MAY 2 2023

MA330 — $\mathbb{C}$-Analysis — HW10 Sample Solns

Q1

$$\int_{x\in\mathbb{R}} e^{-\frac{1}{2}ax^2}\,dx \;\overset{\substack{\frac{1}{2}ax^2 \equiv \pi y^2 \\ y \equiv \sqrt{\frac{a}{2\pi}}\,x \\ dy = \sqrt{\frac{a}{2\pi}}\,dx}}{=}\; \sqrt{\frac{2\pi}{a}}\underbrace{\int_{y\in\mathbb{R}} e^{-\pi y^2}\,dy}_{=1 \text{ by Example 6.36}} = \sqrt{\frac{2\pi}{a}}.$$

Q2

$$\int_{z\in\mathbb{C}} e^{-\frac{1}{2}a|z|^2}\,d^2z \underset{\text{polar coord.}}{=} 2\pi\int_{r=0}^{\infty} e^{-\frac{1}{2}ar^2}\,r\,dr$$

$$\overset{\substack{s:=\frac{1}{2}r^2 \\ ds = r\,dr}}{=} 2\pi\underbrace{\int_{s=0}^{\infty} e^{-as}\,ds}_{\left.\frac{e^{-as}}{-a}\right|_{s=0}^{\infty}} = \frac{2\pi}{a}.$$

Q3

$$I := \int_{x\in\mathbb{R}^n} \exp\left(-\tfrac{1}{2}\langle x, Ax\rangle\right)\,dx$$

$$A \succ 0 \Rightarrow \exists \ \mathcal{U} \in O(n) : \ \mathcal{U}^T \mathcal{U} = \mathbb{1}$$
$$A = \mathcal{U}^T D \mathcal{U}$$

w/ $D \in \mathrm{Mat}_n(\mathbb{R})$ diagonal w/ strictly positive entries.

$$\Rightarrow \quad I = \int_{x \in \mathbb{R}^n} \exp(-\tfrac{1}{2} \underbrace{\langle x, Ax \rangle}_{\underbrace{\langle x, \mathcal{U}^T D \mathcal{U} x \rangle}_{\langle \mathcal{U}x, D\, \mathcal{U}x \rangle}}) \, dx$$

$y := \mathcal{U}x$

$$\overset{\curvearrowright}{=} \int_{y \in \mathbb{R}^n} \exp(-\tfrac{1}{2} \langle y, Dy \rangle) \, |\det(\mathcal{D}(y \mapsto \mathcal{U}^T y))|$$

For any linear map L, $\mathcal{D}L = L$.

$$\Rightarrow \quad \mathcal{D}(y \mapsto \mathcal{U}^T y) = \mathcal{U}^T$$

$$\det(\mathcal{U}^T) = \det(\mathcal{U}) \underset{\substack{\uparrow \\ \mathcal{U} \in O(n)}}{=} \pm 1$$

$$\Rightarrow \quad I = \int_{y \in \mathbb{R}^n} \exp(-\tfrac{1}{2} \underbrace{\langle y, Dy \rangle}_{\sum_{i=1}^{n} y_i D_{ii} y_i}) \, dy$$

$$= \int_{y \in \mathbb{R}^n} \left(\prod_{i=1}^{n} \exp(-\tfrac{1}{2} D_{ii} y_i^2) \right) dy$$

$$= \prod_{i=1}^{n} \int_{y_i \in \mathbb{R}} \exp(-\tfrac{1}{2} D_{ii} y_i^2)\, dy_i$$

$$\boxed{Q1} \quad = \prod_{i=1}^{n} \sqrt{\frac{2\pi}{D_{ii}}}$$

$$= \frac{(2\pi)^{n/2}}{\sqrt{\prod_{i=1}^{n} D_{ii}}} = \frac{(2\pi)^{n/2}}{\sqrt{\det(A)}} .$$

$\boxed{Q4}$ $$I(v) := \int_{x \in \mathbb{R}^n} \exp(-\tfrac{1}{2}\langle x, Ax\rangle + \langle v, x\rangle)\, dx$$

Complete the square: $(A \succ 0 \Rightarrow A^T = A \;\&\; A^{-1} \exists)$

$$-\tfrac{1}{2}\langle x, Ax\rangle + \langle v, x\rangle = -\tfrac{1}{2}\left(\langle x, Ax\rangle - 2\langle A^{-1}v, Ax\rangle\right)$$

$$= -\tfrac{1}{2}\left(\langle x - A^{-1}v, A(x - A^{-1}v)\rangle - \langle A^{-1}v, AA^{-1}v\rangle\right)$$

$$= -\tfrac{1}{2}\langle x - A^{-1}v, A(x - A^{-1}v)\rangle + \tfrac{1}{2}\langle v, A^{-1}v\rangle .$$

$$\Rightarrow I(v) = \int_{x\in\mathbb{R}^n} \exp\left(-\frac{1}{2}\langle x-A^{-1}v, A(x-A^{-1}v)\rangle\right) \exp\left(\frac{1}{2}\langle v, A^{-1}v\rangle\right) dx$$

$y := x - A^{-1}v$

$$\overset{!}{=} \exp\left(\frac{1}{2}\langle v, A^{-1}v\rangle\right) \int_{y\in\mathbb{R}^n} \exp\left(-\frac{1}{2}\langle y, Ay\rangle\right) dy$$

$$= \exp\left(\frac{1}{2}\langle v, A^{-1}v\rangle\right) \frac{(2\pi)^{n/2}}{\sqrt{\det(A)}} .$$

Note: It would have been possible, but unnecessary to diagonalize A here too.

$\boxed{Q6}$ $$I(v) := \int_{x\in\mathbb{R}^n} \exp\left(-\frac{1}{2}\langle x, Ax\rangle + i\langle v, x\rangle\right) dx$$

$A = U^T D U$

$y := Ux$

$$\overset{!}{=} \int_{y\in\mathbb{R}^n} \exp\left(-\frac{1}{2}\langle y, Dy\rangle + i\langle v, U^T y\rangle\right) dy$$

$$= \prod_{i=1}^{n} \int_{y_i\in\mathbb{R}} \exp\left(-\frac{1}{2}D_{ii}\, y_i^2 + i(Uv)_i y_i\right) dy_i$$

$$\pi m_i^2 \equiv \frac{1}{2} D_{ii} y_i^2$$

$$\Rightarrow m_i = \sqrt{\frac{D_{ii}}{2\pi}}\, y_i$$

$$\overset{\circlearrowright}{=} \prod_{i=1}^{n} \sqrt{\frac{2\pi}{D_{ii}}} \int_{m_i \in \mathbb{R}} dm_i \exp\left(-\pi m_i^2 - 2\pi i \frac{-(\mathcal{U}v)_i}{2\pi} \sqrt{\frac{2\pi}{D_{ii}}}\, m_i\right)$$

Example 6.36

$$\overset{\circlearrowright}{=} \prod_{i=1}^{n} \sqrt{\frac{2\pi}{D_{ii}}} \exp\left(-\cancel{\pi} \underbrace{\left(\frac{(\mathcal{U}v)_i}{2\pi} \sqrt{\frac{2\pi}{D_{ii}}}\right)^2}_{\frac{(\mathcal{U}v)_i^2}{\cancel{4\pi^2}\,2} \frac{\cancel{2\pi}}{D_{ii}}}\right)$$

$$= \frac{(2\pi)^{n/2}}{\sqrt{\det(A)}} \exp\left(-\frac{1}{2} \underbrace{\sum_{i=1}^{n} \underbrace{\frac{1}{D_{ii}} (\mathcal{U}v)_i^2}_{(\mathcal{U}v)_i D_{ii}^{-1} (\mathcal{U}v)_i}}_{\langle \mathcal{U}v, D^{-1}\mathcal{U}v \rangle =}\right)$$

$$= \langle v, \mathcal{U}^T D^{-1} \mathcal{U} v \rangle$$

$$= \langle v, A^{-1} v \rangle$$

$$= \frac{(2\pi)^{n/2}}{\sqrt{\det(A)}} \exp\left(-\frac{1}{2} \langle v, A^{-1} v \rangle\right).$$

$\boxed{Q6}$ See Example 10.4.

$\boxed{Q7}$ Let $K \in \mathrm{Cpt}(\mathbb{R}^n)$ and $f : K \to [0,\infty)$

be twice cont. diff. s.t. $\exists!\, x_0 \in \mathrm{int}(K)$

s.t. x_0 is a global max for f; $f(x_0) \neq 0$.

<u>Claim</u>: $\lim_{p \to \infty} \|f\|_{L^p(K)} = \|f\|_{L^\infty(K)}$

<u>Proof</u>: $\|f\|^p_{L^p(K)} = \int_{x \in K} |f(x)|^p \, dx$

$$\overset{f \geq 0}{=} \int_{x \in K} f(x)^p \, dx$$

$$= \underbrace{\int_{x \in K} \exp(-p[-\log(f(x))]) \, dx}_{\text{do Laplace asymptotics of this w.r.t. } p \to \infty.}$$

$$g(x) := -\log(f(x)) \qquad (x \in K)$$

$$(\nabla g)(x) = -\frac{1}{f(x)} (\nabla f)(x) \overset{!}{=} 0 \underset{x_0 \text{ global max}}{\Rightarrow} x = x_0.$$

$$(\mathbb{H} g)(x) = -\frac{1}{f(x)} (\mathbb{H} f)(x) + \frac{1}{f(x)^2} (\nabla f)(x) \otimes (\nabla f)(x)^*$$

Since x_0 is a max for f,

$$(\mathbb{H}f)(x_0) < 0.$$

Also, $(\nabla f)(x) \otimes (\nabla f)(x)^* \geq 0$ always!

$$\Rightarrow \quad (\mathbb{H}g)(x_0) > 0.$$

Hence by Thm 10.1,

$$\lim_{p\to\infty} \frac{\|f\|^p_{L^p(K)}}{\frac{(2\pi)^{n/2}}{\sqrt{\det((\mathbb{H}g)(x_0))}} \exp(-p\,g(x_0))} = 1.$$

$$\exp(-p\,g(x_0)) = f(x_0)^p.$$

$$\Rightarrow \lim_{p\to\infty} \sqrt[p]{\frac{\|f\|^p_{L^p(K)}}{\frac{(2\pi)^{n/2}}{\sqrt{\det((\mathbb{H}g)(x_0))}} \exp(-p\,g(x_0))}} = 1 \quad \text{as } \sqrt[p]{\cdot} \text{ is cont.}$$

But

$$\sqrt[p]{\frac{\|f\|^p_{L^p(K)}}{\frac{(2\pi)^{n/2}}{\sqrt{\det((\mathbb{H}g)(x_0))}} \exp(-p\,g(x_0))}} = \frac{\|f\|_{L^p(K)}}{f(x_0)} \sqrt[p]{\frac{\sqrt{\det((\mathbb{H}g)(x_0))}}{(2\pi)^{n/2}}}$$

$$\sqrt[p]{\alpha} \equiv \alpha^{1/p} = \exp\left(\frac{1}{p}\log(\alpha)\right) \xrightarrow{p\to\infty} 1.$$

We learn $\boxed{\lim_{p\to\infty} \|f\|_{L^p(K)} = f(x_0).}$

But $f(x_0) = \|f\|_{L^\infty(K)}$ as x_0 is a max. $\blacksquare$

$\boxed{Q8}$ Let $p \in (0,1)$.

Claim: $\binom{n}{k} p^k (1-p)^{n-k} \approx \dfrac{\exp\left(-\frac{(k-np)^2}{2np(1-p)}\right)}{\sqrt{2\pi np(1-p)}}$

if $k, n \to \infty$ s.t. $\frac{k}{n} \to p$.

Proof: By Example 10.2,

$$n! = \Gamma(n+1) \approx \sqrt{\frac{2\pi}{n+1}} \exp\left((n+1)\left[\log(n+1) - 1\right]\right).$$

$$= \sqrt{\frac{2\pi}{n+1}} (n+1)^{n+1} e^{-(n+1)}$$

$$= \sqrt{2\pi} (n+1)^{n+\frac{1}{2}} e^{-(n+1)}$$

$$= \sqrt{2\pi(n+1)} (n+1)^n e^{-(n+1)}$$

$$\approx \sqrt{2\pi n}\, n^n e^{-n}$$

$\binom{n}{k} p^k (1-p)^{n-k} = \dfrac{n!}{k!\,(n-k)!}$ [TODO] . . .

$\boxed{Q9}$

$$I(\lambda) := \int_{t=-1}^{1} \frac{\sin(t)}{t} e^{-\lambda \cosh(t)} dt$$

$\cosh(t)$ has min @ $t=0$:

$$\cosh'(t) = \sinh(t) \overset{!}{=} 0 \Rightarrow t=0$$

$$\cosh''(t) = \cosh(t) \xrightarrow{t\to 0} 1 > 0 \Rightarrow \text{min.}$$

Also $\frac{\sin(t)}{t} \xrightarrow{t\to 0} 1$.

So we may apply Laplace to get:

$$I(\lambda) \sim \sqrt{\frac{2\pi}{\lambda}} \exp(-\lambda).$$

$\boxed{Q10}$

$$I(\lambda) := \int_{t=\lambda}^{\infty} \frac{e^{-t}}{t} dt$$

$$s := \frac{t}{\lambda} \Rightarrow$$

$$I(\lambda) = \int_{s=1}^{\infty} \frac{e^{-\lambda s}}{\lambda s} \lambda ds$$

$$= \int_{s=1}^{\infty} \frac{e^{-\lambda s}}{s} ds$$

max @ bdry. $s=1$!

$\int_0$ can't use Laplace directly.

Instead, use IBP:

$$e^{-\lambda s} = -\frac{1}{\lambda}\partial_s e^{-\lambda s}$$

$$\Rightarrow \int_{s=1}^{\infty} \frac{e^{-\lambda s}}{s} ds = \int_{s=1}^{\infty} \frac{1}{s}\left(-\frac{1}{\lambda}\right)\partial_s e^{-\lambda s} ds$$

$$= \underbrace{\frac{1}{\lambda s} e^{-\lambda s}\Big|_{s=1}^{\infty}}_{\frac{1}{\lambda}e^{-\lambda}} + \underbrace{\frac{1}{\lambda}\int_{s=1}^{\infty}\frac{1}{s^2}\underbrace{e^{-\lambda s}}_{-\frac{1}{\lambda}\partial_s e^{-\lambda s}} ds}_{\lesssim \frac{1}{\lambda^2}e^{-\lambda}}$$

$$\Rightarrow I(\lambda) \lesssim \frac{e^{-\lambda}}{\lambda}.$$

$\boxed{Q11}$

$$I(\lambda) := \int_{t=0}^{\infty} \exp\left(-\frac{1}{t}-\lambda t\right)dt$$

$s := \sqrt{\lambda}\, t$

$$\overset{?}{=} \int_{s=0}^{\infty} \exp\left(-\frac{\sqrt{\lambda}}{s}-\sqrt{\lambda}\, s\right)\frac{1}{\sqrt{\lambda}} ds$$

$$= \frac{1}{\sqrt{\lambda}} \int_{s=0}^{\infty} \exp\Big(-\sqrt{\lambda}\big(\underbrace{\tfrac{1}{s}+s}_{f(s)}\big)\Big)\, ds$$

$$f'(s) = 1 - \frac{1}{s^2} \overset{!}{=} 0 \Rightarrow s = 1$$

$$f''(s) = +2\frac{1}{s^3} \xrightarrow{s\to 1} +2 < 0 \Rightarrow \underline{\text{min}}$$

$\Longrightarrow$ Laplace asym. says

$$\int_{s=0}^{\infty} e^{-\sqrt{\lambda} f(s)}\, ds \approx \sqrt{\frac{2\pi}{\sqrt{\lambda}\, 2}}\, \exp(-\sqrt{\lambda}\, 2).$$

$$\Longrightarrow \boxed{I(\lambda) \approx \sqrt{\frac{\pi}{\lambda}}\, \exp(-2\sqrt{\lambda})} .$$

$\boxed{Q12}$

$$I_n(\lambda) := \frac{1}{\pi} \int_{\theta=0}^{\pi} e^{\lambda \cos(\theta)} \cos(n\theta)\, d\theta$$

$$= \frac{1}{2\pi} \int_{\theta=-\pi}^{\pi} e^{-\lambda[-\cos(\theta)]} \cos(n\theta)\, d\theta$$

$$f(\theta) := -\cos(\theta)$$

$$f'(\theta) = \sin(\theta) \overset{!}{=} 0 \Rightarrow \theta = 0, \pm\pi, \ldots$$

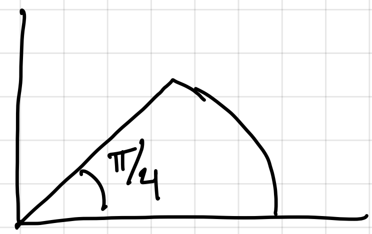
$$f''(\theta) = \cos(\theta), \quad \underbrace{f''(0) = 1}_{\text{min}}, \quad \underbrace{f''(\pm\pi) = -1}_{\text{max}}$$

$\Rightarrow$ Laplace says

$$I_n(\lambda) \sim \frac{1}{2\pi}\sqrt{\frac{2\pi}{\lambda}}\, e^{\lambda} = \frac{e^{\lambda}}{\sqrt{2\pi\lambda}}.$$

$\boxed{\text{Q13}}$

$$\int_{x=0}^{\infty} e^{ix^2}\, dx$$



On arc:

$$\left|\int_{\theta=0}^{\pi/4} e^{i(Re^{i\theta})^2} Re^{i\theta}\, i\, d\theta\right|$$

$$\leq \left|\int_{\theta=0}^{\pi/4} e^{iR^2\cos(2\theta)} e^{-R^2\sin(2\theta)} R e^{i\theta}\, i\, d\theta\right|$$

$$\leq R \underbrace{\int_{\theta=0}^{\pi/4} e^{-R^2\sin(2\theta)}\, d\theta}_{\sim \frac{1}{R^2} \text{ by Lemma A.3}} \lesssim \frac{1}{R} \xrightarrow{R\to 0} 0$$

So we may close the contour to get

$$\int_{x=0}^{\infty} e^{ix^2}\,dx = \int e^{iz^2}\,dz$$

$z = e^{i\frac{\pi}{4}}t$

$$= \int_{t=0}^{\infty} e^{i\left(e^{i\frac{\pi}{4}}t\right)^2} e^{i\frac{\pi}{4}}\,dt$$

$e^{i\frac{\pi}{2}} = i$

$$= e^{i\frac{\pi}{4}} \underbrace{\int_{t=0}^{\infty} e^{-t^2}\,dt}_{\sqrt{\pi}/2}$$

$$= \frac{\sqrt{\pi}}{2} e^{i\frac{\pi}{4}}.$$

$\boxed{Q14}$

$$I(\lambda) := \int_{t\in\mathbb{R}} \exp(i\lambda\cosh(t))\,dt.$$

$$f(z) := -i\cosh(z) \qquad f''(z) = -i\cosh(z)$$

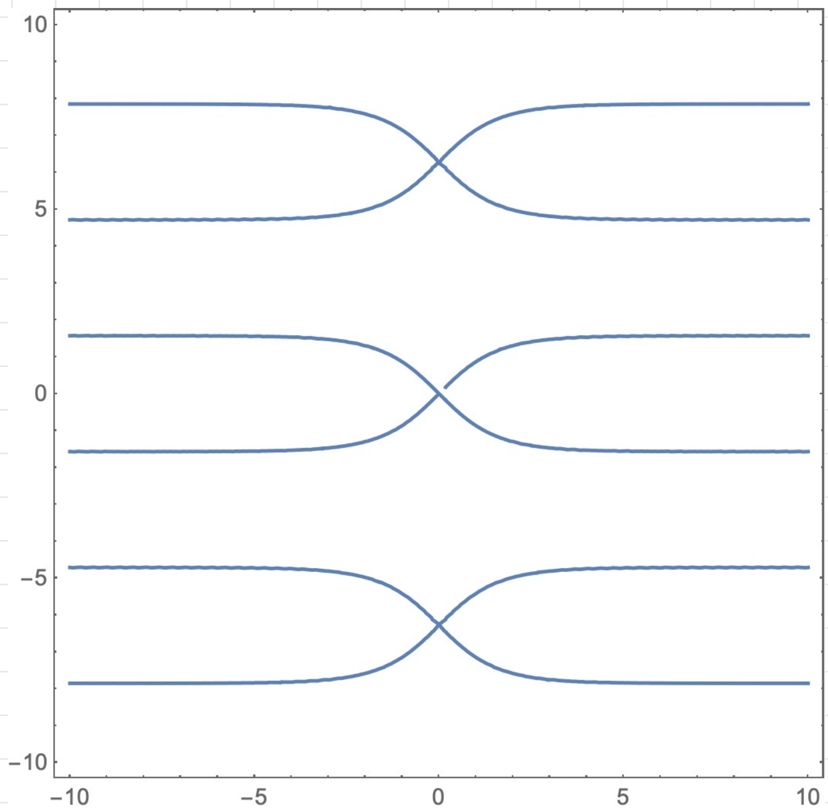
$$f'(z) = -i\sinh(z) \overset{!}{=} 0 \Rightarrow z_* = 0.$$

$$\operatorname{Im}\{f(z)\} = \operatorname{Im}\{-i\cosh(x+iy)\} = -\cos(y)\cosh(x)$$

$$\overset{!}{=} \operatorname{Im}\{f(0)\} = -1.$$

So we're looking for a path (x,y) with an implicit f^n

$$\cos(y)\cosh(x) = 1$$



Defines a family of contours, but only two pass through the origin.

To make the contour def. need vertical legs to decay:

@ $x = \pm\infty$, $\quad y = \lim_{x\to\infty} \arccos\left(\frac{1}{\cosh(x)}\right) = \pm\frac{\pi}{2}$.

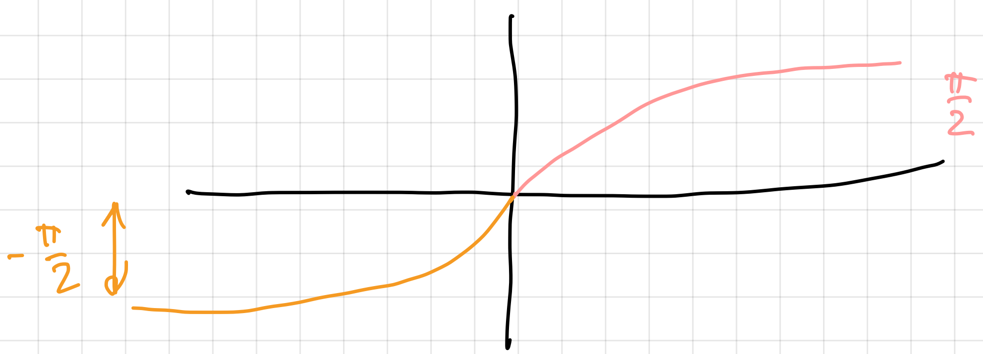
$$\left| \int_{y=0}^{\pi/2} \exp(i\lambda \cosh(R+iy))\, i\,dy \right| \le$$

$$\le \int_{y=0}^{\pi/2} \exp(-\lambda \sin(y) \sinh(R))\, dy$$

$$\lesssim \frac{1}{\sinh(R)} \quad \text{if} \quad R > 0.$$

So we must go up if $R \geq 0$

down $R < 0$.



So we get by Thm 10.7,

$$I(\lambda) \approx \sqrt{\frac{2\pi}{\lambda(-i)}}\, \exp(i\lambda).$$

$\boxed{Q15}$

$$I(\lambda) := \int_{t=0}^{1} \log(t)\, e^{i\lambda t}\, dt$$

$x := \log(t)$

$dx = \frac{1}{t} dt$

$$\overset{?}{=} \int_{x=-\infty}^{0} x\, e^{i\lambda e^{x}}\, e^{x}\, dx$$

$dt = e^{x} dx$

$y := \frac{x}{\log(\lambda)}$

$$\overset{?}{=} \log(\lambda)^2 \int_{y=-\infty}^{0} y\, e^{i\lambda e^{\log(\lambda) y} + \log(\lambda) y}\, dy$$

TODO

$\boxed{Q16}$

$$I(\lambda) := \int_{-\infty}^{\infty} e^{-\lambda t^2} \cos(\lambda t)\, dt$$

$$= \operatorname{Re}\left\{ \int_{t=-\infty}^{\infty} e^{-\lambda t^2 + i\lambda t}\, dt \right\}$$

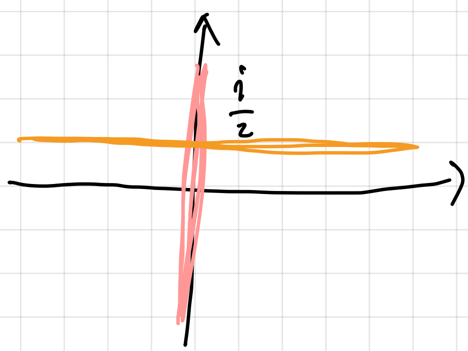
$$f(z) := t^2 - it \qquad f''(t) = 2$$

$$f'(t) = 2t - i \overset{!}{=} 0 \Rightarrow \boxed{t = \frac{i}{2}}.$$

$$\operatorname{Im}\{f(x+iy)\} = -x + 2xy \overset{!}{=} \operatorname{Im}\{f(\tfrac{i}{2})\} = 0$$

$\Leftrightarrow$ Implicit curve $x = 2xy$

$\Leftrightarrow$ $x=0$ or $y = \frac{1}{2}$



Probably deform to orange contour:

Vertical legs:

$$\left| \int_{y=0}^{1/2} \exp(-\lambda(R+iy)^2 + i\lambda(R+iy))\, i\,dy \right|$$

$$\leq \int_{y=0}^{1/2} \exp(-\lambda(R^2+y-y^2))\,dy \xrightarrow{R\to\infty} 0 \quad ✓$$

So Thm 10.7 says:

$$\int_{t=-\infty}^{\infty} e^{-\lambda t^2 + i\lambda t}\,dt \approx \sqrt{\frac{\cancel{2}\pi}{\lambda \cancel{2}}}\exp\left(-\frac{\lambda}{4}\right).$$

$\boxed{Q17}$

$$I(\lambda) := \int_{x\in\mathbb{R}} \exp\left(\lambda\left[\cosh(x-i\pi) - \tfrac{1}{2}(t-i\pi)^2\right]\right)dx$$

TODO

$\boxed{Q18}$

$$\binom{n}{k} = \frac{1}{2\pi i}\oint_{\partial B_1(0)} \frac{(1+z)^n}{z^{k+1}}\,dz$$

$n\to\infty$, $k\to\infty$ but $\mu := \frac{k}{n}$ fixed and $\mu<1$.

$$\frac{(1+z)^n}{z^{k+1}} = \frac{1}{z}\exp\Big(-n\underbrace{\Big(\log\big(\tfrac{1}{1+z}\big) + \mu\log(z)\Big)}_{=:f(z)}\Big)$$

$$f'(z) = -\frac{1}{1+z} + \mu \frac{1}{z} \stackrel{!}{=} 0$$

$$\Rightarrow z_* = \frac{\mu}{1-\mu} > 0 \quad \text{as } \mu < 1.$$

$$f''(z) = \frac{1}{(1+z)^2} - \frac{\mu}{z^2}$$

$$f''(z_*) = -\frac{(1-\mu)^3}{\mu}.$$

$$f(z_*) = \log(1-\mu) + \mu \log\left(\frac{\mu}{1-\mu}\right) > 0$$

Want $\operatorname{Im}\{f(x+iy)\} = 0$ then:

TODO.