Complex Analysis with Applications  
Princeton University MAT330  
HW2, Due Feb 17th 2023

February 14, 2023

1 The complex field \( \mathbb{C} \)

1. Find the principal argument of

(a) \( z_1 = \frac{i}{\sqrt{2}} \).

(b) \( z_2 = (\sqrt{3} - i)^6 \).

2. Prove that any two \( z, w \in \mathbb{C} \setminus \{0\} \) have \( |z| = |w| \) if and only if there exist \( a, b \in \mathbb{C} \) such that \( z = ab \) and \( w = a\overline{b} \).

2 More topology

3. Consider the map \( f : \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C} \) given by the formula \( f(z) = e^z \).

(a) Define the punctured disc of radius \( r \) as \( A_r := \{ z \in \mathbb{C} | 0 < |z| < r \} \). Calculate the image set \( f(A_r) \).

(b) Determine if \( \lim_{z \to 0} f(z) \) exists and if it does calculate it.

4. Prove that any non-constant polynomial \( p : \mathbb{C} \rightarrow \mathbb{C} \) with \( p(z) = \sum_{j=0}^n a_j z^j \) (for some collection of coefficients \( a_0, \ldots, a_n \in \mathbb{C} \) such that not all \( a_1, \ldots, a_n \) are zero) has \( \|p\|_{\infty} = \infty \)
i.e., is not a bounded function.

3 \( \mathbb{C} \)-Differentiability and the Cauchy-Riemann equations

5. Prove that if \( f, g : \mathbb{C} \rightarrow \mathbb{C} \) are \( \mathbb{C} \)-differentiable then \( f \circ g : \mathbb{C} \rightarrow \mathbb{C} \) is \( \mathbb{C} \)-differentiable with derivative \( (f \circ g)' = (f' \circ g)' \).

6. For a given function \( f : \mathbb{C} \rightarrow \mathbb{C} \) with \( f_R = \Re \{f\}, f_I = \Im \{f\} \), the Cauchy-Riemann equations are partial-differential equations connecting \( \partial_x f_R, \partial_x f_I, \partial_y f_R, \partial_y f_I \) as follows:

\[
\begin{align*}
\partial_x f_R &= \partial_y f_I \\
\partial_x f_I &= -\partial_y f_R .
\end{align*}
\]

Defining the polar coordinates \( r = \sqrt{x^2 + y^2} \) and \( \theta = \arctan \left( \frac{y}{x} \right) \), re-write the Cauchy-Riemann equations in terms of \( \partial_r f_R, \partial_r f_I, \partial_\theta f_R, \partial_\theta f_I \) instead of \( \partial_x f_R, \partial_x f_I, \partial_y f_R, \partial_y f_I \).

7. Show that \( f : \{ z = x + iy \in \mathbb{C} | x > 0 \} \rightarrow \mathbb{C} \) given by

\[
f(z) = e^{-\theta} \cos (\log (r)) + ie^{-\theta} \sin (\log (r))
\]
is \( \mathbb{C} \)-differentiable.
8. Prove that \( f : \mathbb{C} \to \mathbb{C} \) given by \( f(z) = x^2 + ixy \) is not \( \mathbb{C} \)-differentiable.

9. Suppose that \( f(z_0) = g(z_0) = 0 \) and \( f'(z_0), g'(z_0) \) exist with \( g'(z_0) \neq 0 \). Show

\[
\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}.
\]

10. Provide an example (with proof) of a function \( f : \mathbb{C} \to \mathbb{C} \) where there’s at least one point \( z_0 \in \mathbb{C} \) such that the Cauchy-Riemann equations hold for \( f \) at \( z_0 \), yet the function is not \( \mathbb{C} \)-differentiable at \( z_0 \in \mathbb{C} \). Conclude that the Cauchy-Riemann equations are necessary, but not sufficient for \( \mathbb{C} \)-differentiability.