# Complex Analysis with Applications
# Princeton University MAT330
# HW2, Due Feb 17th 2023

February 14, 2023

## 1 The complex field $\mathbb{C}$

1. Find the principal argument of
   (a) $z_1 = \frac{\mathrm{i}}{-2-2\mathrm{i}}$.
   (b) $z_2 = \left(\sqrt{3}-\mathrm{i}\right)^6$.

2. Prove that any two $z, w \in \mathbb{C} \setminus \{ 0 \}$ have $|z| = |w|$ iff $\exists a, b \in \mathbb{C}$ such that $z = ab$ and $w = a\bar{b}$.

## 2 More topology

3. Consider the map $f : \mathbb{C} \setminus \{ 0 \} \to \mathbb{C}$ given by the formula $f(z) = \mathrm{e}^{\frac{1}{z}}$.
   (a) Define the punctured disc of radius $r$ as $A_r := \{ z \in \mathbb{C} \mid 0 < |z| < r \}$. Calculate the image set $f(A_r)$.
   (b) Determine if $\lim_{z\to 0} f(z)$ exists and if it does calculate it.

4. Prove that any non-constant polynomial $p : \mathbb{C} \to \mathbb{C}$ with $p(z) = \sum_{j=0}^{n} a_j z^j$ (for some collection of coefficients $a_0, \dots, a_n \in \mathbb{C}$ such that not all $a_1, \dots, a_n$ are zero) has

$$\|p\|_\infty = \infty$$

i.e., is not a bounded function.

## 3 $\mathbb{C}$-Differentiability and the Cauchy-Riemann equations

5. Prove that if $f, g : \mathbb{C} \to \mathbb{C}$ are $\mathbb{C}$-differentiable then $f \circ g : \mathbb{C} \to \mathbb{C}$ is $\mathbb{C}$-differentiable with derivative

$$(f \circ g)' = (f' \circ g)\, g' .$$

6. For a given function $f : \mathbb{C} \to \mathbb{C}$ with $f_R = \mathbb{Re}\{f\}, f_I = \mathbb{Im}\{f\}$, the Cauchy-Riemann equations are partial-differential equations connecting $\partial_x f_R, \partial_x f_I, \partial_y f_R, \partial_y f_I$ as follows:

$$\begin{cases} \partial_x f_R & = \partial_y f_I \\ \partial_x f_I & = -\partial_y f_R \end{cases} .$$

Defining the polar coordinates $r = \sqrt{x^2+y^2}$ and $\theta = \arctan\left(\frac{y}{x}\right)$, re-write the Cauchy-Riemann equations in terms of $\partial_r f_R, \partial_r f_I, \partial_\theta f_R, \partial_\theta f_I$ instead of $\partial_x f_R, \partial_x f_I, \partial_y f_R, \partial_y f_I$.

7. Show that $f : \{ z = x + \mathrm{i}y \in \mathbb{C} \mid x > 0 \} \to \mathbb{C}$ given by

$$f(z) = \mathrm{e}^{-\theta} \cos(\log(r)) + \mathrm{i}\mathrm{e}^{-\theta} \sin(\log(r))$$

is $\mathbb{C}$-differentiable.

8. Prove that $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = x^2 + \mathrm{i}xy$ is *not* $\mathbb{C}$-differentiable.

9. Suppose that $f(z_0) = g(z_0) = 0$ and $f'(z_0), g'(z_0)$ exist with $g'(z_0) \neq 0$. Show

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}.$$

10. Provide an example (with proof) of a function $f : \mathbb{C} \to \mathbb{C}$ where there's at least one point $z_0 \in \mathbb{C}$ such that the Cauchy-Riemann equations hold for $f$ at $z_0$, *yet* the function is not $\mathbb{C}$-differentiable at $z_0 \in \mathbb{C}$. Conclude that the Cauchy-Riemann equations are necessary, but not sufficient for $\mathbb{C}$-differentiability.