FEB 20 2023

MAT 330: Complex Analysis — HW3 Sol-ns

1. **Claim:** $\mathbb{C}$ is simply-connected.

Proof: Clearly $\mathbb{C}$ is path-conn.

Next, given any two cont. paths $\gamma, \tilde{\gamma} : [0,1] \to \mathbb{C}$ s.t. $\gamma(t) = \tilde{\gamma}(t)$ for $t \in \{0,1\}$, define the homotopy $\Gamma : [0,1]^2 \to \mathbb{C}$ via

$$\Gamma(s,t) := (1-s)\gamma(t) + s\tilde{\gamma}(t) \qquad \left((s,t)\in[0,1]^2\right)$$

Claim: Γ is cont.

Proof:
$$|\Gamma(s,t) - \Gamma(s_0,t_0)| = |(1-s)\gamma(t) + s\tilde{\gamma}(t) - (1-s_0)\gamma(t_0) + s_0\tilde{\gamma}(t_0)|$$

$$= |(1-s)\gamma(t) - (1-s_0)\gamma(t) + (1-s_0)\gamma(t) - (1-s_0)\gamma(t_0) + s\tilde{\gamma}(t) - s_0\tilde{\gamma}(t) + s_0\tilde{\gamma}(t) - s_0\tilde{\gamma}(t_0)|$$

triangle ineq.

$$\leq |s-s_0||\gamma(t)| + |1-s_0||\gamma(t)-\gamma(t_0)| + |s-s_0||\tilde{\gamma}(t)| + |s_0||\tilde{\gamma}(t)-\tilde{\gamma}(t_0)|$$

$$\leq |s-s_0||\gamma(t_0)| + |s-s_0||\gamma(t)-\gamma(t_0)| + |1-s_0||\gamma(t)-\gamma(t_0)| + |s-s_0||\tilde{\gamma}(t_0)| +$$

$+|s-s_0||\gamma(t)-\gamma(t_0)|+|s_0||\tilde{\gamma}(t)-\tilde{\gamma}(t_0)|$.

Now, since $\gamma, \tilde{\gamma}$ are both cont., $|\gamma(t)-\gamma(t_0)|$, $|\tilde{\gamma}(t)-\tilde{\gamma}(t_0)|$ can be made arbitrarily small, as can $|s-s_0|$.

Moreover, $\Gamma([0,1]^2) \subseteq \mathbb{C}$ and

$$\Gamma(\cdot, t) = \gamma(t) \quad t \in \{0,1\}$$

$$\Gamma(0, \cdot) = \gamma$$

$$\Gamma(1, \cdot) = \tilde{\gamma}.$$

$\Rightarrow \Gamma$ is the desired homotopy.

Since $\gamma, \tilde{\gamma}$ were arbitrary, $\mathbb{C}$ is simply-conn. ∎

Def.: A convex set $C \subseteq \mathbb{C}$ is a set s.t. $\forall\ z, w \in C$, the path

$$\gamma(t) = (1-t)z + tw \qquad (t \in [0,1])$$

lies entirely within C.

Claim: If $G \subseteq \mathbb{C}$ is convex then it is simply connected.

Proof: Use same homotopy as above. Convexity guarantees straight line from $\gamma(t)$ to $\tilde{\gamma}(t)$ lies in G for any $t \in [0,1]$.

<u>Claim</u>: $\mathbb{C} \setminus \{0\}$ is NOT simply-connected.

<u>Proof</u>: Assume otherwise.

Let $V : \mathbb{R}^2 \setminus \{0,0\} \to \mathbb{R}^2$ be given by

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} -y/x^2+y^2 \\ x/x^2+y^2 \end{bmatrix} .$$

<u>Claim</u>: V is cont. diff.

By the Poincaré lemma, if $\mathbb{R}^2 \setminus \{0\}$ were simply-conn., there would be some $G : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$: $V = \text{grad}(G)$.

But then, $\oint \langle V \circ \gamma, \gamma' \rangle = 0$ for <u>any</u> closed path

$$\gamma : [0,1] \to \mathbb{R}^2 \setminus \{0\}.$$

However, a direct calc. shows:

$$\gamma(t) := \begin{bmatrix} \cos(2\pi t) \\ \sin(2\pi t) \end{bmatrix}$$

$$\gamma'(t) = \begin{bmatrix} -2\pi \sin(2\pi t) \\ 2\pi \cos(2\pi t) \end{bmatrix}$$

$$\langle V(\gamma(t)), \gamma'(t) \rangle =$$

$$= \left\langle \begin{bmatrix} -\sin(2\pi t) \\ \cos(2\pi t) \end{bmatrix}, \begin{bmatrix} -2\pi \sin(2\pi t) \\ 2\pi \cos(2\pi t) \end{bmatrix} \right\rangle$$

$$= 2\pi .$$

$$\Rightarrow \oint_0^1 \langle V \circ \gamma, \gamma' \rangle = 2\pi \neq 0 \, !$$

$$\Rightarrow \boxed{\perp} .$$

$\blacksquare$

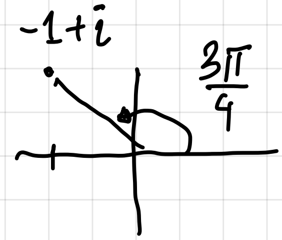
Claim: $\mathbb{C} \setminus B_{1/2}(0)$ is NOT simply-conn.

Proof: Same as above.

2. See Section 4.6 in the lecture notes.
3. See Example 4.26 in the lecture notes.
4. $\mathrm{Log}(-1+i) =$

$$= \log(|-1+i|) + i\,\mathrm{Arg}(-1+i).$$

$$|-1+i| = \sqrt{1^2+1^2} = \sqrt{2}.$$



$$\mathrm{Arg}(-1+i) = \mathrm{arctg2}(1,-1)$$

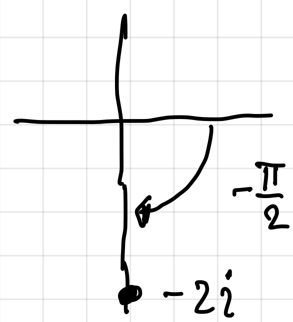
$$= 3\pi/4$$

$$\Rightarrow \mathrm{Log}(-1+i) = \tfrac{1}{2}\log(2) + i\,\frac{3\pi}{4}.$$

$$(-1+i)^2 = 1 - 1 - 2i = -2i$$

$$\mathrm{Log}((-1+i)^2) = \mathrm{Log}(-2i)$$

$$= \underbrace{\log(\underbrace{|-2i|}_{2})}_{\log(2)} + i\,\mathrm{Arg}(-2i)$$



$$\mathrm{Arg}(-2i) = -\frac{\pi}{2}$$

$$\Rightarrow \mathrm{Log}((-1+i)^2) - 2\,\mathrm{Log}((-1+i)) =$$

$$= \log(2) - i\frac{\pi}{2} - 2\left(\tfrac{1}{2}\log(2) + i\,\frac{3\pi}{4}\right)$$

$$= -i\frac{\pi}{2} - i\frac{3\pi}{2} = -i\,2\pi \neq 0$$

5. $\mathrm{Log}(i^3) = \mathrm{Log}(-i) =$

$$= \log(1) - i\frac{\pi}{2} = -i\frac{\pi}{2}.$$

$$\mathrm{Log}(i) = i\frac{\pi}{2}.$$

$$\Rightarrow \mathrm{Log}(i^3) - 3\mathrm{Log}(i) = -i\frac{\pi}{2} - 3i\frac{\pi}{2} = -i\,2\pi$$

$\neq 0$

6. Want sol-ns $z \in \mathbb{C}$: $\log(z) = i\frac{\pi}{2}$.

multi-valued

$$\log(z) = \log(|z|) + i\arg(z) \overset{!}{=} i\frac{\pi}{2}.$$

$\Rightarrow |z| = 1$ and $\arg(z) = \frac{\pi}{2}$.

$\Rightarrow z = i$ is the only sol-n.

7. $i^i = \exp(\log(i^i))$

$$= \exp(i \log(i))$$

principal val. $\overset{\text{principal val.}}{=} \exp(i \operatorname{Log}(i))$

$$= \exp\Big(i \big(\underbrace{\log(1)}_{=0} + i \tfrac{\pi}{2} \big)\Big)$$

$$= \exp\left(-\tfrac{\pi}{2}\right).$$

8. (a) Claim: If $f : \Omega \to \mathbb{C}$ is holomorphic and $\operatorname{Im}\{f\} = 0$ then f is const.

Proof: CRE say $\begin{cases} \partial_x f_R = \partial_y f_I \\ \partial_x f_I = -\partial_y f_R \end{cases}$.

Plug in $f_I = 0$ to get

$$\begin{cases} \partial_x f_R = 0 \\ \partial_y f_R = 0 \end{cases} \iff \operatorname{grad}(f_R) = 0$$

$$\Rightarrow f_R \text{ is const.}$$

$\blacksquare$

(b) Claim: If $f : \Omega \to \mathbb{C}$ is holomorphic and $|f|$ is a const. then f is a a const.

<u>Proof</u>: Write $f = |f| e^{i\theta}$ $\exists$ $\theta : \mathbb{R}^2 \to \mathbb{R}$.

W.T.S. θ is the const. f^n.

CRE for f imply for θ!

$$\partial_x f_R = -|f| \sin(\theta) \partial_x \theta$$
$$\partial_y f_I = |f| \cos(\theta) \partial_y \theta$$
$$\partial_x f_I = |f| \cos(\theta) \partial_x \theta$$
$$\partial_y f_R = -|f| \sin(\theta) \partial_y \theta$$

$$\Rightarrow \begin{cases} -|f| \sin(\theta) \partial_x \theta = |f| \cos(\theta) \partial_y \theta \\ |f| \cos(\theta) \partial_x \theta = +|f| \sin(\theta) \partial_y \theta \end{cases}$$

If $|f| = 0$ we're finished anyway.

Otherwise, cancel it and get:

$$\begin{cases} -\operatorname{tg}(\theta) \partial_x \theta = \partial_y \theta \\ \partial_x \theta = +\operatorname{tg}(\theta) \partial_y \theta \end{cases}$$

Adding them yields:

$$\left[\operatorname{tg}(\theta) + \frac{1}{\operatorname{tg}(\theta)}\right] \partial_x \theta = 0$$

and same for $\partial_y \theta$.

Claim: $\left| tg(\alpha) + \frac{1}{tg(\alpha)} \right| \geq 2 \quad \forall\, \alpha \in \mathbb{R}$

Proof: $\left| \frac{\sin(\alpha)}{\cos(\alpha)} + \frac{\cos(\alpha)}{\sin(\alpha)} \right| = \frac{1}{|\sin(\alpha)\cos(\alpha)|}$

$$= \frac{2}{|\sin(2\alpha)|}$$

But $|\sin(2\alpha)| \leq 1$. ∎

$$\Rightarrow \left[tg(\theta) + \frac{1}{tg(\theta)} \right] \partial_x \theta = 0$$

implies $\underbrace{\left| tg(\theta) + \frac{1}{tg(\theta)} \right|}_{\geq 2} |\partial_x \theta| = 0$

$$\Rightarrow 2|\partial_x \theta| \leq 0.$$

$$\Rightarrow |\partial_x \theta| \leq 0 \Rightarrow |\partial_x \theta| = 0$$

$$\Rightarrow \partial_x \theta = 0$$

and same for $\partial_y \theta$.

$$\Rightarrow \text{grad}(\theta) = 0 \Rightarrow \theta = 0.$$ ∎

9. The following ratios of polynomials may fail to be holomorphic if the denominator is zero:

(a) $$f(z) = \frac{2z+1}{z(z^2+1)}$$

$z=0$, $z=i$ are roots of denom.

(b) $$g(z) = \frac{z^3+i}{z^2-3z+2}$$ $z=1,2$

$$z^2 - 3z + 2 = (z-1)(z-2)$$

(c) $$h(z) = \frac{z^2+1}{(z+2)(z^2+2z+2)}$$

$$z^2+2z+2 = (z+1+i)(z+1-i)$$

$z=-1-i$
$z=-1+i$
$z=-2$