FEB 4 2023

MAT 330 – HW 1 – Sample Solutions

1. (a) $z = (1-i)^3$. Its real and imaginary parts are obtained by expanding $(a+b)^3 = a^3+b^3+3[a^2b+ab^2]$:

$$z = \underbrace{1^3}_{1} + \underbrace{\underbrace{(-i)^3}_{(-1)(-i)}}_{i} + \underbrace{3\cdot 1\cdot \underbrace{(-i)^2}_{-1}}_{-3} + \underbrace{3\cdot 1^2\cdot(-i)}_{-3i}$$

$$= 1-3+i(1-3) = -2-2i.$$

Hence $\mathrm{Re}\{z\} = -2$ and $\mathrm{Im}\{z\} = -2$.

(b) $$z = \frac{x+iy}{a+ib} = \frac{x+iy}{a+ib}\cdot\underbrace{\frac{a-ib}{a-ib}}_{1} = \frac{xa+yb+i(ya-xb)}{a^2+b^2}.$$

Hence $\mathrm{Re}\{z\} = \dfrac{xa+yb}{a^2+b^2}$ and

$$\mathrm{Im}\{z\} = \frac{ya-xb}{a^2+b^2}.$$

(c) $z = \dfrac{\pi i}{(1-i)(2-i)}$ is given.

Its denom. is $(1-i)(2-i) = 2-1-i-2i = +1-3i$.

Hence $$z = \frac{\pi i}{+1-3i} = \frac{\pi i}{+1-3i}\cdot\frac{+1+3i}{+1+3i} = \frac{\pi i(1+3i)}{1^2+3^2} =$$

$$= \frac{\pi i - 3\pi}{10} = \frac{-3\pi}{10} + \frac{\pi}{10} i .$$

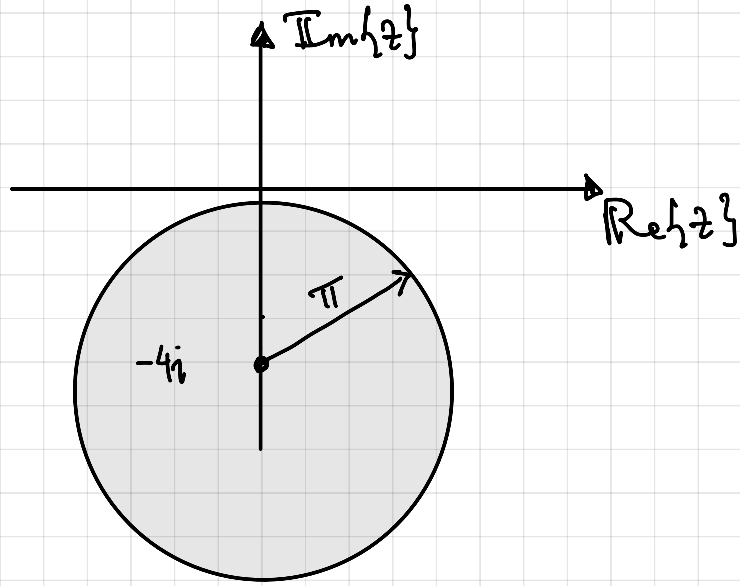
We find $\mathrm{Re}\{z\} = \frac{-3\pi}{10}$ and $\mathrm{Im}\{z\} = \frac{\pi}{10}$.

2. (a) We want to sketch the set

$$S := \{ z \in \mathbb{C} \mid |z + 4i| \leq \pi \} .$$

We interpret $|z+4i| \leq \pi$ as z being distance at most π away from $-4i$.

I.e. it is a <u>closed</u> ball of radius π around $-4i$.



Set includes boundary circle.

(b) Now we have the set

$$S = \{ z \in \mathbb{C} \mid |z+2| < |z+1| \} .$$

We re-write the constraint as follows:

$$|z+2| < |z+1|$$

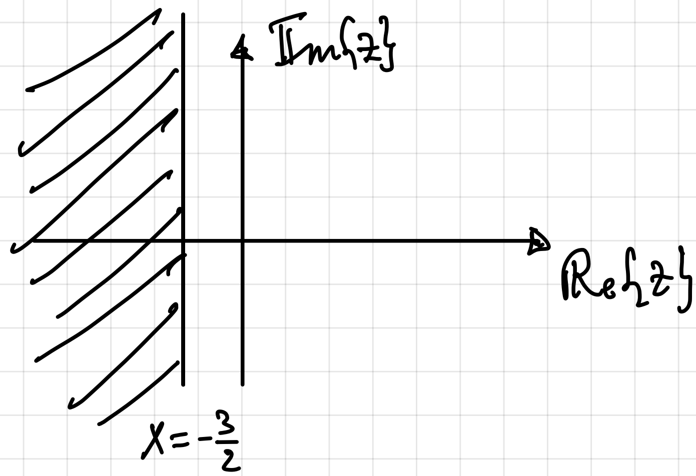
$$\iff \quad |z+2|^2 < |z+1|^2 \qquad |z+w|^2 = |z|^2 + |w|^2 + 2\,\mathrm{Re}\{z\bar{w}\}$$

$$\cancel{|z|^2} + 4 + 4\,\mathrm{Re}\{z\} < \cancel{|z|^2} + 1 + 2\,\mathrm{Re}\{z\}$$

$$3 + 2\,\mathrm{Re}\{z\} < 0$$

$$\mathrm{Re}\{z\} < -3/2 .$$

We find a half-plane to the left of the vertical line $x = -\frac{3}{2}$, not including that boundary line.



3. We seek the minimal const. $c > 0$ s.t.

$$c|z| \geq |\mathrm{Re}\{z\}| + |\mathrm{Im}\{z\}| .$$

Rewrite this using $z = x + iy$ to get:

$$c\sqrt{x^2+y^2} \geqslant |x| + |y| .$$

Square the ineq. (both sides are positive) to get:

$$c^2(x^2+y^2) \geq (|x|+|y|)^2 = x^2+y^2+2|x||y|$$

$$(c^2-1)(x^2+y^2) \geq 2|x||y| .$$

Now complete the square:

$$x^2+y^2 = |x|^2+|y|^2 = (|x|-|y|)^2 + 2|x||y|$$

to get the equivalent ineq.

$$(c^2-1)\left[(|x|-|y|)^2 + 2|x||y|\right] \geq 2|x||y| .$$

For what c is this <u>always</u> true (i.e., $\forall x,y \in \mathbb{R}$)?

1. Clearly $c>1$ is necessary.

2. $(|x|-|y|)^2$ is always positive and by choosing x,y appropriately can be made arbitrarily small, so we can't "rely" on it.
   In other words, we may ask instead when
   $$2(c^2-1)|x||y| \geq 2|x||y| .$$

3. WLOG $|x||y| \neq 0$ so we get
   $$c^2-1 \geq 1$$
   $$\Leftrightarrow \quad c^2 \geq 2$$
   $$\Leftrightarrow \quad \boxed{c \geq \sqrt{2}} .$$

4. (a) W.T.S. $\displaystyle\sum_{j=0}^{n} z^j = \frac{1-z^{n+1}}{1-z}$ $(z \in \mathbb{C} \setminus \{1\})$.

We start from $\displaystyle S_n(z) := \sum_{j=0}^{n} z^j$.

Calculate
$$z S_n(z) = z \sum_{j=0}^{n} z^j$$
$$= \sum_{j=0}^{n} z^{j+1}$$
$$= \sum_{j=1}^{n+1} z^j = -1 + \sum_{j=0}^{n+1} z^j = -1 + S_n(z) + z^{n+1}.$$

Hence $(1-z) S_n(z) = 1 - z^{n+1}$. Since $z \neq 1$ this implies the result.

(b) W.T.S $\displaystyle\sum_{j=0}^{n} \cos(j\theta) = \frac{1}{2} + \frac{\sin\left(\frac{2n+1}{2}\theta\right)}{2\sin(\theta/2)}$.

We use the following "trick":
$$\sum_{j=0}^{n} \cos(j\theta) = \sum_{j=0}^{n} \mathrm{Re}\{e^{ij\theta}\}$$
$\mathrm{Re}\{z+w\} = \mathrm{Re}\{z\} + \mathrm{Re}\{w\}$
$$\overset{\uparrow}{=} \mathrm{Re}\left\{\sum_{j=0}^{n} e^{ij\theta}\right\}$$
$$= \mathrm{Re}\left\{\sum_{j=0}^{n} (e^{i\theta})^j\right\}$$

part (a)

Ok as $\theta \in 2\pi\mathbb{Z}$ so $e^{i\theta} \neq 1$. $= \mathrm{Re}\left\{ \dfrac{1-e^{i(n+1)\theta}}{1-e^{i\theta}} \right\}$.

$$= \mathrm{Re}\left\{ \frac{e^{i(n+1)\theta}-1}{e^{i\theta}-1} \cdot \frac{e^{-i\theta/2}}{e^{-i\theta/2}} \right\}$$

$$= \mathrm{Re}\left\{ \frac{e^{i(n+1)\theta - i\theta/2} - e^{-i\theta/2}}{e^{i\theta/2}-e^{-i\theta/2}} \right\}$$

$$= \mathrm{Re}\left\{ \frac{e^{i(n+\frac{1}{2})\theta} - e^{-i\theta/2}}{2i\sin(\theta/2)} \right\}$$

$$= \frac{1}{2\sin(\theta/2)} \mathrm{Im}\left\{ e^{i(n+\frac{1}{2})\theta} - e^{-i\theta/2} \right\}$$

$$= \frac{1}{2\sin(\theta/2)} \left[\sin((n+\tfrac{1}{2})\theta) + \sin(\theta/2) \right]$$

$$= \frac{1}{2}\left[1 + \frac{\sin(\frac{2n+1}{2}\theta)}{\sin(\theta/2)} \right]. \quad \blacksquare$$

5. We seek the zeros of the eq-n $z^5+32=0$. Noting $2^5=32$ we get

$$z^5 = -2^5 \iff \left(\frac{z}{2}\right)^5 = -1.$$

$$\iff \frac{z}{2} = \sqrt[5]{-1}.$$

$$\Leftrightarrow \quad z = 2\sqrt[5]{-1}.$$

The problem with this is $\sqrt[5]{\cdot}$ is a multi-valued function. One way to understand this is via the polar form:

$$\sqrt[5]{z} = \sqrt[5]{re^{i\theta}}$$
$$= \sqrt[5]{r}\, e^{i\theta/5}.$$

Since $\theta \in \mathbb{R}$ has a $2\pi\mathbb{Z}$ ambiguity, once we divide by 5 we get 5 possibilities:

$$\tilde{\theta} = \theta + 2\pi n \, ; \quad \theta \in [0, 2\pi) \, , \; n \in \mathbb{Z}$$
$$\frac{\tilde{\theta}}{5} = \frac{\theta}{5} + \frac{2\pi}{5} n.$$

So the five possibilities for the root are:

$$\left\{ \frac{\theta}{5}, \frac{\theta}{5} + \frac{2\pi}{5}, \frac{\theta}{5} + \frac{2\pi}{5}\cdot 2, \frac{\theta}{5} + \frac{2\pi}{5}\cdot 3, \frac{\theta}{5} + \frac{2\pi}{5}\cdot 4 \right\}.$$

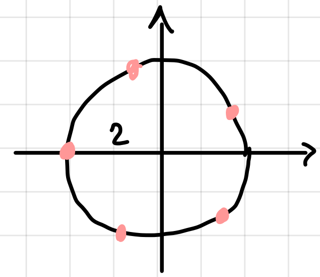
We need $\sqrt[5]{-1}$ so we have $-1 = 1 \cdot e^{i\pi}$,

i.e., $\theta = \pi$ and so

$$\sqrt[5]{-1} = \left\{ \frac{\pi}{5}, \frac{3\pi}{5}, \frac{5\pi}{5}, \frac{7\pi}{5}, \frac{9\pi}{5} \right\}$$

Hence we get the five sol-ns of the eq-n:

$$z \in 2\cdot\left\{e^{i\frac{\pi}{5}}, e^{i\frac{3\pi}{5}}, \dots, e^{i\frac{9\pi}{5}}\right\}.$$



6. **Claim:** $$|z| = \max_{\theta\in[-\pi,\pi]} \operatorname{Re}\{z e^{i\theta}\}.$$

Proof: Write $z = re^{i\alpha}$ so we W.T.S.

$$|re^{i\alpha}| = \max_{\theta\in[-\pi,\pi]} \operatorname{Re}\{re^{i(\alpha+\theta)}\}.$$

Since $r>0$ and indep. of θ, this is equiv. to

$$1 = \max_{\theta\in[-\pi,\pi]} \operatorname{Re}\{e^{i(\alpha+\theta)}\}$$

$$= \max_{\theta\in[-\pi,\pi]} \cos(\alpha+\theta)$$

which is a manifestly correct statement. ∎

Claim: $|z+w| \leq |z| + |w|$

Proof: Using the above,

$$|z+w| = \max_{\theta} \operatorname{Re}\{(z+w)e^{i\theta}\}$$

$$= \max_\theta \operatorname{Re}\{ze^{i\theta} + we^{i\theta}\}$$

$$= \max_\theta \left(\operatorname{Re}\{ze^{i\theta}\} + \operatorname{Re}\{we^{i\theta}\} \right).$$

<u>FACT</u>: $\max_\theta [f(\theta) + g(\theta)] \leq \left(\max_\theta f(\theta)\right) + \left(\max_\theta g(\theta)\right)$

(Prove it...)

Using this fact,

$$|z+w| \leq \left(\max_\theta \operatorname{Re}\{ze^{i\theta}\}\right) + \left(\max_\theta \operatorname{Re}\{we^{i\theta}\}\right)$$

$$\overset{\text{part (a)}}{=} |z| + |w|.$$

∎

8. Looking for accum. pts. of

$$S = \{ (2i)^n \mid n \in \mathbb{N}_{\geq 1} \} \subseteq \mathbb{C}.$$

Recall $i^n = \begin{cases} 1 & n \in 4\mathbb{Z} \\ i & n \in 4\mathbb{Z}+1 \\ -1 & n \in 4\mathbb{Z}+2 \\ -i & n \in 4\mathbb{Z}+3 \end{cases}$

whereas $2^n \longrightarrow \infty$. So this set has no accum. pts. since its elements become increasingly scattered.

9. (1) Define $S_1 = \mathbb{R} \subseteq \mathbb{C}$.

FACT: $\mathbb{R}^c \equiv \mathbb{C} \setminus \mathbb{R} \in \mathrm{Open}(\mathbb{C})$
(e.g. via open ball def.)

Hence $\mathbb{R} \in \mathrm{Closed}(\mathbb{C})$.

Thus: ⊛ $\mathrm{clo}(\mathbb{R}) = \mathbb{R}$ (closure of closed set is the set itself).

⊛ $\mathrm{int}(\mathbb{R}) = \emptyset$ bcs. $\nexists$ open balls within $\mathbb{R}$.

⊛ $\partial\mathbb{R} \equiv \mathrm{clo}(\mathbb{R}) \setminus \mathrm{int}(\mathbb{R}) = \mathbb{R}$.

$\Rightarrow$ $\mathbb{R}$ is closed in $\mathbb{C}$.

(2) $S_2 = B_{1/2}(1)$

⊛ $\mathrm{clo}(B_{1/2}(1)) = \{ z \in \mathbb{C} \mid |z-1| \leq \frac{1}{2} \}$

i.e. a closed ball including its boundary circle.

⊛ $\text{int}(B_{1/2}(1)) = B_{1/2}(1)$ since $B_{1/2}(1) \in \text{Open}(\mathbb{C})$.

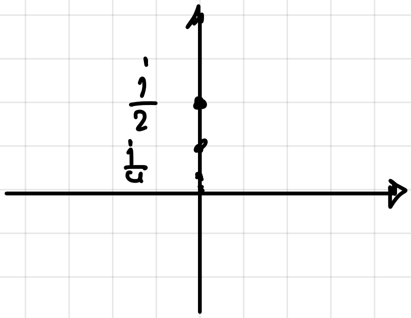
⊛ $\partial B_{1/2}(1) = \text{clo}(B_{1/2}(1)) \setminus \text{int}(B_{1/2}(1))$

$$= \{ z \in \mathbb{C} \mid |z-1| = \tfrac{1}{2} \}$$

= boundary circle.

$B_{1/2}(1)$ is open in $\mathbb{C}$.

(3) $S_3 = \left\{ \frac{i}{2} \mid n \in \mathbb{N}_{\geq 1} \right\}$



⊛ $\text{clo}(S_3) = S_3 \cup \{0\}$ using the characterization that a closed set contains all its limit points, which 0 is for S_3.

⊛ $\text{int}(S_3) = \emptyset$ since S_3 contains no open balls.

⊛ $\partial S_3 \equiv \text{clo}(S_3) \setminus \text{int}(S_3) = S_3 \cup \{0\}$.

Hence S_3 is not closed nor open.

(4) $S_4 = \{x+iy \in \mathbb{C} \mid x, y \in \mathbb{Q}\}$ rational #'s

Claim: $clo(S_4) = \mathbb{C}$

Proof: Any pts. $z \in \mathbb{C}$ is a limit pt. of S_4.

Claim: $int(S_4) = \emptyset$

Proof: S_4 contains no open balls!

$$\Rightarrow \quad \partial S_4 = clo(S_4) \setminus int(S_4) = \mathbb{C}.$$

S_4 is not closed neither open.

10. We're given $f, g: \mathbb{C} \to \mathbb{C}$ and $z_0 \in \mathbb{C}$, s.t.

$$\lim_{z \to z_0} g(z) = g(z_0), \quad \lim_{w \to g(z_0)} f(w) = f(g(z_0)) \quad \exists.$$

Let $\delta_f, \delta_g > 0$ be the relevant δ's from the existence of these limits resp.

W.T.S. $\lim_{z \to z_0} f(g(z)) = f(g(z_0)) \quad \exists.$

I.e. want $\delta_*(\varepsilon) > 0$: if $z \in B_{\delta_*(\varepsilon)}(z_0)$

then $f(g(z)) \in B_\varepsilon(f(g(z_0)))$.

Pick $\boxed{\delta_*(\varepsilon) := \delta_g(\delta_f(\varepsilon))}$.

Then, if $|z - z_0| < \delta_*(\varepsilon) = \delta_g(\delta_f(\varepsilon))$

$$\Downarrow \lim g \ \exists$$

$$|g(z) - g(z_0)| < \delta_f(\varepsilon)$$

$$\Downarrow \lim f \ \exists$$

$$|f(g(z)) - f(g(z_0))| < \varepsilon.$$

$\blacksquare$

11. (a) Consider $B_1(0) \ni z \xmapsto{f} z^2 \in \mathbb{C}$.

We seek a modulus of cont. for it.

Start backwards:

$$\begin{aligned} |f(z) - f(w)| &= |z^2 - w^2| \\ &= |z-w||z+w| \\ &\leq |z||z-w| + |w||z-w| \\ &\overset{z,w \in B_1(0)}{\leq} 2|z-w|. \end{aligned}$$

Hence $w(\alpha) := 2\alpha$ will do the job.

(b) Now we have

$$B_2(-3) \ni z \xmapsto{g} \exp(z) \in \mathbb{C}$$

Since $B_2(-3) \subseteq B_5(0)$, we'll work w/ $|z| < 5$ instead.

Following the same procedure, we have

$$|g(z) - g(w)| = |e^z - e^w|$$
$$= |e^z||1 - e^{w-z}|$$
$$\leq e^{|z|} |1 - e^{w-z}|$$
$$\overset{|z|\leq 5}{\leq} e^5 |1 - e^{w-z}|$$

Now, $|1 - e^{a+ib}|^2 = |1 - e^a \cos(b) - ie^a \sin(b)|^2$ $a, b \in \mathbb{R}$

$$= (1 - e^a \cos(b))^2 + e^{2a} \sin(b)^2$$
$$= 1 + e^{2a} - 2e^a \cos(b)$$
$$= 1 + e^{2a} - 2e^a + 2e^a [1 - \cos(b)]$$
$$= (1 - e^a)^2 + 4e^a \sin(\tfrac{b}{2})^2.$$

From the mean value thm. on $\exp : \mathbb{R} \to \mathbb{R}$

We have $\underbrace{e^a - 1}_{e^0} = (a-0)\, e^c \quad \exists\ c \in [-a,a].$

$$\implies \quad |1 - e^a| \leqslant |a|\, e^{|c|} \leqslant |a| e^{|a|}.$$

For the sine, we have $|\sin(\alpha)| \leqslant |\alpha|$.

Combining the two estimates we get

$$|1 - e^{a+ib}|^2 \leqslant |a|^2 e^{2|a|} + e^{|a|}\, |b|^2$$

$$\leqslant e^{2|a|} (|a|^2 + |b|^2).$$

e increasing

Hence $$|1 - e^{z-w}| \leqslant e^{|z-w|}\, |z-w|$$

$$\leqslant e^{|z|+|w|}\, |z-w|$$

$|z|, |w| \leqslant 5$

$$\leqslant e^{10}\, |z-w|.$$

We find $|g(z) - g(w)| \leqslant e^{15} |z-w|$ so that

$w(\alpha) := e^{15}\alpha$ would do the job.

Since f, g are NOT uniformly cont., extending their domains to $\mathbb{C}$ would eliminate their moduli of cont. since it implies uniform cont.