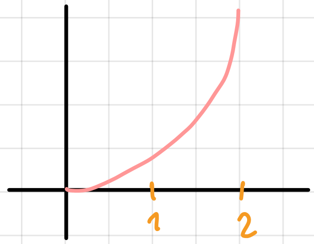
MAR 5 2023

MAT330 — HW4 Sample Sol-ns

$\boxed{Q1}$ Define $f: \mathbb{C} \to \mathbb{C}$ by $f(z) = x^2 - iy^2$.

(a) $\Gamma = \{ z \in \mathbb{C} \mid y = 2x^2 \wedge x \in [1,2] \}$



Parametrization of Γ:

$$\gamma : [1,2] \to \mathbb{C}$$
$$\gamma(t) := t + i2t^2$$
$$\gamma'(t) = 1 + i4t.$$

$$\int_\Gamma f \equiv \int_{t=1}^{2} f(\gamma(t))\gamma'(t)\,dt$$
$$= \int_{t=1}^{2} \left[\gamma_R(t)^2 - i\gamma_I(t)^2\right]\gamma'(t)\,dt$$
$$= \int_{t=1}^{2} \left(t^2 - i4t^4\right)\left(1 + i4t\right)dt$$
$$= \frac{1}{3}(511 - 49i).$$

Other orientation will have a minus sign.

(b) Γ is the straight line from $(1,8)$ to

$(2,8)$.

Parametrize Γ as

$$\gamma(t) = (1-t)(1+8i) + t(2+8i) \qquad t\in[0,1]$$

collect

$$\stackrel{?}{=} (1-t) + 2t + 8i = 1+t+8i$$

$$\gamma'(t) = 1$$

$$\int_\Gamma f = \int_{t=0}^{1} f(\gamma(t))\,\gamma'(t)\,dt$$

$$= \int_{t=0}^{1} \left[(1+t)^2 - i\,64\right] dt$$

$$= \frac{7}{3} - 64i.$$

(c) Γ is the straight line from $(1,2)$ to $(2,8)$. So

$$\gamma(t) = (1-t)(1+2i) + t(2+8i)$$

$$= (1-t) + 2t + \left[(1-t)2 + t8\right] i$$

$$= 1+t + (2+6t)i$$

$$\gamma'(t) = 1+6i.$$

So

$$\int_\Gamma f \equiv \int_{t=0}^{1} f(\gamma(t))\,\gamma'(t)\,dt$$

$$= \int_{t=0}^{1} \left[(1+t)^2 - i\,(2+6t)^2\right](1+6i)\,dt$$

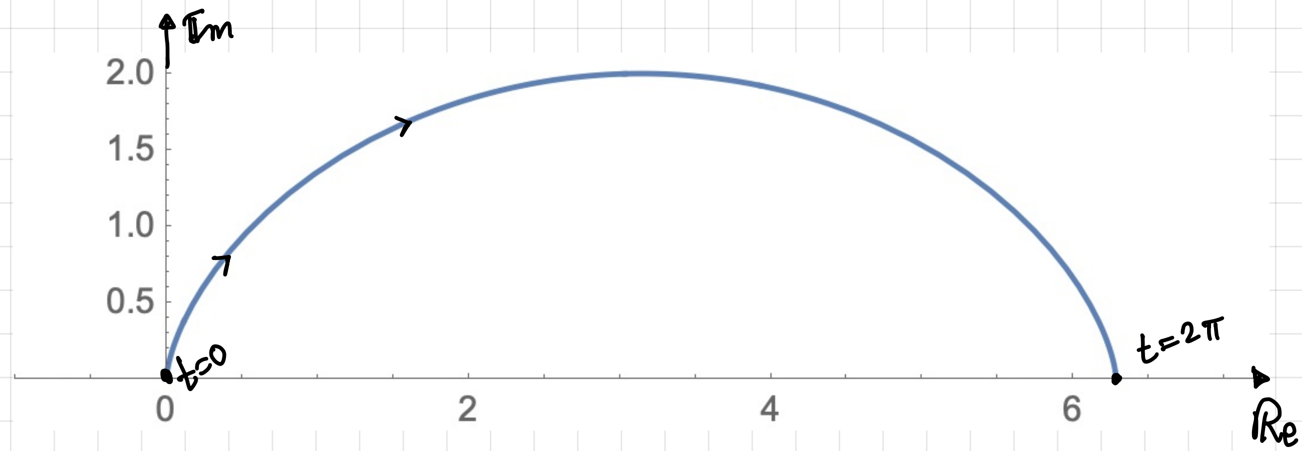
$$= \frac{511}{3} - 14i\,.$$

$\boxed{Q2}$ $\gamma : [0, 2\pi] \longrightarrow \mathbb{C}$

$$t \longmapsto (i+t) - i e^{-it}$$

$$= i + t - i\left[\cos(t) - i\sin(t)\right]$$

$$= t - \sin(t) + i\left[1 - \cos(t)\right]$$

(a) 

(b) It is simple and NOT closed. Indeed, γ is an injective f^n:

If $\exists\ t, s \in [0, 2\pi]$: $\gamma(t) = \gamma(s)$

$$\Updownarrow$$

$$\begin{cases} t - \sin(t) = s - \sin(s) \\ 1 - \cos(t) = 1 - \cos(s) \end{cases}$$

$\Longrightarrow$ $t = s + 2\pi n$ $\exists\ n \in \mathbb{Z}$ from 2^{nd} eq-n.

Plug into 1^{st} eq-n to get

$$\cancel{s}+2\pi n-\cancel{\sin(s+2\pi n)}=\cancel{s}-\cancel{\sin(s)}$$

$$\Rightarrow 2\pi n=0 \Rightarrow n=0$$

$$\Rightarrow s=t.$$

<u>Claim</u>: γ is piecewise smooth: it fails to be smooth @ $t\in 2\pi\mathbb{Z}$.

<u>Proof</u>: Consider the point $t=0$:

Calculate the one-sided derivative:

$$\lim_{\varepsilon\to 0^+}\frac{\gamma(\varepsilon)-\gamma(0)}{\varepsilon}=$$

$$=\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\Big[i+\varepsilon-i\underbrace{e^{-i\varepsilon}}_{1-i\varepsilon+\frac{1}{2}(-i\varepsilon)^2+\dots}-\underbrace{(i+0-ie^{-i0})}_{=0}\Big]$$

$$=\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\Big[\frac{i}{2}\varepsilon^2+O(\varepsilon^3)\Big]=0$$

$$\Rightarrow \gamma'(0^+)=0.$$

But we have defined (Def. 6.7) γ to be smooth iff $\gamma'\neq 0$. Similarly @ $t=2\pi$.

$\blacksquare$

To gain a bit more intuition, write the curve implicitly rather than parametrically (i.e. y as a f^n of x):

$$x = t - \sin(t)$$

$$\Rightarrow \quad t = f(x) \quad \text{for some function } f.$$

(inverse of $t \mapsto t - \sin(t)$)

Plug into $y = 1 - \cos(t)$ to get y as a f^n of x:

$$y(x) = 1 - \cos(f(x))$$

Calculate the derivative:

$$y'(x) = \sin(f(x))\, f'(x)$$

To find $f'(x)$, differentiate

$$x = t - \sin(t) \equiv f^{-1}(t)$$

w.r.t. t to get:

$$1 - \cos(t) = \partial f^{-1}(t)$$

Now, since $f \circ f^{-1} \equiv \mathbb{1}$

$$[(\partial f) \circ f^{-1}]\, \partial f^{-1} = 1$$

$$\Leftrightarrow \quad \partial f^{-1} = \frac{1}{(\partial f) \circ f^{-1}}$$

We find: $$\frac{1}{[(\partial f) \circ f^{-1}](t)} = 1 - \cos(t)$$

$$\Leftrightarrow \quad ((\partial f) \circ f^{-1})(t) = \frac{1}{1 - \cos(t)}$$

Hence

$$y'(x) = \frac{\sin(f(x))}{1-\cos(f(x))} .$$

It is clear from this eq-n that y' explodes when $t \equiv f(x) \in 2\pi\mathbb{Z}$, since then $\cos(t) = 1$!

(This has been a long-winded but elementary explanation to the inverse f^n thm., which is sometimes explained in Leibniz notation as

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \equiv \frac{\gamma_I'}{\gamma_R'}$$

and

$$\gamma_R(t) = t - \sin(t)$$

$$\gamma_R'(t) = 1 - \cos(t) \stackrel{!}{=} 0 \Rightarrow t \in 2\pi\mathbb{Z}.)$$

(c) Define $f: \mathbb{C} \to \mathbb{C}$ via $f(z) = 1 + z^2$.

$$\int_\Gamma f \equiv \int_{t=0}^{2\pi} \left(1 + \gamma(t)^2\right) \gamma'(t)\, dt$$

$$= \int_{t=0}^{2\pi} \left(1 + \left\{(i+t) - i e^{-it}\right\}^2\right)\left(1 - e^{-it}\right) dt$$

$$= 2\pi + \frac{8\pi^3}{3}.$$

$\boxed{Q3}$ Define $f(z) := \frac{1}{z} - 1$ on $\mathbb{C} \setminus \{0\}$.

(a) $$\oint_{\partial B_1(2)} f = ?$$

Parametrize $\gamma(t) = 2 + e^{it}$ $\quad t \in [0, 2\pi]$.

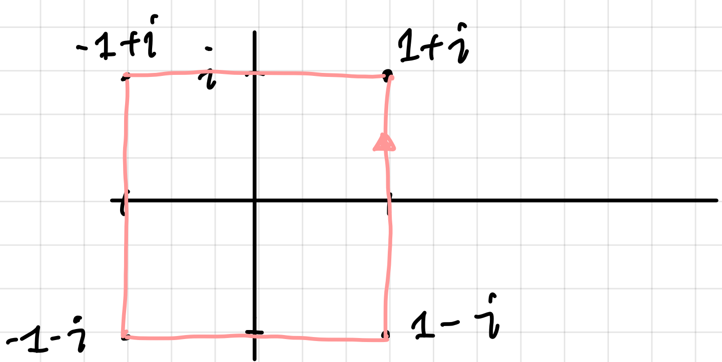
$\gamma'(t) = i e^{it}$

$$\Rightarrow \oint_{\partial B_1(2)} f = \int_{t=0}^{2\pi} f(\gamma(t))\, \gamma'(t)\, dt$$

$$= \int_{t=0}^{2\pi} \left(\frac{1}{2 + e^{it}} + 1 \right) i e^{it}\, dt = 0.$$

This agrees w/ Cauchy's Thm. since f is holomorphic in $\operatorname{int}(\partial B_1(2))$!

(b)



Now we expect that the int. need not be zero.

Parametrize the four legs:

$$\begin{aligned}
\gamma_1(t) &= 1 + ti & t &\in [-1,1] \\
\gamma_2(t) &= i - t & t &\in [-1,1] \\
\gamma_3(t) &= -1 - ti & t &\in [-1,1] \\
\gamma_4(t) &= -i + t & t &\in [-1,1]
\end{aligned}$$

$$I_1 = \int_{t=-1}^{1} \left(\frac{1}{1+ti} - 1 \right) i \, dt = \frac{i}{2}(\pi - 4)$$

$$I_2 = \int_{t=-1}^{+1} \left(\frac{1}{i-t} - 1 \right) (-1) dt = 2 + \frac{i\pi}{2}$$

$$I_3 = \int_{t=-1}^{1} \left(\frac{1}{-1-it} - 1 \right) (-i) \, dt = \frac{i}{2}(4+\pi)$$

$$I_4 = \int_{t=-1}^{1} \left(\frac{1}{-i+t} - 1 \right) dt = -2 + \frac{i\pi}{2} .$$

$$I_1 + I_2 + I_3 + I_4 = 2\pi i .$$

This could have been forseen w/ Cauchy's int. formula: $z \mapsto 1$ is holomorphic so its int. would be zero and $z \mapsto \frac{1}{z}$ obeys the formula $\frac{1}{2\pi i} \oint \frac{1}{z}\, dz = 1$.

$\boxed{Q4}$ (a) Claim: If γ is a simple closed curve then

$$\underbrace{|\mathrm{int}(\gamma)|}_{\text{area}} = \frac{1}{2i} \oint_{\Gamma} \bar{z}\, dz .$$

Proof: Recall Green's Thm.:

If $V : \mathbb{R}^2 \to \mathbb{R}^2$ is cont. diff. and $\Omega \subseteq \mathbb{R}^2$ is simply-conn. then

$$\oint_{\partial\Omega} V = \int_{\Omega} \mathrm{curl}(V) .$$

We know (Eqn (6.7, 6.8) in lecture notes) that

$$\oint_{\Gamma} f = \oint_{\Gamma} \mathcal{U} + i \oint_{\Gamma} V$$

with $\mathcal{U} = \begin{bmatrix} f_R \\ -f_I \end{bmatrix}$ $V = \begin{bmatrix} f_I \\ f_R \end{bmatrix}$.

For us $f(z) = \bar{z}$, i.e., $f_R(x,y) = x$
$f_I(x,y) = -y$

and so $\mathcal{U}(x,y) = \begin{bmatrix} x \\ y \end{bmatrix}$ $V(x,y) = \begin{bmatrix} -y \\ x \end{bmatrix}$.

Now, $\quad \mathrm{curl}(\mathcal{U})(x,y) = \partial_x \mathcal{U}_2 - \partial_y \mathcal{U}_1 = 0.$

$$\mathrm{curl}(V)(x,y) = \partial_x V_2 - \partial_y V_1 = 1+1=2.$$

We find: $\displaystyle \oint_\Gamma f = \oint_\Gamma \mathcal{U} + i \oint_\Gamma V$

$$\overset{\text{Green}}{=} \int_{\mathrm{int}(\gamma)} \mathrm{curl}(\mathcal{U}) + i \int_{\mathrm{int}(\gamma)} \mathrm{curl}(V)$$

$$= 2i \int_{\mathrm{int}(\gamma)}$$

$$= 2i \, |\mathrm{int}(\gamma)| .$$

$\blacksquare$

(b) <u>Claim</u>: The area of the ellipse given by

$$\begin{cases} x = a\cos(\theta) \\ y = b\sin(\theta) \end{cases} \qquad \theta \in [0, 2\pi]$$

$\exists\ a,b>0$ is πab.

<u>Proof</u>: Using the formula above we parametrize

$$\gamma(t) = a\cos(t) + i b \sin(t)$$
$$\gamma'(t) = -a\sin(t) + ib\cos(t)$$

$$\text{Area} = \frac{1}{2i} \int_{t=0}^{2\pi} \left[\overline{a\cos(t) + ib\sin(t)} \right] \times$$
$$\times \left[-a\sin(t) + ib\cos(t) \right] dt$$

$$= \frac{1}{2i}\left[2i\pi ab\right] = \pi ab.$$

$\blacksquare$

$\boxed{Q5}$ Let $\varphi : [\alpha, \beta] \to [a,b]$ be smooth w/ $\varphi' > 0$.

We know that $\int_\Gamma f \equiv \int_{[a,b]} (f\circ\gamma)\gamma'$

and furthermore under reparam. $\gamma \mapsto \gamma\circ\varphi$, we have
$$\int_\Gamma f = \int_{[\alpha,\beta]} (f\circ\gamma\circ\varphi)(\gamma'\circ\varphi)\varphi'.$$

w/o the factor γ' the eq-n
$$\int_{[a,b]} f\circ\gamma = \int_{[\alpha,\beta]} f\circ\gamma\circ\varphi$$

is clearly false!

<u>Example</u>: Take $f=1$, γ arbitrary,
$$\varphi : [0,1] \to [0,2]$$
$$t \mapsto 2t$$
$\varphi' = 2 > 0$.

Then $\int_{[0,1]} 1 = 1$ yet

$\int_{[0,2]} 1 = 2$!

$\boxed{Q6}$ $\gamma : [0, 2\pi] \to \mathbb{C}$ given by $\gamma(t) = z_0 + Re^{it}$.

<u>Claim</u>: $\oint_\Gamma (z - z_0)^k \, dz = 2\pi i \, \delta_{k,-1}$.

<u>Proof</u>: By Cauchy, if $k \geq 0$, int $= 0$.

If $k = -1$ get $2\pi i$ explicitly.

If $k < -1$, apply the Cauchy form. for derivations:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint \frac{f(z)}{(z - z_0)^{n+1}} \, dz$$

w/ $f(z) = 1$, whose derivatives are all zero. ∎

As we know, $z \mapsto \frac{1}{z - z_0}$ is NOT holomorphic @ $z = z_0$ and hence $\nexists$ contradiction w/ Cauchy's Thm.

$\boxed{Q7}$ <u>Claim</u>: $\frac{1}{2\pi i} \oint_\Gamma \frac{z^2}{z - 1} \, dz = 0$ if

$\Gamma = \partial B_{1/2}(0)$.

Proof: $z \mapsto \frac{1}{z-1}$ is holomorphic on $\overline{B_{1/2}(0)}$.

$\blacksquare$

Claim: $\frac{1}{2\pi i} \oint_\Gamma \frac{z^2}{z-1}\, dz = 1$ if

$\Gamma = \partial B_5(0)$.

Proof: Apply Cauchy w/ $f(z) = z^2$.

Then

$$\frac{1}{2\pi i} \oint_\Gamma \frac{z^2}{z-1}\, dz = f(1) = 1^2 = 1.$$

$\blacksquare$

$\boxed{Q8}$

Claim: $\frac{1}{2\pi i} \oint_{\partial B_4(0)} \frac{e^{2z}}{z}\, dz = 1$

Proof: Apply Cauchy w/ $f(z) = \exp(2z)$

whence $\frac{1}{2\pi i} \oint_{\partial B_4(0)} \frac{e^{2z}}{z}\, dz = e^{2\cdot 0} = 1.$

Q9 **Claim:** If $F, G : \mathbb{R}^2 \to \mathbb{R}$ are a pair of harmonic conj. f^n's then

$$\oint_\Gamma V = 0$$

where $V = \begin{bmatrix} G \\ F \end{bmatrix}$ and Γ is any closed contour.

Proof: Since F, G are harmonic conjugates,

$$f := F + iG$$

is holomorphic (Prop. 4.15).

Hence by Cauchy, $\oint_\Gamma f = 0$.

As we have seen (lecture notes eqns (6.7, 6.8))

$$\oint_\Gamma f = \oint_\Gamma \mathcal{U} + i \oint_\Gamma V$$

w/ $\mathcal{U} = \begin{bmatrix} F \\ -G \end{bmatrix}$ $V = \begin{bmatrix} G \\ F \end{bmatrix}$.

$\blacksquare$

Q10 Claim: If $f: \mathbb{C} \to \mathbb{C}$ is holomorphic on $\overline{B_R(z_0)}$ then

$$f(z_0) = \frac{1}{2\pi} \int_{\theta=0}^{2\pi} f(z_0 + Re^{i\theta})\, d\theta .$$

Proof: Start w/ eq-n 6.12

$$f(z_0) = \frac{1}{2\pi i} \oint_\Gamma \frac{f(z)}{z - z_0}\, dz$$

w/ $\Gamma = \partial B_R(z_0)$:

$$\gamma(t) = z_0 + Re^{it} \qquad t \in [0, 2\pi]$$
$$\gamma'(t) = Re^{it} i$$

$$\Rightarrow \oint_\Gamma \frac{f(z)}{z - z_0}\, dz = \int_{t=0}^{2\pi} \frac{f(z_0 + Re^{it})}{Re^{it}} Re^{it} i\, dt .$$

∎

By taking Re, Im of this formula we arrive at

$$\begin{cases} f_R(z_0) = \dfrac{1}{2\pi} \displaystyle\int_{\theta=0}^{2\pi} f_R(z_0 + Re^{i\theta})\, d\theta \\ f_I(z_0) = \dfrac{1}{2\pi} \displaystyle\int_{\theta=0}^{2\pi} f_I(z_0 + Re^{i\theta})\, d\theta \end{cases} .$$

Since f_R, f_I are harmonic (Prop. 4.15)

We learn a general fact about harmonic functions $F: \mathbb{R}^2 \to \mathbb{R}$:

$$\boxed{-\Delta F = 0 \implies F(z_0) = \frac{1}{2\pi}\int_{\theta=0}^{2\pi} F(z_0 + Re^{i\theta})\, d\theta .}$$

Q11

Claim: Let $f: \Omega \to \mathbb{C}$ be holomorphic and non-const w/ Ω path-connected.
Then $|f|: \Omega \to [0,\infty)$ has no max pts. within $\mathrm{int}(\Omega)$ (but may have them on $\partial\Omega$).

Proof: Assume otherwise. Then $\exists\, z \in \mathrm{int}(\Omega)$: $|f(z)| \geq |f(w)| \quad \forall \quad w \in \Omega$.

Let $r > 0$: $B_r(z) \subseteq \mathrm{int}(\Omega)$ (by openness).
Then $f(z) = \frac{1}{2\pi}\int_{\theta=0}^{2\pi} f(z + re^{i\theta})\, d\theta$.

$$\Rightarrow \quad |f(z)| \leq \frac{1}{2\pi}\int_{\theta=0}^{2\pi} \underbrace{|f(z+re^{i\theta})|}_{\leq |f(z)| \text{ by hypothesis}}\, d\theta$$

$$= |f(z)|.$$

$$\Rightarrow \quad \frac{1}{2\pi}\int_0^{2\pi}\left[|f(z)| - |f(z+re^{i\theta})|\right] d\theta = 0$$

But $\int_a^b g = 0$ for $g \geq 0$ cont.

implies $g = 0$.

Since $\theta \mapsto |f(z)| - |f(z+re^{i\theta})|$ is cont.,

it must therefore be zero.

I.e., $|f(z)| = |f(z+re^{i\theta})| \quad \forall \ \theta \in [0, 2\pi]$.

Since r was any radius s.t. $B_r(z) \subseteq \mathrm{int}(\Omega)$,

we find

$$|f(z)| = |f(z+re^{i\theta})| \quad \forall \ \begin{array}{l}\theta \in [0,2\pi] \\ r \in [0, r_0]\end{array}$$

where $r_0 > 0$: $B_{r_0}(z) \subseteq \mathrm{int}(\Omega)$.

I.e., $|f|$ is const. on $B_{r_0}(z)$.

$\Rightarrow$ By HW 3 Q 8 (b), f is const. on

$B_{r_0}(z)$.

This yields the claim if $\Omega = B_R(z_0)$

$\exists \ z_0 \in \mathbb{C}$, $R > 0$.

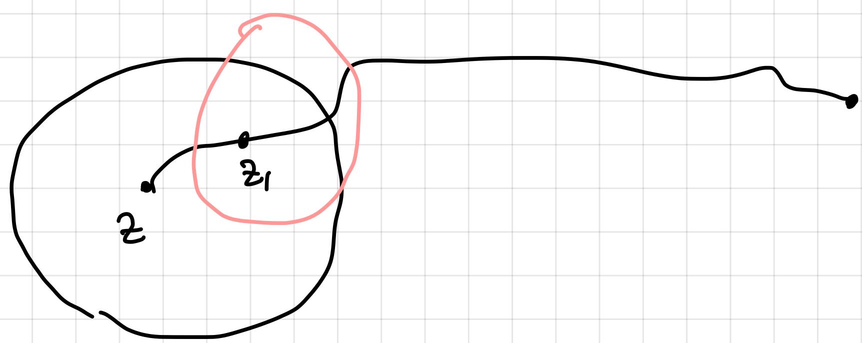
Now if Ω in NOT a disc:

Still assume $z \in int(\Omega)$ is a max for $|f|$.

Let $w \in int(\Omega)$ and $\eta : [0,1] \to int(\Omega)$ any cont. path $\eta(0) = z$
$\eta(1) = w$.

Possible via path-conn.

Let $R_0 > 0$ be the largest : $B_{R_0}(z) \subseteq int(\Omega)$.
Pick $z_1 \in im(\eta) \cap B_{R_0}(z)$.
Then $|f(z_1)| = |f(z)|$ by the above.



Let $R_1 > 0$ be the largest: $B_{R_1}(z_1) \subseteq int(\Omega)$.

Pick $z_2 \in im(\eta) \cap B_{R_1}(z_1)$.

$\vdots$

Like that we may recursively continue along η to get $|f(w)| = |f(z)|$ for any $w \in int(\Omega)$.

$\blacksquare$

$\boxed{Q12}$

$$\frac{1}{2\pi}\int_0^{2\pi} \cos\left(\frac{\pi}{3} + 2e^{i\theta}\right) d\theta = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}.$$

$\boxed{Q13}$

$$I(t) := \frac{1}{2\pi i} \int_{\partial B_3(0)} \frac{e^{zt}}{z^2+1} \, dz =$$

$$= \frac{1}{2\pi i} \int_{\partial B_3(0)} \frac{e^{zt}}{(z-i)(z+i)} \, dz$$

Note the identity

$$(z-i)^{-1} - (z+i)^{-1} = (z-i)^{-1}(z+i)^{-1} \, 2i.$$

$$\Rightarrow I(t) = \frac{1}{2\pi i} \int_{\partial B_3(0)} e^{zt} \frac{1}{2i}\left[(z-i)^{-1} - (z+i)^{-1}\right] dz$$

$$\overset{\text{Cauchy}}{=} \frac{1}{2i}\left[e^{it} - e^{-it}\right] = \sin(t).$$

$\boxed{Q14}$

Let f be a holomorphic function satisfying

$$|f(z)| \le 1 + |z|^{1.5} \qquad (z \in \mathbb{C}).$$

Cl.: f is a poly. w/ $\deg(f) \le 1$.

Pf.: If f were a poly, it could not have deg 2 or higher since then $|f(z)| \leq 1+|z|^{1.5}$ would be violated at $|z|$ arbit. large.

But couldn't f be some smooth f^n which grows like $|z|^{1.5}$ at ∞? No:

By Cauchy's estimate, $\forall R$

$$|f^{(n)}(z)| \leq \frac{n!}{R^n} \sup_{z \in B_R(0)} |f(z)|$$

$$\leq \frac{n!}{R^n}(1+R^{1.5})$$

Hence for all $n \geq 2$, $f^{(n)} = 0$.

This implies $f(z) = a+bz$ $\exists\, a,b \in \mathbb{C}$.

$\blacksquare$

$\boxed{Q15}$ See Thm. 6.33.