Complex Analysis with Applications
Princeton University MAT330
HW1, Due Feb 10th 2023

1 The field $\mathbb{C}$

1. Find the real and imaginary parts of the following:
   (a) $z = (1 - \mathrm{i})^3$.
   (b) $z = \frac{x+\mathrm{i}y}{a+\mathrm{i}b}$ for real x, y, a, b.
   (c) $z = \frac{\pi \mathrm{i}}{(1-\mathrm{i})(2-\mathrm{i})}$.

2. Sketch the regions of the complex plane defined by the following constraints:
   (a) $|z + 4\mathrm{i}| \leq \pi$.
   (b) $|z + 2| < |z + 1|$.

3. Find the minimal constant $c > 0$ such that

$$c\,|z| \geq |\mathbb{Re}\{z\}| + |\mathbb{Im}\{z\}| \qquad (z \in \mathbb{C}) .$$

4. Show that

$$\sum_{j=0}^{n} z^j = \frac{1 - z^{n+1}}{1 - z} \qquad (z \in \mathbb{C} \setminus \{ 1 \}) .$$

   Using this, prove Lagrange's trigonometric identity

$$\sum_{j=0}^{n} \cos(j\theta) \quad = \quad \frac{1}{2} + \frac{\sin\left((2n+1)\,\frac{1}{2}\theta\right)}{2\sin\left(\frac{1}{2}\theta\right)} \qquad (\theta \notin 2\pi\mathbb{Z}) .$$

   Note that an earlier version of the HW had a type in the above equation (it had $\cos(n\theta)$ instead of $\cos(j\theta)$ on the LHS).

5. Find the zeros of the equation $z^5 + 32 = 0$ and mark where they are positioned in the complex plane.

6. Prove that

$$|z| \quad = \quad \max_{\theta \in (-\pi, \pi]} \mathbb{Re}\left\{z\mathrm{e}^{\mathrm{i}\theta}\right\} \qquad (z \in \mathbb{C})$$

   and use this to prove

$$|z + w| \quad \leq \quad |z| + |w| \qquad (z, w \in \mathbb{C}) .$$

2 A bit of topology

For reference, see the corresponding section of the lecture notes titled "a bit of topology".

7. Find the accumulation points (if any) of

$$S = \{ (2\mathrm{i})^n \mid n \in \mathbb{N}_{\geq 1} \} \subseteq \mathbb{C} .$$

8. Find the closure, interior and boundary of the following sets within $\mathbb{C}$:

$$\begin{aligned} S_1 &= \mathbb{R} \\ S_2 &= \left\{ w \in \mathbb{C} \,\middle|\, |w-1| < \frac{1}{2} \right\} \equiv B_{\frac{1}{2}}(1) \\ S_3 &= \left\{ \frac{\mathrm{i}}{n} \,\middle|\, n \in \mathbb{N}_{\geq 1} \right\} \\ S_4 &= \{ x + \mathrm{i}y \mid x, y \in \mathbb{Q} \} . \end{aligned}$$

Conclude if these sets are open, closed, neither or both.

9. Let $f, g : \mathbb{C} \to \mathbb{C}$ be two functions and $z_0 \in \mathbb{C}$ be given. Say g has a limit $\lim_{z \to z_0} g(z) = g(z_0)$ and f has a limit $\lim_{w \to g(z_0)} f(w) = f(g(z_0))$. This means that for any $\varepsilon > 0$, there are some $\delta_f(\varepsilon), \delta_g(\varepsilon) > 0$ such that

$$w \in B_{\delta_f(\varepsilon)}(g(z_0)) \Longrightarrow f(w) \in B_\varepsilon(f(g(z_0)))$$

and

$$z \in B_{\delta_g(\varepsilon)}(z_0) \Longrightarrow g(z) \in B_\varepsilon(g(z_0)) .$$

For a given $\varepsilon > 0$, find some $\delta_\star(\varepsilon) > 0$ (in terms of δ_f, δ_g) which would satisfy the $\varepsilon - \delta$ definition of existence of the limit of $f \circ g : \mathbb{C} \to \mathbb{C}$ at z_0, i.e.,:

$$\lim_{z \to z_0} f(g(z)) = f(g(z_0)) .$$

10. A *modulus of continuity* is a function $\omega : [0, \infty] \to [0, \infty]$ which is continuous at zero and obeys $\omega(0) = 0$. A function $f : A \to \mathbb{C}$ (where $A \subseteq \mathbb{C}$) admits a modulus of continuity ω iff

$$|f(z) - f(w)| \leq \omega(|z - w|) \qquad (z, w \in A) .$$

Find a modulus of continuity for the function $f : B_1(0) \to \mathbb{C}$ given by the formula $f(z) = z^2$. What about $g : B_2(-3) \to \mathbb{C}$ given by the formula $g(z) = \exp(z)$? What happens to the moduli of continuity of these functions if their domains were extended to the entire complex plane?