Complex Analysis with Applications
Princeton University MAT330
HW10, Due May 5th 2023 (but not to be submitted)

May 3, 2023

Note: when doing asymptotic analysis, the word "fixed" means that a parameter is *not* taken to infinity and does not depend on the asymptotic parameter.

1 Gaussian integrals

1. Calculate, for some $a > 0$,
$$\int_{x\in\mathbb{R}} \mathrm{e}^{-\frac{1}{2}ax^2}\mathrm{d}x\,.$$

2. Calculate, for some $a > 0$,
$$\int_{z\in\mathbb{C}} \mathrm{e}^{-\frac{1}{2}a|z|^2}\mathrm{d}^2 z$$
(i.e., a two-dimensional surface integral throughout the complex plane).

3. Let now $n \in \mathbb{N}$. Recall that a matrix $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ is called "positive" and denoted
$$A \geq 0$$
iff $A = B^*B$ for some $B \in \mathrm{Mat}_{n\times n}(\mathbb{R})$. Equivalent and convenient conditions are: (i) $A \geq 0$ iff A is Hermitian and $\langle v, Av\rangle_{\mathbb{R}^n} \geq 0$ for any $v \in \mathbb{R}^n$ and (ii) $A \geq 0$ if A is Hermitian and all eigenvalues of A are non-negative. Let $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ and assume $A > 0$ (i.e. A is positive and non-singular).
Calculate
$$\int_{x\in\mathbb{R}^n} \mathrm{e}^{-\frac{1}{2}\langle x,Ax\rangle_{\mathbb{R}^n}}\mathrm{d}x\,.$$
Hint: Since A is self-adjoint, it is unitarily diagonalizable with $A = O^*DO$ for some orthogonal O and diagonal D whose entries all strictly positive. Make a change of variable $y := Ox$ (whose Jacobian is ... what?) to factorize into n independent Gaussian integrals.

4. Again assuming $A > 0$, calculate
$$\int_{x\in\mathbb{R}^n} \mathrm{e}^{-\frac{1}{2}\langle x,Ax\rangle+\langle v,x\rangle}\mathrm{d}x$$
for some $v \in \mathbb{R}^n$.
Hint: Complete the square on the expression $-\frac{1}{2}\langle x, Ax\rangle + \langle v, x\rangle$ and make a shift change of variable.

5. Again assuming $A > 0$, calculate
$$\int_{x\in\mathbb{R}^n} \mathrm{e}^{-\frac{1}{2}\langle x,Ax\rangle+\mathrm{i}\langle v,x\rangle}\mathrm{d}x$$
for some $v \in \mathbb{R}^n$.
Hint: Diagonalize A again and then calculate n Fourier transforms of n independent Gaussians, each of which we have already calculated using contour integration in the past.

2 Laplace asymptotics

6. The complementary error function is defined as as

$$\operatorname{erfc}(x) \quad := \quad \frac{2}{\sqrt{\pi}} \int_x^\infty \mathrm{e}^{-t^2} \mathrm{d}t \qquad (x \in \mathbb{R}) \ .$$

   Calculate $\operatorname{erfc}(-\infty)$ and do Laplace asymptotics for erfc as large positive x.

7. [extra] Let $K \subseteq \mathbb{R}^n$ be compact and $f : K \to [0, \infty)$ have continuous second derivative with a unique global maximizer at $x_0 \in \operatorname{interior}(K)$. Prove that

$$\lim_{p\to\infty} \|f\|_{L^p(K)} \quad = \quad \|f\|_{L^\infty(K)} \, .$$

8. [extra] Using the Stirling approximation of $n!$ which we saw in class, prove the De Moivre–Laplace theorem, which states that the binomial distribution may be approximated via the normal distribution, in the following sense.
   Let $p \in (0, 1)$ be the probability of "heads" in an unfair coin-toss game. The Bernoulli distribution gives the probability that we find $k = 0, \dots, n$ heads in n independent consecutive coin-tosses, as

$$\mathbb{P}[\{ \ k \text{ heads } \}] \equiv \binom{n}{k} p^k (1-p)^{n-k} \ .$$

   The De Moivre–Laplace theorem says that at large n this roughly behaves like a normal distribution with mean np and variance $np(1-p)$, i.e.,

$$\mathbb{P}[\{ \ k \text{ heads } \}] \sim \frac{1}{\sqrt{2\pi np(1-p)}} \exp\left(-\frac{(k-np)^2}{2np(1-p)} \right) \ .$$

   To derive this leading order asymptotic, do asymptotics of the binomial factor $\binom{n}{k}$ assuming that both n is large and $k = \alpha n$ for some fixed $\alpha \in (0, 1)$, so you can do asymptotics of $k!$ and $(n-k)!$ too. Then also use

$$\log(1+x) \sim x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots \ .$$

9. Determine the leading order asymptotics of

$$I(\lambda) \quad := \quad \int_{t=-1}^1 \frac{\sin(t)}{t} \mathrm{e}^{-\lambda \cosh(t)} \mathrm{d}t$$

   as $\lambda \to \infty$.

10. [extra] Determine the leading order asymptotics of

$$I(\lambda) := \int_{t=\lambda}^\infty \frac{\mathrm{e}^{-t}}{t} \mathrm{d}t$$

    as $\lambda \to \infty$ via integration by parts.

11. Determine the leading order asymptotics of

$$I(\lambda) := \int_{t=0}^\infty \mathrm{e}^{-\frac{1}{t}-\lambda t} \mathrm{d}t$$

    as $\lambda \to \infty$. Hint: make a change of variables $t = \frac{s}{\sqrt{\lambda}}$.

12. Determine the leading order asymptotics of the modified Bessel function

$$I_n(x) = \frac{1}{\pi} \int_{\theta=0}^\pi \mathrm{e}^{\lambda \cos(\theta)} \cos(n\theta) \, \mathrm{d}\theta$$

    for fixed $n \in \mathbb{N}$.

3 Steepest descent asymptotics

13. Calculate

$$\int_{x=0}^{\infty} \mathrm{e}^{\mathrm{i}x^2}\mathrm{d}x$$

via contour deformation.

14. Calculate the leading order asymptotics of

$$I(\lambda) := \int_{t\in\mathbb{R}} \mathrm{e}^{\mathrm{i}\lambda\cosh(t)}\mathrm{d}t$$

as $\lambda \to \infty$.

15. [extra] Calculate the leading order asymptotics of

$$I(\lambda) := \int_{t=0}^{1} \log(t)\,\mathrm{e}^{\mathrm{i}\lambda t}\mathrm{d}t$$

as $\lambda \to \infty$ by contour deformation.

16. Calculate the leading order asymptotics of

$$I(\lambda) := \int_{-\infty}^{\infty} \mathrm{e}^{-\lambda t^2}\cos(\lambda t)\,f(t)\,\mathrm{d}t$$

as $\lambda \to \infty$ for some entire f.

17. [extra] Calculate the leading order asymptotics of

$$I(\lambda) := \int_{x\in\mathbb{R}} \mathrm{e}^{\lambda\left[\cosh(x-\mathrm{i}\pi)-\frac{1}{2}(x-\mathrm{i}\pi)^2\right]}\mathrm{d}x$$

as $\lambda \to \infty$.

18. [extra] We have for any $n \in \mathbb{N}$ and $k \in \mathbb{N}_{\leq n}$, the following contour integral representation of the binomial coefficient:

$$\binom{n}{k} = \frac{1}{2\pi\mathrm{i}}\oint_{z\in B_1(0)} \frac{(1+z)^n}{z^{k+1}}\mathrm{d}z\,.$$

Indeed, this follows from Cauchy's integral formula: with $f(z) := (1+z)^n$ we have

$$\begin{aligned} \frac{1}{2\pi\mathrm{i}}\oint_{z\in B_1(0)} \frac{(1+z)^n}{z^{k+1}}\mathrm{d}z &= \frac{1}{k!}f^{(k)}(0) \\ &= \frac{1}{k!}n(n-1)\dots(n-k+1) \\ &= \frac{n!}{k!(n-k)!} \\ &\equiv \binom{n}{k}\,. \end{aligned}$$

Use this contour integral representation to do leading order asymptotics of $\binom{n}{k}$ (again, as above, but now avoiding Stirling) for large n and large $k = \alpha n$ for some fixed $\alpha \in (0,1)$.