Complex Analysis with Applications
Princeton University MAT330
HW3, Due Feb 24th 2023

February 19, 2023

1 Harmonic functions and conjugates

1. Give examples (with proof) of two subsets of $\mathbb{C}$ which are simply connected and two subsets which are not simply connected (but still path-connected).

2. In this question we will prove the two-dimensional Poincaré lemma, which states that if $\Omega \subseteq \mathbb{R}^2$ is a simply-connected set and $V : \Omega \to \mathbb{R}^2$ is a zero-curl vector field (recall that in two-dimensions, the curl $(V) \equiv \partial_1 V_2 - \partial_2 V_1$ is a scalar field) then there is a scalar field $G : \Omega \to \mathbb{R}$ such that

$$V = \operatorname{grad}(G) \equiv \begin{bmatrix} \partial_1 G \\ \partial_2 G \end{bmatrix} .$$

   Equivalently, V is a conservative vector field. Since another definition of a conservative vector field is that its line integral is independent of the path chosen, the general strategy is to define G via the line integral of V and show it is well-defined, which in particular shows the independence of the path and hence that V is conservative. To that end, follow the steps below:

   (a) For any $z \in \mathbb{C}$, define

$$G_\gamma(z) \quad := \quad \int_0^1 \langle V(\gamma(t)), \dot{\gamma}(t) \rangle \, \mathrm{d}t$$

   where $\gamma : [0,1] \to \Omega$ is a path from some fixed initial point $z_0 \in \mathbb{C}$ to $z \in \mathbb{C}$.

   (b) Using the fundamental theorem of calculus

$$\int_{s=0}^1 \partial_s h(s) \, \mathrm{d}s \quad = \quad h(1) - h(0)$$

   and the fact that Ω is simply-connected, re-write

$$G_{\tilde{\gamma}}(z) - G_\gamma(z) = \int_0^1 \left\langle V(\tilde{\gamma}(t)), \dot{\tilde{\gamma}}(t) \right\rangle \mathrm{d}t - \int_0^1 \langle V(\gamma(t)), \dot{\gamma}(t) \rangle \, \mathrm{d}t$$

   (where $\tilde{\gamma} : [0,1] \to \Omega$ is any other choice of path which also ends at $z \in \mathbb{C}$) using a double-integral

$$\int_{t=0}^1 \int_{s=0}^1 \cdots \mathrm{d}s \mathrm{d}t$$

   involving the homotopy $\Gamma : [0,1]^2 \to \Omega$ which interpolates between γ and $\tilde{\gamma}$. What guarantees that such Γ exists?

   (c) Argue why

$$\partial_s \langle V(\Gamma(t,s)), (\partial_t \Gamma)(t,s) \rangle = \partial_t \langle V(\Gamma(t,s)), (\partial_s \Gamma)(t,s) \rangle$$

   using $\partial_s \partial_t \Gamma = \partial_t \partial_s \Gamma$ and a statement about the matrix Jacobian $(V) \equiv \begin{bmatrix} \partial_1 V_1 & \partial_1 V_2 \\ \partial_2 V_1 & \partial_2 V_2 \end{bmatrix}$ which is a consequence of the zero curl condition. This will involve the chain rule from multivariable calculus.

(d) Now do the t integral and conclude that $G_{\tilde{\gamma}}(z) - G_{\gamma}(z) = 0$ using the fact that the end points of any path during the interpolating homotopy Γ has fixed end points.

3. For the function $F : \{ z \in \mathbb{C} \mid x > 0 \} \to \mathbb{R}$ given by $F(x, y) := \log\left(\sqrt{x^2 + y^2}\right)$, show that it is harmonic and find its harmonic conjugate. Show that if F is extended to $\mathbb{C} \setminus \{ 0 \}$ it is still harmonic but its harmonic conjugate fails to be defined.

2 Basic functions

4. Calculate

$$\operatorname{Log}\left((-1 + \mathrm{i})^2\right) - 2\operatorname{Log}(-1 + \mathrm{i}) \ .$$

5. Calculate

$$\operatorname{Log}\left(\mathrm{i}^3\right) - 3\operatorname{Log}(\mathrm{i}) \ .$$

6. Find all roots of the equation

$$\log(z) = \mathrm{i}\frac{\pi}{2} \, .$$

7. Find the principal value of i^{i}.

3 Holomorphic functions

8. Show that if $f : \Omega \to \mathbb{C}$ is holomorphic on Ω then:

   (a) If f is real-valued for all $z \in \Omega$ then f is constant.

   (b) If $|f|$ is constant throughout Ω then f is constant.

9. Find the points $z \in \mathbb{C}$ where the following functions fail to be holomorphic, and explain why:

   (a) $f(z) = \frac{2z+1}{z(z^2+1)}$.

   (b) $g(z) = \frac{z^3+\mathrm{i}}{z^2-3z+2}$.

   (c) $h(z) = \frac{z^2+1}{(z+2)(z^2+2z+2)}$.