FEB 13 2023

HW2 — Sample Solutions

1. (a) $z_1 = \frac{i}{-2-2i}$. Seek its principal arg.:

$$-2-2i = \sqrt{\underbrace{2^2+2^2}_{8}}\, \exp(i \underbrace{\operatorname{arctg}(\underbrace{{}^{-2}/_{-2}}_{1}))}_{\pi/4}$$

$$i = \exp(i\tfrac{\pi}{2})$$

$$\Rightarrow \quad z_1 = \frac{e^{i\frac{\pi}{2}}}{\sqrt{8}\, e^{i\frac{\pi}{4}}} = \frac{1}{\sqrt{8}} e^{i(\frac{\pi}{2}-\frac{\pi}{4})} = \frac{1}{\sqrt{8}} e^{i\frac{\pi}{4}}.$$

$$\frac{\pi}{4} + 2\pi n \in (-\pi, \pi] \quad \text{for} \quad n = 0$$

$\Rightarrow \frac{\pi}{4}$ is the p. arg.

(b) $z_2 = (\sqrt{3} - i)^6$.

First study $\sqrt{3} - i$:

$$\sqrt{3} - i = \sqrt{3+1}\, \exp(i \operatorname{arctg}(\tfrac{-1}{\sqrt{3}}))$$

$= 2\exp(-i\frac{\pi}{6})$.

calculator for $\arctg(-\frac{1}{\sqrt{3}})$.

Raising to power 6 yields

$$z_2 = 2^6 e^{-i\pi} = -64$$

p. arg. $= \pi$.

2. Claim: $\forall\, z, w \in \mathbb{C} \setminus \{0\}$,

$$|z| = |w| \Leftrightarrow \exists\, a, b \in \mathbb{C} : \begin{array}{l} z = ab \\ w = a\bar{b} \end{array}$$

Proof: $\boxed{\Leftarrow}$ $|z| = |ab| = |a||b| = |a||\bar{b}| = |a\bar{b}| = |w|$. ✓

$\boxed{\Rightarrow}$ Write $z = re^{i\alpha}$, $w = re^{i\beta}$

$\exists\, r > 0$, $\alpha, \beta \in \mathbb{R}$.

Then $\frac{z}{w} = e^{i(\alpha-\beta)} = \frac{b}{\bar{b}} = e^{2i\arg(b)}$

$\Rightarrow \arg(b) = \frac{1}{2}(\alpha - \beta)$.

$\frac{z}{\bar{w}} = e^{i(\alpha+\beta)} = \frac{a}{\bar{a}} = e^{2i\arg(a)}$

$\Rightarrow \arg(a) = \frac{1}{2}(\alpha + \beta)$.

Pick $a = \sqrt{r}\exp(\frac{i}{2}(\alpha+\beta))$

$$b = \sqrt{r}\, \exp\left(\tfrac{i}{2}(\alpha - \beta)\right).$$

3. $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$; $f(z) = \exp(\frac{1}{z})$.

(a) $A_r := \{ z \in \mathbb{C} \mid 0 < |z| < r \}$.

$$f(A_r) \equiv \{ f(z) \in \mathbb{C} \mid 0 < |z| < r \}$$

$$= \{ e^{\frac{1}{z}} \in \mathbb{C} \mid 0 < |z| < r \}$$

In polar form, $z = \rho e^{i\theta}$

$$\exp(\tfrac{1}{z}) = \exp(\tfrac{1}{\rho} e^{-i\theta})$$

$$= \exp(\tfrac{1}{\rho}\cos(\theta) - \tfrac{i}{\rho}\sin(\theta))$$

$$= \exp(\tfrac{1}{\rho}\cos(\theta))\, e^{i \frac{-1}{\rho}\sin(\theta)}$$

$$\rho \in (0, r) \Rightarrow \tfrac{1}{\rho} \in (\tfrac{1}{r}, \infty)$$

$$\Rightarrow \exp(\tfrac{1}{\rho}\cos(\theta)) \in (e^{-\infty}, e^{\infty}]$$

$$= (0, \infty).$$

We find $f(A_r) = \mathbb{C} \setminus \{0\}$.

(b) If $\varepsilon > 0$,

$$f(\varepsilon) = e^{\frac{1}{\varepsilon}} \xrightarrow{\varepsilon \to 0^+} +\infty .$$

OTOH,

$$f(-\varepsilon) = e^{-\frac{1}{\varepsilon}} \xrightarrow{\varepsilon \to 0^+} 0 .$$

Since the putative limit depends on the direction, it cannot $\exists$.

4. <u>Claim</u>: If $p : \mathbb{C} \to \mathbb{C}$ is a non-const. poly. Then $\|p\|_\infty = \infty$.

<u>Proof</u>: Let $N \in \mathbb{N}$ be the largest non-zero deg in p, i.e.,

$$p(z) = a_N z^N + \tilde{p}(z)$$

$\exists$ $a_N \in \mathbb{C} \setminus \{0\}$ and $\tilde{p}$ poly. of deg $< N$. Then

$$\lim_{|z| \to \infty} \frac{\tilde{p}(z)}{z^N} = 0 \quad \text{as}$$

$\deg(\tilde{p}) < N.$

$$\Rightarrow \quad p(z) = a_N z^N \underbrace{\left(1 + a_N \frac{\tilde{p}(z)}{z^N}\right)}_{\xrightarrow{|z|\to\infty} 1}.$$

Howeven we can make z^N arbit. large. $\Rightarrow \|p\|_\infty = \infty.$ $\blacksquare$

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5. <u>Claim</u>: If f, g are $\mathbb{C}$-diff, so is $f \circ g$ and $(f\circ g)' = (f' \circ g) g'$.

<u>Proof</u>: Calculate the prelimit entailed in $(f\circ g)'$:

$$\frac{(f\circ g)(z_0+z) - (f\circ g)(z_0)}{z} = \frac{f(g(z_0+z)) - f(g(z_0))}{z}.$$

Since g is $\mathbb{C}$-diff. (Frechet diff.):

$$g(z_0+z) = g(z_0) + g'(z_0) z + O(|z|^2)$$

$$\Rightarrow f(g(z_0+z)) = f(g(z_0) + g'(z_0) z + O(|z|^2))$$

$$= f(g(z_0)) + f'(g(z_0)) g'(z_0) z + O(|g'(z_0) z|^2).$$

We find:

$$\frac{f(g(z_0+z))-f(g(z_0))}{z} = f'(g(z_0))g'(z_0) + O(|g'(z_0)|^2|z|)$$

$$\xrightarrow{z\to 0} f'(g(z_0))g'(z_0) \quad . \quad \blacksquare$$

6. Want to express the CRE in polar coordinates. That means:

$$f: \mathbb{C} \to \mathbb{C}$$

is written as a function of (r, θ) (although we still keep the Cartesian

$$f = f_R + i f_I$$

decomposition — it's just the variables of the domain that change).

Write
$$r(x,y) = \sqrt{x^2+y^2}$$
$$\theta(x,y) = \arctg(y/x)$$

$$\Rightarrow \quad (\partial_x g)(r,\theta) = (\partial_r g)(r,\theta)\,\partial_x r \; +$$
$$+ (\partial_\theta g)(r,\theta)\,\partial_x \theta$$

$$\partial_x r = \frac{x}{\sqrt{x^2+y^2}} = \cos(\theta) \qquad \partial_y r = \sin(\theta)$$

$$\partial_x \theta = -\frac{y}{x^2+y^2} = -\frac{\sin(\theta)}{r}$$

$$\partial_y \theta = \frac{\cos(\theta)}{r}$$

$$\Rightarrow (\partial_x g)(r,\theta) = (\partial_r g)(r,\theta)\cos(\theta) -$$
$$- (\partial_\theta g)(r,\theta)\frac{\sin(\theta)}{r}$$

$$(\partial_y g)(r,\theta) = (\partial_r g)(r,\theta)\sin(\theta) +$$
$$+ (\partial_\theta g)(r,\theta)\frac{\cos(\theta)}{r}.$$

We may now apply this to $g = f_R, f_I$:

$$\partial_x f_R = \partial_r f_R \cos(\theta) - \partial_\theta f_R \frac{\sin(\theta)}{r}$$

$$\overset{(CREI)}{=} \partial_y f_I = \partial_r f_I \sin(\theta) + \partial_\theta f_I \frac{\cos(\theta)}{r}$$

$$\partial_x f_I = \partial_r f_I \cos(\theta) - \partial_\theta f_I \frac{\sin(\theta)}{r}$$

$$\overset{(CRE2)}{=} -\partial_y f_R$$

$$= -\partial_r f_R \sin(\theta) - \partial_\theta f_R \frac{\cos(\theta)}{r} .$$

Collecting everything:

$$\begin{cases} \partial_r f_R \cos(\theta) - \partial_\theta f_R \frac{\sin(\theta)}{r} = \partial_r f_I \sin(\theta) + \partial_\theta f_I \frac{\cos(\theta)}{r} \\ \partial_r f_I \cos(\theta) - \partial_\theta f_I \frac{\sin(\theta)}{r} = -\partial_r f_R \sin(\theta) - \partial_\theta f_R \frac{\cos(\theta)}{r} \end{cases}$$

① Multiply 1^{st} eq-n by $\cos(\theta)$

2^{nd} eq-n by $\sin(\theta)$

add the two resulting eq-n to get:

$$\partial_r f_R = \frac{1}{r} \partial_\theta f_I$$

② Multiply 1^{st} eq-n by $\sin(\theta)$

2^{nd} eq-n by $\cos(\theta)$

add the two resulting eq-n to get:

$$\partial_r f_I = -\frac{1}{r}\partial_\theta f_R .$$

In conclusion:

$$\boxed{\begin{cases} r\,\partial_r f_R = \partial_\theta f_I \\ r\,\partial_r f_I = -\partial_\theta f_R \end{cases}} .$$

7. **Claim:** $f : \{x+iy \in \mathbb{C} \mid x>0\} \to \mathbb{C}$

def. via

$$f(r,\theta) = e^{-\theta}\cos(\log(r)) + i\,e^{-\theta}\sin(\log(r))$$

is $\mathbb{C}$-diff.

Proof: Note: if we knew the def. of $\log : \mathbb{C} \to \mathbb{C}$ we could have used it here since

$$
\begin{aligned}
z^i &= \exp(i \log(z)) \\
&= \exp(i \log(re^{i\theta})) \\
&= \exp(i[\log(r) + i\theta]) \\
&= e^{-\theta} \cos(\log(r)) + i e^{-\theta} \sin(\log(r)) \\
&= f(r,\theta).
\end{aligned}
$$

Instead, verify that f is $\mathbb{R}^2$ diff. and satisfies the polar CRE:

$$
\left.\begin{aligned}
(\partial_r f_R)(r,\theta) &= -e^{-\theta} \sin(\log(r)) \frac{1}{r} \\
(\partial_\theta f_I)(r,\theta) &= -e^{-\theta} \sin(\log(r))
\end{aligned}\right\} \checkmark
$$

Similarly for the other one.

Note $\mathbb{R}^2$ diff. is std. as exp, log, cos, sin are $\mathbb{R}$-diff. (log away from zero).

We had to restrict to $\{x > 0\}$

to keep the limit at zero

path indep. (check).

∎

8. Claim: $\mathbb{C} \ni x+iy \xmapsto{f} x^2+ixy$ is not $\mathbb{C}$-diff.

Proof: Jacobian $= \begin{bmatrix} 2x & y \\ 0 & x \end{bmatrix}$ is

NOT of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

$\Rightarrow$ Not $\mathbb{C}$-linear.

9. Suppose that $f(z_0) = g(z_0) = 0$ and $f'(z_0), g'(z_0)$ exist with $g'(z_0) \neq 0$.

Claim:
$$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}.$$

Proof:
$$f'(z_0) \equiv \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z}$$
$$= \lim_{z\to z_0} \frac{f(z)}{z}$$

and $g'(z_0) = \lim_{z\to z_0} \frac{g(z)}{z}$.

Hence

$$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \lim_{z\to z_0} \frac{f(z)/z}{g(z)/z}$$

lim of ratio = ratio of lim if both $\exists$

$$\overset{\downarrow}{=} \left(\lim_{z\to z_0} f(z)/z\right) \Big/ \left(\lim_{z\to z_0} g(z)/z\right)$$

$$= f'(z_0) \,/\, g'(z_0).$$

$\blacksquare$

10. See Example 4.12 in the notes.