Complex Analysis with Applications
Princeton University MAT330
HW10, Due May 5th 2023 (but not to be submitted)

May 3, 2023

Note: when doing asymptotic analysis, the word “fixed” means that a parameter is not taken to infinity and does not depend on the asymptotic parameter.

1 Gaussian integrals

1. Calculate, for some $a > 0$,
\[
\int_{x \in \mathbb{R}} e^{-\frac{1}{2}ax^2} \, dx.
\]

2. Calculate, for some $a > 0$,
\[
\int_{z \in \mathbb{C}} e^{-\frac{1}{2}a|z|^2} \, d^2z
\]
(i.e., a two-dimensional surface integral throughout the complex plane).

3. Let now $n \in \mathbb{N}$. Recall that a matrix $A \in \text{Mat}_{n \times n}(\mathbb{R})$ is called “positive” and denoted $A \geq 0$ iff $A = B^*B$ for some $B \in \text{Mat}_{n \times n}(\mathbb{R})$. Equivalent and convenient conditions are: (i) $A \geq 0$ iff $A$ is Hermitian and $\langle v, Av \rangle_{\mathbb{R}^n} \geq 0$ for any $v \in \mathbb{R}^n$ and (ii) $A \geq 0$ if $A$ is Hermitian and all eigenvalues of $A$ are non-negative. Let $A \in \text{Mat}_{n \times n}(\mathbb{R})$ and assume $A > 0$ (i.e. $A$ is positive and non-singular).
Calculate
\[
\int_{x \in \mathbb{R}^n} e^{-\frac{1}{2} \langle x, Ax \rangle_{\mathbb{R}^n}} \, dx.
\]
Hint: Since $A$ is self-adjoint, it is unitarily diagonalizable with $A = O^*DO$ for some orthogonal $O$ and diagonal $D$ whose entries all strictly positive. Make a change of variable $y := Ox$ (whose Jacobian is ... what?) to factorize into $n$ independent Gaussian integrals.

4. Again assuming $A > 0$, calculate
\[
\int_{x \in \mathbb{R}^n} e^{-\frac{1}{2} \langle x, Ax \rangle + \langle v, x \rangle} \, dx
\]
for some $v \in \mathbb{R}^n$.
Hint: Complete the square on the expression $-\frac{1}{2} \langle x, Ax \rangle + \langle v, x \rangle$ and make a shift change of variable.

5. Again assuming $A > 0$, calculate
\[
\int_{x \in \mathbb{R}^n} e^{-\frac{1}{2} \langle x, Ax \rangle + i \langle v, x \rangle} \, dx
\]
for some $v \in \mathbb{R}^n$.
Hint: Diagonalize $A$ again and then calculate $n$ Fourier transforms of $n$ independent Gaussians, each of which we have already calculated using contour integration in the past.
2 Laplace asymptotics

6. The complementary error function is defined as as

\[ \text{erfc}(x) := \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} \, dt \quad (x \in \mathbb{R}). \]

Calculate \( \text{erfc}(-\infty) \) and do Laplace asymptotics for \( \text{erfc} \) as large positive \( x \).

7. [extra] Let \( K \subseteq \mathbb{R}^n \) be compact and \( f : K \to [0, \infty) \) have continuous second derivative with a unique global maximizer at \( x_0 \in \text{interior}(K) \). Prove that

\[ \lim_{p \to \infty} \| f \|_{L^p(K)} = \| f \|_{L^\infty(K)}. \]

8. [extra] Using the Stirling approximation of \( n! \) which we saw in class, prove the De Moivre–Laplace theorem, which states that the binomial distribution may be approximated via the normal distribution, in the following sense.

Let \( p \in (0, 1) \) be the probability of “heads” in an unfair coin-toss game. The Bernoulli distribution gives the probability that we find \( k = 0, \ldots, n \) heads in \( n \) independent consecutive coin-tosses, as

\[ \mathbb{P}\left[ \{ k \text{ heads} \} \right] = \binom{n}{k} p^k (1-p)^{n-k}. \]

The De Moivre–Laplace theorem says that at large \( n \) this roughly behaves like a normal distribution with mean \( np \) and variance \( np(1-p) \), i.e.,

\[ \mathbb{P}\left[ \{ k \text{ heads} \} \right] \sim \frac{1}{\sqrt{2\pi np(1-p)}} \exp \left( -\frac{(k-np)^2}{2np(1-p)} \right). \]

To derive this leading order asymptotic, do asymptotics of the binomial factor \( \binom{n}{k} \) assuming that both \( n \) is large and \( k = \alpha n \) for some fixed \( \alpha \in (0, 1) \), so you can do asymptotics of \( k! \) and \( (n-k)! \) too. Then also use

\[ \log(1+x) \sim x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \ldots. \]

9. Determine the leading order asymptotics of

\[ I(\lambda) := \int_{t=-1}^{1} \frac{\sin(t)}{t} e^{-\lambda \cosh(t)} \, dt \]

as \( \lambda \to \infty \).

10. [extra] Determine the leading order asymptotics of

\[ I(\lambda) := \int_{t=\lambda}^{\infty} \frac{e^{-t}}{t} \, dt \]

as \( \lambda \to \infty \) via integration by parts.

11. Determine the leading order asymptotics of

\[ I(\lambda) := \int_{t=0}^{\infty} e^{-\frac{1}{t} - \lambda t} \, dt \]

as \( \lambda \to \infty \). Hint: make a change of variables \( t = \frac{s}{\sqrt{\lambda}} \).

12. Determine the leading order asymptotics of the modified Bessel function

\[ I_n(x) = \frac{1}{\pi} \int_{\theta=0}^{\pi} e^{\lambda \cos(\theta)} \cos(n\theta) \, d\theta \]

for fixed \( n \in \mathbb{N} \).
3 Steepest descent asymptotics

13. Calculate
\[ \int_{x=0}^{\infty} e^{ix^2} \, dx \]
via contour deformation.

14. Calculate the leading order asymptotics of
\[ I(\lambda) := \int_{t \in \mathbb{R}} e^{i\lambda \cosh(t)} \, dt \]
as \( \lambda \to \infty \).

15. [extra] Calculate the leading order asymptotics of
\[ I(\lambda) := \int_{t=0}^{1} \log(t) e^{i\lambda t} \, dt \]
as \( \lambda \to \infty \) by contour deformation.

16. Calculate the leading order asymptotics of
\[ I(\lambda) := \int_{-\infty}^{\infty} e^{-\lambda t^2} \cos(\lambda t) f(t) \, dt \]
as \( \lambda \to \infty \) for some entire \( f \).

17. [extra] Calculate the leading order asymptotics of
\[ I(\lambda) := \int_{x \in \mathbb{R}} e^{i\lambda [\cosh(x-i\pi)-\frac{1}{2}(x-i\pi)^2]} \, dx \]
as \( \lambda \to \infty \).

18. [extra] We have for any \( n \in \mathbb{N} \) and \( k \in \mathbb{N}_{\leq n} \), the following contour integral representation of the binomial coefficient:
\[ \binom{n}{k} = \frac{1}{2\pi i} \oint_{z \in B_1(0)} \frac{(1+z)^n}{z^{k+1}} \, dz \, . \]
Indeed, this follows from Cauchy’s integral formula: with \( f(z) := (1+z)^n \) we have
\[ \frac{1}{2\pi i} \oint_{z \in B_1(0)} \frac{(1+z)^n}{z^{k+1}} \, dz = \frac{1}{k!} f^{(k)}(0) \]
\[ = \frac{1}{k!} n(n-1)\ldots(n-k+1) \]
\[ = \frac{n!}{k!(n-k)!} \]
\[ = \binom{n}{k} \, . \]

Use this contour integral representation to do leading order asymptotics of \( \binom{n}{k} \) (again, as above, but now avoiding Stirling) for large \( n \) and large \( k = \alpha n \) for some fixed \( \alpha \in (0,1) \).