x V_2 - \partial_y V_1 = 1 + 1 = 2 \).

We find:  
\[
\oint \oint \text{curl}(U) + i \oint \text{curl}(V) = 2i \int \int_{\text{int}(\gamma)} \text{curl}(U) + i \oint \text{curl}(V) = 2i \int \text{int}(\gamma)  
\]

\( \text{Green's Theorem} \)  

\[
\text{Claim: The area of the ellipse given by } \left\{ \begin{array}{l} x = a \cos(\theta) \\ y = b \sin(\theta) \end{array} \right. \quad \theta \in [0,2\pi] \\
\text{Proof: Using the formula above we parametrize} \\
\gamma(t) = a \cos(t) + ib \sin(t) \\
\gamma'(t) = -a \sin(t) + ib \cos(t) \\
\text{Area} = \frac{1}{2} \int_{t=0}^{2\pi} \frac{\left[ a \cos(t) + ib \sin(t) \right] x \left[ -a \sin(t) + ib \cos(t) \right] dt}{2} 
\]
Let \( \psi : [a, b] \to [a, b] \) be smooth with \( \psi' > 0 \).

We know that
\[
\int_{[a, b]} \psi' \psi = \int_{[a, b]} (\psi \circ \psi') \psi',
\]
and furthermore under reparam. \( \gamma \to \gamma' \), we have
\[
\int_{[a, b]} \psi' \psi = \int_{[a, b]} (\psi' \circ \psi') \psi'.
\]

Without the factor \( \psi' \), the equation
\[
\int_{[a, b]} \psi \circ \gamma = \int_{[a, b]} \psi \circ \gamma'
\]
is clearly false!

**Example:** Take \( f = 1 \), \( N \) arbitrary,
\[\psi : [0, 1] \to [0, 2] \quad t \mapsto 2t\]
then
\[
\int_{[0, 1]} 1 = 1 \quad \text{yet} \quad \int_{[0, 2]} 1 = 2.
\]
Claim: $\oint \frac{1}{z-z_0} \, dz = 2\pi i \text{Re} \, \gamma$.

Proof: By Cauchy, if $k \geq 0$, $\text{Im} \, \gamma = 0$.

If $k = -1$, get $2\pi i$ explicitly.

If $k < -1$, apply the Cauchy integral formula for derivatives:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint \frac{f(z)}{(z-z_0)^{n+1}} \, dz$$

If $f(z) = 1$, whose derivatives are all zero.

As we know, $2 \to \frac{1}{2-z_0}$ is not holomorphic at $z = z_0$ and hence contradicts Cauchy’s theorem.

Claim: $\frac{1}{2\pi i} \oint \frac{z^2}{z-1} \, dz = 0$ if $\gamma = \partial B_{\frac{1}{2}}(0)$. 
Proof: \( z \rightarrow \frac{1}{z-1} \) is holomorphic on \( \bar{B}_{1/2}(0) \).  

Claim: \( \frac{1}{2\pi i} \oint_{\Gamma} \frac{z^2}{z-1} \, dz = 1 \) if \( \Gamma = \partial B_5(0) \).

Proof: Apply Cauchy w/ \( f(z) = z^2 \).

Then \( \frac{1}{2\pi i} \oint_{\Gamma} \frac{z^2}{z-1} \, dz = f(1) = 1^2 = 1 \).

Claim: \( \frac{1}{2\pi i} \oint_{\partial B_4(0)} e^{2z} \, dz = 1 \)

Proof: Apply Cauchy w/ \( f(z) = \exp(2z) \)

whence \( \frac{1}{2\pi i} \oint_{\partial B_4(0)} e^{2z} \, dz = e^{2 \cdot 0} = 1 \).
Claim: If $F, G : \mathbb{R}^2 \to \mathbb{R}$ are a pair of harmonic conjugates, $F$ is then

$$\oint_{C} V = 0$$

where $V = \begin{bmatrix} G \\ F \end{bmatrix}$ and $C$ is any closed contour.

Proof: Since $F, G$ are harmonic conjugates,

$$f := F + i \cdot G$$

is holomorphic (Prop. 4.15).

Hence by Cauchy, $\oint_{C} f = 0$.

As we have seen (lecture notes eqns (6.3, 6.8))

$$\oint_{C} f = \oint_{C} U + i \cdot \oint_{C} V$$

with $U = \begin{bmatrix} F \\ -G \end{bmatrix}$ and $V = \begin{bmatrix} G \\ F \end{bmatrix}$. 

Claim: If \( f : \mathbb{C} \to \mathbb{C} \) is holomorphic on \( \mathbb{B}_R(z_0) \), then
\[
\begin{align*}
\hat{f}(z_0) &= \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(z)}{z - z_0} \, dz \\
\hat{f}(z_0) &= \frac{1}{2\pi} \int_{0}^{2\pi} f(z_0 + R e^{i\theta}) \, d\theta.
\end{align*}
\]

Proof: Start with eqn 6.12
\[
\hat{f}(z_0) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(z)}{z - z_0} \, dz
\]

with \( \Gamma = \partial \mathbb{B}_R(z_0) \):
\[
\begin{align*}
\gamma(t) &= z_0 + R e^{it}, \quad t \in [0, 2\pi] \\
\gamma'(t) &= R e^{it}
\end{align*}
\]

\[\Rightarrow \oint_{\Gamma} \frac{f(z)}{z - z_0} \, dz = \int_{0}^{2\pi} \hat{f}(z_0 + R e^{it}) \, d\theta\]

By taking \( \text{Re}, \text{Im} \) of this formula we arrive at
\[
\begin{align*}
\hat{f}_R(z_0) &= \frac{1}{2\pi} \int_{\theta=0}^{2\pi} \hat{f}_R(z_0 + R e^{i\theta}) \, d\theta \\
\hat{f}_I(z_0) &= \frac{1}{2\pi} \int_{\theta=0}^{2\pi} \hat{f}_I(z_0 + R e^{i\theta}) \, d\theta.
\end{align*}
\]
Since \( f_{1, f_2} \) are harmonic (Prop 4.15), we learn a general fact about harmonic functions \( F : \mathbb{R}^2 \to \mathbb{R} \):

\[
-\Delta F = 0 \implies F(z_0) = \frac{1}{2\pi} \int_{\theta=0}^{2\pi} F(z_0 + \text{Re}^i\theta) \, d\theta.
\]

---

**Claim:** Let \( f : \Omega \to \mathbb{C} \) be holomorphic and non-constant with \( \Omega \) path-connected. Then \( \text{If}(\Omega) \to [0, \infty) \) has no max pts. within \( \text{int}(\Omega) \) (but may have them on \( \partial \Omega \)).

**Proof:** Assume otherwise. Then \( \exists z \in \text{int}(\Omega) : \text{If}(z) = 1 \) and \( \forall w \in \Omega \).

Let \( r > 0 : B_r(z) \subseteq \text{int}(\Omega) \) (by openness).

Then \( \text{If}(z) = \frac{1}{2\pi} \int_{\theta=0}^{2\pi} \text{If}(z + \text{Re}^i\theta) \, d\theta \leq \text{If}(z) \) by hypothesis.

\[
\Rightarrow \text{If}(z) \leq \frac{1}{2\pi} \int_{\theta=0}^{2\pi} \text{If}(z + \text{Re}^i\theta) \, d\theta = \text{If}(z) \cdot \text{If}(z).
\]

So \( \text{If}(z) = 1 \).
\[ \Rightarrow \quad \frac{1}{2\pi} \int_0^{2\pi} \left| f(z) \right|^2 |1 - f(z + re^{i\theta})| \, d\theta = 0 \]

But \[ \int_a^b g = 0 \quad \text{for} \quad g \geq 0 \quad \text{cont.} \]

implies \( g = 0 \).

Since \( \theta \mapsto |f(z)| - |f(z + re^{i\theta})| \) is cont., it must therefore be zero.

i.e., \( |f(z)| = |f(z + re^{i\theta})| \quad \forall \quad \theta \in [0, 2\pi] \).

Since \( r \) was any radius s.t. \( B_r(z) \subseteq \text{int}(\Omega) \), we find

\[ |f(z)| = |f(z + re^{i\theta})| \quad \forall \quad \theta \in [0, 2\pi] \quad \forall \quad r \in [0, r_0] \]

where \( r_0 > 0 : \quad B_{r_0}(z) \subseteq \text{int}(\Omega) \).

i.e., \( |f| \) is const. on \( B_{r_0}(z) \).

\[ \Rightarrow \quad \text{By HW 2.8(b),} \quad f \text{ is const. on } B_{r_0}(z). \]

This yields the claim if \( \Omega = B_R(z_0) \)

\[ \exists \quad z_0 \in \Omega, \quad r > 0. \]

Now if \( \Omega \) is NOT a disc:
Still assume 2 \in \text{int}(\Omega) is a max for 1|\Omega|.

Let \( w \in \text{int}(\Omega) \) and \( \eta : [0,1] \to \text{int}(\Omega) \) any continuous path \( \eta(0) = 2 \)
\[ \eta(1) = w. \]

Possible via path-conn.

Let \( R_0 > 0 \) be the largest: \( B_{R_0}(2) \subseteq \text{int}(\Omega) \)

Pick \( z_1 \in \text{im}(\eta) \cap B_{R_0}(2) \).

Then \( \|z_1 - 2\| \leq \|z_1 - w\| \) by the above.

Let \( R_1 > 0 \) be the largest: \( B_{R_1}(z_1) \subseteq \text{int}(\Omega) \).

Pick \( z_2 \in \text{im}(\eta) \cap B_{R_1}(z_1) \).

Like that we may recursively continue along \( \eta \) to get \( \|w - z_1\| = \|z_2 - z_1\| \)
for any \( w \in \text{int}(\Omega) \).
\[ Q12 \]
\[ \frac{1}{2\pi i} \int_{\partial B_\rho(0)} e^{2t} \frac{1}{(z-i)(z+i)} \, dz = \]
\[ \int_0^{2\pi} \cos \left( \frac{\pi}{2} + 2e^{i\theta} \right) \, d\theta = \cos \left( \frac{\pi}{2} \right) = \frac{1}{2}. \]

\[ Q13 \]
\[ I(t) = \frac{1}{2\pi i} \int_{\partial B_\rho(0)} \frac{e^{2t}}{z^2 + 1} \, dz = \]
\[ = \frac{1}{2\pi i} \int_{\partial B_\rho(0)} \frac{e^{2t}}{(z-i)(z+i)} \, dz \]

Note the identity
\[ (z-i)^{-1} - (z+i)^{-1} = (z-i)^{-1} (2+i)^{-1} 2i. \]

\[ \Rightarrow I(t) = \frac{1}{2\pi i} \int_{\partial B_\rho(0)} e^{2t} \frac{1}{2i} [(z-i)^{-1} - (z+i)^{-1}] \, dz \]

\[ \text{Cauchy} \quad J = \frac{1}{2i} \left[ e^{it} - e^{-it} \right] = \sin(t). \]

Let \( f \) be a holomorphic function satisfying
\[ |f(z)| \leq 1 + |z|^{1.5} \quad (z \in \mathbb{C}). \]

\[ Q14 \]
\[ \text{Cl.: } \text{if } f \text{ is a poly. w/ } \deg(f) \leq 1. \]
\[ \text{Pf.: } \text{If } f \text{ were a poly, it could not have } \deg \geq 2 \text{ or higher. } \]
then \( |f(z)| \leq 1 + |z|^{1.5} \) would be violated at \( |z| \) arbit. large.

But couldn’t \( f \) be some smooth \( f \) which grows like \( |z|^{1.5} \) at \( \infty \)? No:

By Cauchy’s estimate, \( \forall R > 0 \)

\[
|f^n(z)| \leq \frac{n!}{R^n} \sup_{z \in B_R(0)} |f(z)|
\]

\[
\leq \frac{n!}{R^n} (1 + R^{1.5})
\]

Hence for all \( n \geq 2 \), \( f^n = 0 \).

This implies \( f(z) = a + \bar{a}z \exists a, b \in \mathbb{C} \).

\[ \square \]

See Thm. 6.33.