1. (a) \( z = (1-i)^3 \). Its real and imaginary parts are obtained by expanding \((a+b)^3 = a^3 + b^3 + 3ab(a+b)\):

\[
\begin{align*}
z &= 1^3 + (-i)^3 + 3 \cdot 1 \cdot (-i)^2 + 3 \cdot 1^2 \cdot (-i) \\
&= 1 \cdot 1 + (-1) \cdot i + 3 \cdot 1 \cdot i - 3 \cdot 1 \cdot i \\
&= 1 - 3 + i(1-3) = -2 - 2i.
\end{align*}
\]

Hence \( \text{Re} \{z\} = -2 \) and \( \text{Im} \{z\} = -2 \).

(b) \( z = \frac{x+iy}{a+ib} = \frac{x+iy}{a+ib} \cdot \frac{a-ib}{a-ib} = \frac{xa+iyb + i(ya-xb)}{a^2 + b^2} \).

Hence \( \text{Re} \{z\} = \frac{xa+by}{a^2 + b^2} \) and

\( \text{Im} \{z\} = \frac{ya-xb}{a^2 + b^2} \).

(c) \( z = \frac{\pi i}{(1-i)(2-i)} \) is given.

Its denom. is \( (1-i)(2-i) = 2 - 1 - i - 2i = 1 - 3i \).

Hence \( z = \frac{\pi i}{1-3i} \cdot \frac{1+3i}{1+3i} = \frac{\pi i (1+3i)}{1^2 + 3^2} = \)
\[ \frac{\pi i - 3\pi}{10} = \frac{-2\pi}{10} + \frac{\pi}{10} i. \]

We find \( \text{Re}(z^3) = \frac{-2\pi}{10} \) and \( \text{Im}(z^3) = \frac{\pi}{10}. \)

2. (a) We want to sketch the set

\[ S := \left\{ z \in \mathbb{C} \mid |z + 4i| \leq \pi \right\}. \]

We interpret \( |z + 4i| \leq \pi \) as \( z \) being distance at most \( \pi \) away from \( -4i. \)

I.e. it is a \underline{closed} ball of radius \( \pi \) around \( -4i. \)

\[ z \text{ includes boundary circle.} \]

(b) Now we have the set

\[ S = \left\{ z \in \mathbb{C} \mid |z + 2| < |2 + 1| \right\}. \]

We re-write the constraint as follows:
\[ 12 + 21 < 12 + 11 \]
\[ \Rightarrow 12 + 21^2 < 12 + 11^2 \]
\[ 12 + w^2 = 12 + 1 + 2 \text{Re}(z \bar{w}) \]
\[ \frac{1}{2} + \frac{1}{2} \text{Re}(z \bar{w}) < 12 + 1 + 2 \text{Re}(z \bar{w}) \]
\[ 3 + 2 \text{Re}(z \bar{w}) < 0 \]
\[ \text{Re}(z \bar{w}) < -\frac{3}{2} . \]

We find a half-plane to the left of the vertical line \( x = -\frac{3}{2} \), not including that boundary line.

\[ x = -\frac{3}{2} \]

3. We seek the minimal const. \( c > 0 \) s.t.
\[ c |z| \geq |\text{Re}(z \bar{w})| + |\text{Im}(z \bar{w})| . \]

Rewrite this using \( z = x + iy \) to get:
\[ c \sqrt{x^2 + y^2} \geq |x| + |y| . \]

Square the ineq. (both sides are positive) to get:

\[ c^2 (x^2+y^2) \geq (|x|+|y|)^2 = x^2 + y^2 + 2 |x| |y| \]
\[ (c^2-1)(x^2+y^2) \geq 2 |x| |y| . \]

Now complete the square:

\[ x^2 + y^2 = |x|^2 + |y|^2 = (|x| - |y|)^2 + 2 |x| |y| \]

to get the equivalent inequ.

\[ (c^2-1)[(|x| - |y|)^2 + 2 |x| |y|] \geq 2 |x| |y| . \]

For what \( c \) is this always true (i.e., \( \forall x \in \mathbb{R} \))?

1. Clearly \( c > 1 \) is necessary.

2. \((|x| - |y|)^2 \) is always positive and by choosing \( x, y \) appropriately can be made arbitrarily small, so we can't "rely" on it.

In other words, we may ask instead when

\[ 2(c^2-1) |x| |y| \geq 2 |x| |y| . \]

3. WLOG \( |x| |y| \neq 0 \) so we get

\[ c^2 - 1 \geq 1 \]

\[ \iff c^2 \geq 2 \]

\[ \iff c \geq \sqrt{2} . \]
4. (a) W.T.S. \[ \sum_{j=0}^{n} 2^j = \frac{1 - 2^{n+1}}{1 - 2} \quad (z \in \mathbb{C} \setminus \{2\}). \]

We start from \( S_n(z) := \sum_{j=0}^{n} 2^j. \)

Calculate \( 2S_n(z) = \sum_{j=0}^{n} 2^j \)

\[ = \sum_{j=0}^{n} 2^j = 1 + \sum_{j=1}^{n} 2^j = 1 + S_n(z) = 1 - 2^{n+1}. \]

Hence \( (1 - z)S_n(z) = 1 - 2^{n+1}. \) Since \( z \neq 1, \) this implies the result.

(b) W.T.S. \[ \sum_{j=0}^{n} \cos(j \theta) = \frac{1}{2} + \frac{\sin\left(\frac{2n+1}{2}\theta\right)}{2 \sin(\theta/2)}. \]

We use the following "trick":

\[ \sum_{j=0}^{n} \cos(j \theta) = \sum_{j=0}^{n} \Re\{e^{ij \theta}\} = \Re\left\{ \sum_{j=0}^{n} e^{ij \theta}\right\} = \Re\left\{ \sum_{j=0}^{n} (e^{i \theta})^j\right\} \]
part (a) \[ \text{Oh as } \theta \in 2\pi \mathbb{Z} \Rightarrow \Re \left\{ \frac{1 - e^{i(\theta + n)\theta}}{1 - e^{i\theta}} \right\} . \]

so \( e^{i\theta} \neq 1. \)

\[ \Rightarrow \Re \left\{ \frac{e^{i(\theta + n)\theta} - 1}{e^{i\theta} - 1} \cdot \frac{e^{-i\theta/2}}{e^{-i\theta/2}} \right\} \]

\[ = \Re \left\{ \frac{e^{i(\theta + n)\theta} - e^{-i\theta/2}}{e^{i\theta/2} - e^{-i\theta/2}} \right\} \]

\[ = \Re \left\{ \frac{e^{i(\theta + n)\theta} - e^{-i\theta/2}}{2i \sin(\theta/2)} \right\} \]

\[ = \frac{1}{2 \sin(\theta/2)} \left[ \delta \sin((\theta + n)\theta) + \delta \sin(\theta/2) \right] \]

\[ = \frac{1}{2} \left[ 1 + \frac{\delta \sin(n\theta/2)}{\sin(\theta/2)} \right]. \]

5. We seek the zeros of the eqn \( z^5 + 32 = 0. \)

Noting \( 2^5 = 32 \) we get

\[ z^5 = -2^5 \iff \left( \frac{z}{2} \right)^5 = -1 \]

\[ \iff \frac{z}{2} = \sqrt[5]{-1} . \]
\[ \Rightarrow z = 2^{\frac{5}{1}}. \]

The problem with this is \( \sqrt[5]{1} \) is a multi-valued function. One way to understand this is via the polar form:

\[
\sqrt[5]{z} = \sqrt[5]{r e^{i\theta}}
\]

\[
= \sqrt[5]{r} e^{i\theta/5}.
\]

Since \( \theta \in \mathbb{R} \) has a \( 2\pi \mathbb{Z} \) ambiguity, once we divide by 5 we get 5 possibilities:

\[
\theta = \theta + 2\pi n ; \quad \theta \in [0, 2\pi), \quad n \in \mathbb{Z}
\]

\[
\frac{\theta}{5} = \frac{\theta}{5} + \frac{2\pi}{5} n.
\]

So the five possibilities for the root are:

\[
\left\{ \frac{\theta}{5}, \frac{\theta + 2\pi}{5}, \frac{\theta + 2\pi}{5} + 2, \frac{\theta + 2\pi}{5} + 3, \frac{\theta + 2\pi}{5} + 4 \right\}
\]

We need \( \sqrt[5]{-1} \) so we have \( -1 = 1 \cdot e^{i\pi} \), i.e., \( \theta = \pi \) and so

\[
\sqrt[5]{-1} = \left\{ \frac{\pi}{5}, \frac{3\pi}{5}, \frac{5\pi}{5}, \frac{7\pi}{5}, \frac{9\pi}{5} \right\}
\]

Hence we get the five solutions of the eqn:
6. Claim: \[ |z| = \max_{\theta \in [-\pi, \pi]} \Re(e^{i\theta}) \]

Proof: Write \( z = r e^{i\alpha} \) so we write.

\[
|e^{i\alpha}| = \max_{\theta \in [-\pi, \pi]} \Re\{re^{i(x+\theta)}\}
\]

Since \( r > 0 \) and \( x \) and \( \theta \) are independent, this is equivalent to

\[
1 = \max_{\theta \in [-\pi, \pi]} \Re\{e^{i(x+\theta)}\}
\]

\[
= \max_{\theta \in [-\pi, \pi]} \cos(x+\theta)
\]

which is a manifestly correct statement.

Claim: \[ |z + w| \leq |z| + |w| \]

Proof: Using the above,

\[
|z + w| = \max_{\theta} \Re\{(z + w)e^{i\theta}\}
\]
\[
\text{max} \quad \Re f(z) + \Re g(z)
\]

\[
= \left( \max_{\theta} \Re f(z) \right) + \left( \max_{\theta} \Re g(z) \right)
\]

**FACT:** \(\max_{\theta} [f(\theta) + g(\theta)] \leq \left( \max_{\theta} f(\theta) \right) + \left( \max_{\theta} g(\theta) \right)\)

(Prove it...)

Using this fact,

\[
|z + w| \leq \left( \max_{\theta} \Re e^{i\theta} \right) + \left( \max_{\theta} \Re we^{i\theta} \right)
\]

Part (a) \[
= |z| + |w|.
\]

8. Looking for accum. pts. of

\[
S = \left\{ (2i)^n \right\} \quad n \in \mathbb{N}_1 \subseteq \mathbb{C}
\]

Recall \[
i^n = \begin{cases} 
1 & n \equiv 0 \pmod{4} \\
i & n \equiv 1 \pmod{4} \\
-1 & n \equiv 2 \pmod{4} \\
-i & n \equiv 3 \pmod{4}
\end{cases}
\]
 whereas \( 2^n \to \infty \). So this set has no accum. pts. since its elements become increasingly scattered.

9. (a) Define \( S_1 = R \subseteq C \).

**FACT:** \( R^c = C \setminus R \in \text{Open}(C) \) (e.g. via open ball def.)

Hence \( R \in \text{Closed}(C) \).

Thus: \( \overset{\circ}{\text{clo}}(R) = R \) (closure of closed set is the set itself).

\( \overset{\circ}{\text{int}}(R) = \emptyset \) b/c \( \emptyset \) open balls within \( R \).

\( \partial R = \text{clo}(R) \setminus \text{int}(R) = R \).

\( \Rightarrow R \) is closed in \( C \).

(2) \( S_2 = B_{y_2}(1) \)

\( \overset{\circ}{\text{clo}}(B_{y_2}(1)) = \left\{ z \in C \mid 12-11 \leq \frac{1}{2} \right\} \)
i.e. a closed ball including its boundary circle.

\[ \text{int}(B_{y_2}(1)) = B_{y_2}(1) \text{ since } B_{y_2}(1) \text{ is open}. \]

\[ \partial B_{y_2}(1) = \text{clo}(B_{y_2}(1)) \setminus \text{int}(B_{y_2}(1)) \]

\[ = \left\{ z \in C \mid 1z - 1 = \frac{1}{2} \right\} \]

= boundary circle.

\( B_{y_2}(1) \) is open in \( C \).

\[ S_3 = \left\{ \frac{i}{2} \mid n \in \mathbb{N} \right\} \]

\[ \text{clo}(S_3) = S_3 \cup \{0\} \text{ using the characterization that a closed set contains all its limit points, which } 0 \text{ is for } S_3. \]

\[ \text{int}(S_3) = \emptyset \text{ since } S_3 \text{ contains no open balls.} \]

\[ \partial S_3 \equiv \text{clo}(S_3) \setminus \text{int}(S_3) = S_3 \cup \{0\}. \]
Hence $S_3$ is not closed nor open.

\[(4) \quad S_1 = \{x + iy \in \mathbb{C} \mid x, y \in \mathbb{Q} \} \]  

rational #s

Claim: $\text{clo}(S_4) = \mathbb{C}$

Proof: Any pts. $z \in \mathbb{C}$ is a limit pt. of $S_4$.

Claim: $\text{int}(S_4) = \emptyset$

Proof: $S_4$ contains no open balls!

$\Rightarrow D_{S_4} = \text{clo}(S_4) \setminus \text{int}(S_4) = \mathbb{C}.$

$S_4$ is not closed neither open.

10. We're given $f, g: \mathbb{C} \to \mathbb{C}$ and $z_0 \in \mathbb{C}$, s.t.

\[ \lim_{z \to z_0} g(z) = g(z_0), \quad \lim_{w \to g(z_0)} f(w) = f(g(z_0)) \ \exists. \]

Let $S_f, S_g$ to be the relevant $S$'s from the existence of those limits resp.

W.R.T.S. \[ \lim_{z \to z_0} f(g(z)) = f(g(z_0)) \ \exists. \]
I.e. want $\delta_0(\varepsilon) > 0$; if $z \in B_{\delta_0}(z_0)$

then $f(g(z)) \in B_{\varepsilon}(f(g(z_0)))$.

Pick $\delta_0(\varepsilon) := \delta_g(\delta_f(\varepsilon))$.

Then, if $\|z - z_0\| < \delta_0(\varepsilon) = \delta_g(\delta_f(\varepsilon))$

$\lim_{z \to z_0} g \in \mathbb{C}.$

$|g(z) - g(z_0)| < \delta_f(\varepsilon)$

$\lim_{z \to z_0} f \in \mathbb{C}.$

$|f(g(z)) - f(g(z_0))| < \varepsilon.$

\[11. \text{(a) Consider } B_1(0) \ni z \xrightarrow{f} z^2 \in \mathbb{C}.\]

We seek a modulus of cont. for it.

Start backwards:

$|f(z) - f(w)| = |z^2 - w^2|$

$= |z - w||z + w|$

$\leq 2|z - w|^{1/2} + |w||z - w|$

$\leq 2|z - w|.$

Hence $w(x) := 2x$ will do the job.
(b) Now we have
\[
B_2(-3) \ni z \rightarrow \exp(z) \in C
\]
Since \( B_2(-3) \subseteq B_5(0) \), we'll work w/ \( |z| < 5 \) instead.

Following the same procedure, we have
\[
|g(2) - g(w)| = |e^z - e^w|
\]
\[
= |e^z||1 - e^{w-z}|
\]
\[
\leq |z| |1 - e^{w-z}|
\]
\[
\leq e^{|z|} |1 - e^{w-z}|
\]
\[
|z| \leq 5 \Rightarrow e^{|z|} |1 - e^{w-z}|
\]

Now, \[
|1 - e^{z}|^2 = |1 - e^a \cos(b) - ie^a \sin(b)|^2
\]
\[
\leq (1 - e^a \cos(b))^2 + e^{2a} \sin(b)^2
\]
\[
= 1 + e^{2a} - 2e^a \cos(b)
\]
\[
= 1 + e^{2a} - 2e^a + 2e^a |1 - \cos(b)|
\]
\[
= (1 - e^a)^2 + 4e^a \sin(b)^2.
\]

From the mean value thm. on \( \exp : \mathbb{R} \rightarrow \mathbb{R} \).
we have \( e^{a-\frac{1}{e^x}} = (a-0) e^x \in \mathbb{C} \in [-a,a] \).

\[ \Rightarrow \quad |1 - e^{a-\frac{1}{e^x}}| \leq |a| e^{a-\frac{1}{e^x}} . \]

For the since, we have \(|\sin(x)| \leq 1|a| \).

Combining the two estimates we get

\[ |1 - e^{a + ib|2} \leq |a|^2 e^{2|a|} + e^{a|1|} |b|^2 \]

\[ \leq e^{2|a|} (|a|^2 + |b|^2) . \]

\( e \) increasing

Hence

\[ |1 - e^{2-w^1} | \leq e^{12-w^1} 12-w^1 \]

\[ \leq e^{12-w^1} 12-w^1 \]

\[ \leq e^{10} 12-w^1 . \]

We find \(|g(2) - g(1)| \leq e^{15} (2-w^1) \) so that

\[ w(x) := e^{15} x \] would do the job.

Since \( f, g \) are NOT uniformly cont., extending their domains to \( \mathbb{C} \) would eliminate their moduli of cont. Since it implies uniform cont.