Complex Analysis with Applications
Princeton University MAT330
Spring 2023 Final Exam

May 14, 2023

Note: the following is the final exam held on May 14th 2023 at 9am-noon in Fine 314. Please PRINT your first and last name in the box:

JACOB SHAPIRO

state the honor code pledge:

SAMPLE SOLUTIONS

and sign it

Now please wait, without turning the page, until you are told to start the exam, at which point you shall have three hours.

Please write legibly and neatly. In the long questions part you are expected to justify your answer in full sentences. In the short questions part merely providing a correct succinct one-word answer will suffice.

1 Relevant formulas

In the following items, I remind you of relevant formulas but *not* of their scope of validity, which I expect you to know.

- The Cauchy-Riemann equations

$$\begin{aligned} \partial_x f_R &= \partial_y f_I \\ \partial_x f_I &= -\partial_y f_R \,. \end{aligned}$$

- Cauchy's integral formula:

$$f^{(n)}(z_0) = \frac{n!}{2\pi \mathrm{i}} \oint_\Gamma \frac{f(z)}{(z-z_0)^{n+1}} \mathrm{d}z \qquad (n \in \mathbb{N}_{\geq 0}) \,.$$

- Taylor's theorem:

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(z_0)(z-z_0)^n \,.$$

- The residue formula:

$$\mathrm{residue}_{z_0}(f) = \lim_{z \to z_0} \frac{1}{(n-1)!} \partial_z^{n-1} (z-z_0)^n f(z) \,.$$

- The argument principle:

$$\mathrm{index}_D(f) = \frac{1}{2\pi \mathrm{i}} \oint_\Gamma \frac{f'(z)}{f(z)} \mathrm{d}z \,.$$

- The Krammers-Kronig relation:

$$\lim_{\varepsilon \to 0^+} \frac{1}{x \pm \mathrm{i}\varepsilon} = \mp \mathrm{i}\pi\delta(x) + \mathscr{P}\left(\frac{1}{x}\right) \,.$$

- The Fourier series

$$(\mathcal{F}\psi)(n) \equiv \hat{\psi}(n) = \frac{1}{2\pi} \int_{\theta=0}^{2\pi} \mathrm{e}^{-\mathrm{i}n\theta} \psi(\theta) \mathrm{d}\theta$$

  and

$$\left(\mathcal{F}^{-1}\hat{\psi}\right)(\theta) = \sum_{n \in \mathbb{Z}} \mathrm{e}^{\mathrm{i}n\theta} \hat{\psi}(n) \,.$$

- The Fourier transform

$$(\mathcal{F}f)(\xi) \equiv \hat{f}(\xi) = \int_{x \in \mathbb{R}} \mathrm{e}^{-2\pi \mathrm{i}\xi x} f(x) \mathrm{d}x \,.$$

- Laplace and steepest descent asymptotics

$$\int_{x\in\mathbb{R}} \mathrm{e}^{-\lambda f(x)} g\left(x\right) \mathrm{d}x \overset{\lambda\to\infty}{\sim} \sqrt{\frac{2\pi}{\lambda \tilde{f}''\left(z_\star\right)}} \mathrm{e}^{-\lambda \tilde{f}(z_\star)} \tilde{g}\left(z_\star\right) .$$

- Jordan's inequality:

$$\frac{2}{\pi}\alpha \quad \leq \sin\left(\alpha\right) \leq \quad \alpha \qquad \left(\alpha \in \left[0, \frac{\pi}{2}\right]\right) ,$$

Kober's inequality

$$1 - \frac{2}{\pi}\left|x\right| \leq \cos\left(x\right) \leq 1 - \frac{x^2}{\pi} \qquad \left(x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right) ,$$

and the "big arc lemma":

$$\int_{\theta=0}^{\pi} \mathrm{e}^{-R\sin(\theta)}\mathrm{d}\theta \quad \lesssim \quad \frac{1}{R} .$$

2 Short questions [20 points]

In the following questions, *no* justification is necessary. Simply provide a short as possible *correct* response.

1. If f is analytic and injective, then f' can take on the value zero. Correct or incorrect?

   Incorrect.

   Lemma 9.2

2. If $F : \mathbb{R}^2 \to \mathbb{R}$ is a given harmonic function, provide an explicit expression for a harmonic conjugate $G : \mathbb{R}^2 \to \mathbb{R}$ to F at some $(x, y) \in \mathbb{R}^2$:

   $G(x,y) = \int_0^1 (\nabla F \circ \gamma) \wedge \dot{\gamma}$; $\gamma : [0,1] \to \mathbb{R}^2$ path : $\gamma(1) = (x,y)$

   Eq-n (4.7)

$u \wedge v \equiv$
$u_1 v_2 - u_2 v_1$

3. Provide an example $z \in \mathbb{C}$, $\alpha \in [0, 2\pi)$ for which $\mathbb{Im}\{\mathrm{Log}_\alpha(\overline{z})\} \neq -\mathbb{Im}\{\mathrm{Log}_\alpha(z)\}$, where Log_α is the complex logarithm with branch cut at α:

   $\alpha = 0$ and $z = \exp(i\pi/4)$.

   Def. 5.5

4. Let $\Omega \subseteq \mathbb{C}$ be open and bounded, and assume that $f : \Omega \to \mathbb{C}$ is analytic and extends continuously to $\overline{\Omega}$ such that $|f| \leq 1$ on $\partial\Omega$. What is $\sup_{z \in \Omega} |f(z)|$?

   ≤ 1. ⊛

   Cor. 7.52

5. If an analytic function $f : \Omega \to \mathbb{C}$ with Ω open and connected vanishes on an open subset of Ω, does the function *have* to be the zero function?

   Yes. ⊛⊛

   Thm. 7.16

⊛ Actually all we can say is $\sup_{z \in \Omega} |f(z)| \leq 1$, because from the given information $\sup_{z \in \Omega} |f(z)|$ cannot be determined, only an upper bound on it.

⊛⊛ If the open set on which f vanishes is empty (remember $\emptyset \in \mathrm{Open}(\mathbb{C})$) then nothing can be said. If this is what you had in mind and you did not recieve points for this you may appeal.

3 Long questions [80 points]

In the following questions, you must justify your work and convince me that you not only know what the correct answer is, but also *why* it is so. You may freely invoke any result from the lecture notes, homework or various textbooks just so long as you properly cite and explain in what way you're invoking it. Note that if you're being asked about a result that appeared in the HW or lecture notes you can't verbatim invoke that very result: that would be silly.

6. Find a Taylor series at $z = 0$ for

$$f(z) = \frac{z^2}{(2+z)^2}$$

and indicate its domain of convergence.

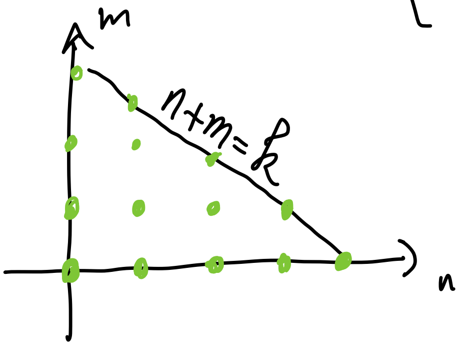
One pole @ $z=-2$ of 2^{nd} order.

$\Rightarrow$ domain of conv. is $B_2(0)$.

$$\frac{1}{2+z} = \frac{1}{2}\,\frac{1}{1+\frac{z}{2}} = \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{z}{2}\right)^n$$

geometric series

$$\Rightarrow f(z) = z^2\left(\frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{z}{2}\right)^n\right)^2 = \frac{1}{4}\sum_{n,m=0}^{\infty}\frac{(-1)^{n+m}\,z^{n+m+2}}{2^{n+m}}$$



$n+m=k$

Change var: $(n,m) \longmapsto (\underbrace{n+m}_{k}, n)$

$$= \frac{1}{4}\sum_{k=0}^{\infty}\left(\sum_{n=0}^{k} 1\right)(-1)^k\,\frac{z^{k+2}}{2^k}$$

$$= \sum_{k=0}^{\infty}(k+1)(-1)^k\left(\frac{z}{2}\right)^{k+2}.$$

Note: $\exists$ alt. sol-n.

7. Let $\gamma : [0,1] \to \mathbb{C}$ be a CCW contour such that $\Gamma \equiv \operatorname{im}(\gamma)$ is the *square* of side-length $2a > 0$ centered at the origin. Calculate $\oint_\Gamma \bar{z} \mathrm{d}z$ and $\oint_\Gamma \frac{\mathrm{e}^z}{z^2}$.

By HW4Q4, $\frac{1}{2i}\int_\Gamma \bar{z}\, dz = $ area enclosed in $\operatorname{int}(\Gamma)$.

(Note: to get full credit one had to carry out this calculation ⊛).

$$\Longrightarrow \frac{1}{2i}\int_\Gamma \bar{z}\, dz = 4a^2 \Longleftrightarrow \boxed{\int_\Gamma \bar{z}\, dz = 8ia^2}.$$

By residue integral formula,

$$\oint_\Gamma \frac{e^z}{z^2} dz = 2\pi i \operatorname{res}_0\left(z \mapsto \frac{e^z}{z^2}\right)$$

as $z \mapsto \frac{e^z}{z^2}$ has one pole of order 2 at the origin (inside of $\operatorname{int}(\Gamma)$).

$$\operatorname{res}_0\left(z \mapsto \frac{e^z}{z^2}\right) = \lim_{z\to 0} \partial_z z^2 e^z = 1.$$

$$\Longrightarrow \boxed{\oint_\Gamma \frac{e^z}{z^2} dz = 2\pi i}.$$

⊛ Explicit calc. of 1^{st} integral:

$$
\begin{aligned}
\gamma_1(t) &= a + it & t &\in [-a,a] \\
\gamma_2(t) &= ai - t & t &\in [-a,a] \\
\gamma_3(t) &= -a - it & t &\in [-a,a] \\
\gamma_4(t) &= -ai + t & t &\in [-a,a]
\end{aligned}
$$

$$
\left.
\begin{aligned}
I_1 &= \int_{-a}^{a} (a - it)\, i\, dt = 2ia^2 \\
I_2 &= \int_{-a}^{a} (-ai - t)(-1)\, dt = 2ia^2 \\
I_3 &= \int_{-a}^{a} (-a + it)(-i)\, dt = 2ia^2 \\
I_4 &= \int_{-a}^{a} (ai + t)\, dt = 2ia^2
\end{aligned}
\right\} I = 8ia^2 .
$$

8. Suppose p is a polynomial of degree $n \in \mathbb{N}_{\geq 0}$ and $R > 0$ such that $p \neq 0$ in $\mathbb{C} \setminus B_R(0)$. Calculate

$$\oint_{\partial B_R(0)} p, \oint_{\partial B_R(0)} \frac{1}{p}, \text{ and } \oint_{\partial B_R(0)} \frac{p'}{p}.$$

$\boxed{\mathrm{I}}$ p is entire $\Rightarrow$ $\boxed{\int_{\partial B_R(0)} p = 0}$.

$\boxed{\mathrm{II}}$ $\frac{1}{p}$ is meromorphic $\Rightarrow$ $\oint_{\partial B_R(0)} \frac{1}{p} = 2\pi i \sum_{\lambda_j \in \text{poles}} \operatorname{res}_{\lambda_j}\left(\frac{1}{p}\right)$.

Case 1: $n = 0 \Rightarrow p$ is const

$\Rightarrow \frac{1}{p}$ is entire

$\Rightarrow \oint_{\partial B_R(0)} \frac{1}{p} = 0$.

Write $p(z) = a(z-\lambda)$ $\exists\, a \in \mathbb{C} \setminus \{0\}$ $\lambda \in B_R(0)$

Case 2: $n = 1$ whence $\operatorname{res}_\lambda \left(z \mapsto \frac{1}{a(z-\lambda_j)}\right) = \frac{1}{a}$.

$$\Rightarrow \oint_{\partial B_R(0)} \frac{1}{a(z-\lambda)}\, dz = \frac{2\pi i}{a}.$$

Case 3: $n \geq 2$. Two possible answers would be accepted.

Claim: $\oint_{\partial B_R(0)} \frac{1}{p} = 0$.

Note: $B_R(0)$ is open. Since $\lambda_j \in B_R(0)$ by assumption, we don't have to worry about $\lambda_j \in \partial B_R(0)$ (this would have amounted to half a residue).

write $p(z) = a \prod_{j=1}^{m} (z-\lambda_j)^{\alpha_j}$ with

m # of distinct roots

$\{\alpha_j\}_{j=1}^{m} \subseteq \mathbb{N}_{\geq 1}$ multiplicity of each root

$\{\lambda_j\}_{j=1}^{m} \subseteq B_R(0)$ distinct roots

$a \in \mathbb{C} \setminus \{0\}$ overall const.

Proof that $\oint \frac{1}{p} = 0$:

By Cauchy's thm.,

$[0,\infty) \ni R \mapsto \int_{\partial B_R(0)} \frac{1}{p}$ is a const.

for all R large enough.

Hence that const. fn should equal its limit as $R \to \infty$. But that limit is zero:

$$\left| \oint_{\partial B_R(0)} \frac{1}{p} \right| = \left| \int_{\theta=0}^{2\pi} \frac{1}{a \prod_{j=1}^{m} (Re^{i\theta} - \lambda_j)^{\alpha_j}} Re^{i\theta} i \, d\theta \right|$$

$$\leq \frac{2\pi R}{|a| \prod_{j=1}^{m} (R - |\lambda_j|)^{\alpha_j}}$$

$$\lesssim R^{1-\sum_{j=1}^{m} \alpha_j} \xrightarrow{R\to\infty} 0$$

as soon as $\sum_{j=1}^{m} \alpha_j = n \geq 2$.

If somehow you did not realize this you could still earn full credit on this question by providing the following calculation:

$$\oint_{\partial B_R(0)} \frac{1}{p} = 2\pi i \sum_{j=1}^{m} \mathrm{res}_{\lambda_j}\left(\frac{1}{p}\right)$$

and

$$\mathrm{res}_{\lambda_j}\left(\frac{1}{p}\right) = \lim_{z\to\lambda_j} \frac{1}{(\alpha_j-1)!}\,\partial_z^{\alpha_j-1} \frac{(z-\lambda_j)^{\alpha_j}}{p(z)}$$

$$= \lim_{z\to\lambda_j} \frac{1}{(\alpha_j-1)!}\,\partial_z^{\alpha_j-1} \frac{1}{a\prod_{\substack{l=1\\ l\neq j}}^{m} (z-\lambda_e)^{\alpha_e}}$$

$$= \frac{1}{a(\alpha_j-1)!}\,\partial_z^{\alpha_j-1}\Big|_{z=\lambda_j} \prod_{\substack{l=1\\ l\neq j}}^{m} (z-\lambda_e)^{-\alpha_e}$$

$$\Rightarrow \oint_{\partial B_R(0)} \frac{1}{p} = \frac{2\pi i}{a} \sum_{j=1}^{m} \frac{1}{(\alpha_j-1)!}\,\partial_z^{\alpha_j-1}\Big|_{z=\lambda_j} \prod_{\substack{l=1\\ l\neq j}}^{m} (z-\lambda_e)^{-\alpha_e}$$

(If you did not take multiplicities into account you would lose points).

May be simplified e.g. via partial fraction decomposition to obtain zero, but not nec. for full credit.

$\boxed{\text{III}}$ Finally, for the last integral,

$$\oint_{\partial B_R(0)} \frac{p'}{p} \underset{\uparrow}{=} 2\pi i \operatorname{index}_{B_R(0)}(p) = 2\pi i n.$$

argument principle

Thm. 7.43

$$\boxed{\oint_{\partial B_R(0)} \frac{p'}{p} = 2\pi i n}.$$

Note: One did NOT have to carry out the calculation explicitly. It sufficed to cite the argument principle.

9. Calculate
$$\int_{x=0}^{\infty} \frac{\sin(x)}{x(1+x^2)} \mathrm{d}x \,.$$

$$I := \int_{x=0}^{\infty} \frac{\sin(x)}{x(1+x^2)}\, dx = \frac{1}{2}\int_{x=-\infty}^{\infty} \frac{\sin(x)}{x(1+x^2)}\, dx$$

$$= \frac{1}{2}\mathrm{Im}\left\{ \int_{x=-\infty}^{\infty} \frac{e^{ix}}{x(1+x^2)}\, dx \right\}$$

$f(z) := \dfrac{e^{iz}}{z(1+z^2)}$ meromorphic w/ poles @ $z=0$ and $z=\pm i$.

Close contour upwards w/ a semicircle:

$$\left| \int_{\theta=0}^{\pi} \frac{\exp(iRe^{i\theta})}{Re^{i\theta}(1+R^2e^{2i\theta})} Re^{i\theta} i\, d\theta \right| \leq$$

$$\leq \frac{1}{R^2-1} \underbrace{\int_{\theta=0}^{\pi} \exp(-R\sin(\theta))\, d\theta}_{\text{big arc } \lesssim \frac{1}{R}} \xrightarrow{R\to\infty} 0$$

$$\Longrightarrow \int_{\frown} \frac{e^{iz}}{z(1+z^2)}\, dz = 2\pi i \operatorname{res}_i \left(z \mapsto \frac{e^{iz}}{z(1+z^2)}\right) +$$

$$+ \pi i \operatorname{res}_0 \left(z \mapsto \frac{e^{iz}}{z(1+z^2)}\right)$$

half since pole ON contour Lemma 7.56

Calculate residues:

$$\operatorname{res}_0\left(z \mapsto \frac{e^{iz}}{z(1+z^2)}\right) = 1$$

$$\operatorname{res}_i\left(z \mapsto \frac{e^{iz}}{z(1+z^2)}\right) = \frac{e^{-1}}{i(2i)} = \frac{-1}{2e}.$$

$$\Rightarrow I = \frac{1}{2}\operatorname{Im}\left\{\pi i\left(2\left(-\frac{1}{2e}\right)+1\right)\right\}$$

$$= \frac{\pi}{2}\left(1-\frac{1}{e}\right).$$

$$\boxed{I = \frac{\pi}{2}\left(1-\frac{1}{e}\right)}.$$

<u>Note</u>: Instead of invoking Lemma 7.56 one could have used a small semi-circle arc to avoid hitting the pole. This would've been harder.

10. Find a conformal equivalence $c : B_1(0) \cap \mathbb{H} \to \mathbb{H}$ (and prove it is so) where $\mathbb{H}$ is the open upper half plane and $B_1(0)$ the open unit disc.

$$c(z) := -\frac{1}{2}\left(z + \frac{1}{z}\right) \qquad \text{(Jukowski map).}$$

Clearly analytic on $B_1(0) \cap \mathbb{H}$, so need to show it is a bijection $B_1(0) \cap \mathbb{H} \to \mathbb{H}$ (this is, by using Def. 9.1).

To do so, one had to produce a proof analogous to, e.g., the claim in the sample solutions of HW8Q5 (but $\exists$ other ways to prove it differently). We attach it here below for the reader's convenience:

Claim: $f: B_1(0) \cap \mathbb{H} \to \mathbb{H}$ (Jukowski map from HW6Q6)

$z \mapsto -\frac{1}{2}(z + \frac{1}{z})$

is a conformal equivalence.

Proof: Note:

$$\frac{\left(\frac{z-1}{z+1}\right)^2 + 1}{\left(\frac{z-1}{z+1}\right)^2 - 1} = \frac{(z-1)^2 + (z+1)^2}{(z-1)^2 - (z+1)^2} = \frac{2z^2+2}{-4z}$$

$$= -\frac{1}{2}\left(z + \frac{1}{z}\right) = f(z).$$

Define $g: \mathbb{H} \cap B_1(0) \to Q_2$ ← 2nd quadrant

$z \mapsto \frac{z-1}{z+1}$

$S: Q_2 \to Q_3 \cup Q_4$ ← lower half plane

$z \mapsto z^2$

$h: Q_3 \cup Q_4 \to \mathbb{H}$

$z \mapsto \frac{z+1}{z-1}$

$f = h \circ S \circ g$

So will show each of the three maps is a conformal equiv.

Claim: $g: B_1(0) \cap \mathbb{H} \to Q_2$ is a conf. equiv.

Proof: We've seen g in Example 9.4 that $\mathbb{H} \cong B_1(0)$ via

$z \mapsto \frac{i-z}{i+z}$

with inverse $w \mapsto i\frac{1-w}{1+w} = -i \underbrace{\frac{w-1}{w+1}}_{g(w)}$

rotation by $90°$

restriction of injective map is injective

Hence by that same proof when restricting the inverse from $B_1(0)$ to $\mathbb{H} \cap B_1(0)$ we obtain a map whose range is Q_2 and is still inj.

We've seen there e.g. $\operatorname{Im}\{-i\frac{w-1}{w+1}\} > 0$

$$\Leftrightarrow \operatorname{Im}\{-i\, g(w)\} > 0$$

$$\Leftrightarrow -\operatorname{Re}\{g(w)\} > 0$$

$$\Leftrightarrow \operatorname{Re}\{g(w)\} < 0$$

$\Rightarrow \operatorname{im}(g) \subseteq$ left half plane.

Moreover,

$$\operatorname{Im}\{g(w)\} = \operatorname{Im}\left\{\frac{w-1}{w+1}\right\} = \operatorname{Im}\left\{\frac{(w-1)(\overline{w+1})}{|w+1|^2}\right\}$$

$$= |w+1|^{-2}\operatorname{Im}\{|w|^2 - 1 + 2i\operatorname{Im}\{w\}\}$$

$$= |w+1|^{-2}\, 2\operatorname{Im}\{w\} > 0$$

$\Rightarrow \operatorname{im}(g) \subseteq Q_2$. ∎

Claim: $S : Q_2 \to Q_3 \cup Q_4$ is a conf. equiv.

Proof: Properly defining $\widetilde{\mathrm{Arg}}$ yields the correct inverse. $\blacksquare$

Claim: $h : Q_3 \cup Q_4 \longrightarrow \mathbb{H}$ is a conf. equiv.

Proof:

$$\mathrm{Im}\left\{\frac{z+1}{z-1}\right\} = \mathrm{Im}\left\{\frac{(z+1)(\bar{z}-1)}{|z-1|^2}\right\}$$

$$= |z-1|^{-2}\,\mathrm{Im}\left\{|z|^2 - 1 - 2i\,\mathrm{Im}\{z\}\right\}$$

$$= |z-1|^{-2}\,(-2)\,\underbrace{\mathrm{Im}\{z\}}_{<0} > 0 \checkmark$$

$\Rightarrow$ well-def.

inverse is $w \mapsto \dfrac{w+1}{w-1}$. $\blacksquare$

$\blacksquare$

11. Let $\alpha \in \mathbb{C}$ with $\mathbb{Im}\{\alpha\} > 0$. Calculate

$$\oint_{-\infty}^{\infty} \frac{1}{(x-1)(x-\alpha)^2} \mathrm{d}x \, .$$

$$\boxed{I = \frac{-i\pi}{(1-\alpha)^2}} \, .$$

This was presented in Example 7.60, which one had to explicitly carry out to get full credit.

12. Let $f : \mathbb{R} \to \mathbb{C}$ be given such that it extends to an analytic function $\tilde{f} : S_\varepsilon \to \mathbb{C}$ where S_ε is a horizontal strip of width 2ε about the real axis and such that

$$\sup_{y\in(-\varepsilon,\varepsilon)} |f(x+\mathrm{i}y)| \leq \frac{1}{1+x^2} \qquad (x\in\mathbb{R}) \ .$$

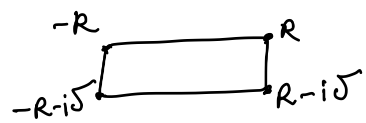
Show that there exists some $B \in (0,\infty)$ such that for any $\delta \in [0,\varepsilon)$,

$$|(\mathscr{F}f)(\xi)| \quad \leq \quad B\mathrm{e}^{-2\pi\delta|\xi|} \qquad (\xi\in\mathbb{R})$$

and give an explicit expression for B (i.e., if you find an integral, calculate it).

This was an easier version of Lemma 8.11, and one could NOT have invoked that result to get full credit.

Case 1: $\xi > 0$.



$$\hat{f}(\xi) \equiv \int_{x\in\mathbb{R}} e^{-2\pi i x\xi} f(x)\, dx$$

Add vertical legs:

$$\left| \int_{t=-\delta}^{0} e^{-2\pi i (R+it)} f(R+it)\, i\, dt \right| \leq$$

$$\leq \frac{1}{1+R^2} \int_{t=-\delta}^{0} e^{2\pi t}\, dt \leq \frac{\delta}{1+R^2} \xrightarrow{R\to\infty} 0$$

$$\Longrightarrow \quad \hat{f}(\xi) = \int_{x\in\mathbb{R}} e^{-2\pi i (x-i\delta)\xi} f(x-i\delta)\, dx \quad \forall\, \delta\in(0,\varepsilon).$$

$$\Rightarrow \quad |\hat{f}(\xi)| \leq e^{-2\pi\delta\xi} \underbrace{\int_{x\in\mathbb{R}} |f(x-i\delta)|\, dx}$$

$$\leq \int_{x\in\mathbb{R}} \frac{1}{1+x^2}\, dx = \frac{\pi}{2} < \infty.$$

$\forall\, \delta < \varepsilon$

need to calculate this, as in Example 7.31.

<u>Case 2</u>: $\xi < 0$. Go up instead of down.

Follow analogous calc. to get

$$|\hat{f}(\xi)| \leq e^{+2\pi\delta\xi} \cdot \frac{\pi}{2}.$$

$$\Longrightarrow \boxed{|\hat{f}(\xi)| \leq \frac{\pi}{2} \exp(-2\pi\delta|\xi|)} \quad (\xi \in \mathbb{R}, \ \delta \in (0, \varepsilon)).$$

13. Calculate the leading order asymptotics as $\lambda \to \infty$ of

$$I(\lambda) = \int_{x\in\mathbb{R}} \exp(\mathrm{i}\lambda\cosh(x))\,\mathrm{d}x\,.$$

This was presented in HW10Q14 which one had to present explicitly to obtain full credit:

$$\boxed{I(\lambda) \overset{\lambda\to\infty}{\sim} \sqrt{\frac{2\pi}{\lambda(-i)}}\exp(i\lambda)}\,.$$

We include that solution here for convenience:

$$I(\lambda) := \int_{t\in\mathbb{R}} \exp(i\lambda\cosh(t))\,dt\,.$$

$$f(z) := -i\cosh(z) \qquad f''(z) = -i\cosh(z)$$

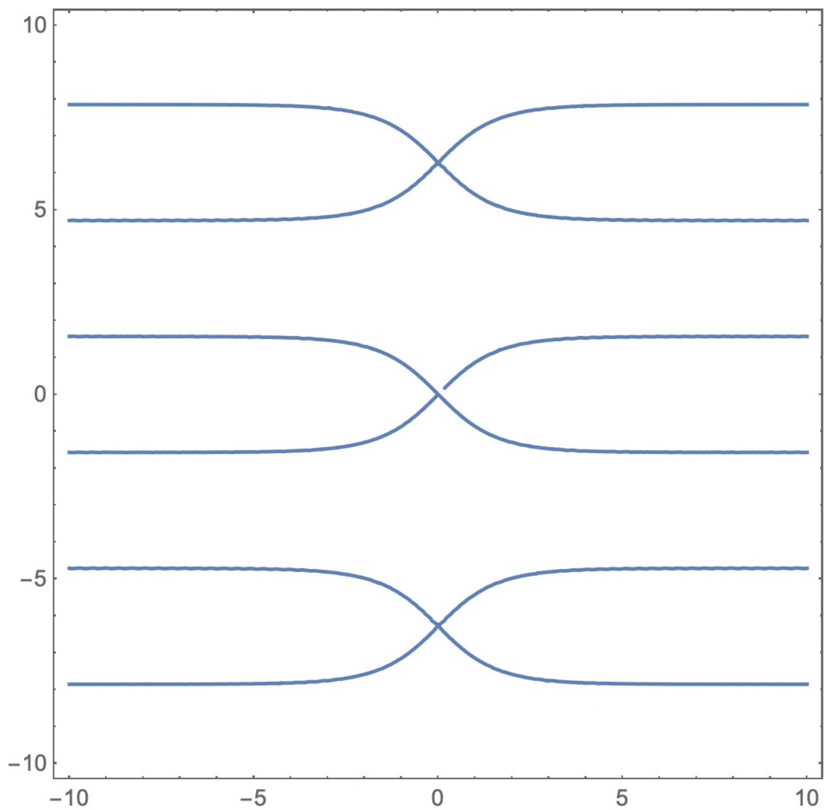
$$f'(z) = -i\sinh(z) \overset{!}{=} 0 \Rightarrow z_* = 0\,.$$

$$\mathrm{Im}\{f(z)\} = \mathrm{Im}\{-i\cosh(x+iy)\} = -\cos(y)\cosh(x)$$

$$\overset{!}{=} \mathrm{Im}\{f(0)\} = -1\,.$$

So we're looking for a path (x,y) with an implicit f^n

$$\cos(y)\cosh(x) = 1$$



Defines a family of contours, but only two pass through the origin.

To make the contour def. need vertical legs to decay:

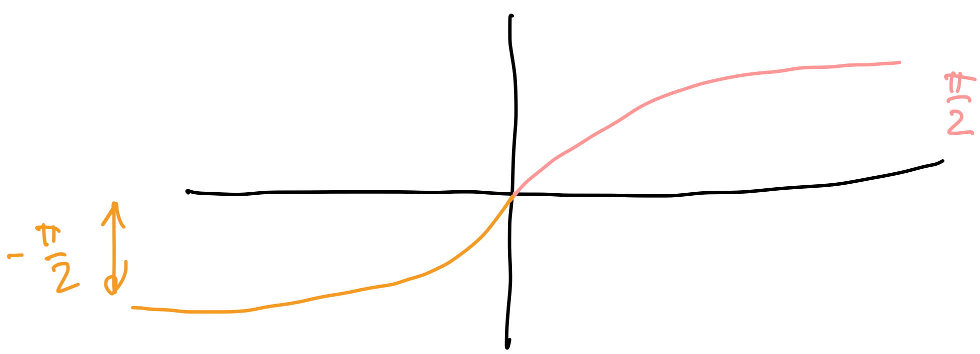
$$@\ x = \pm\infty, \quad y = \lim_{x\to\infty} \arccos\left(\frac{1}{\cosh(x)}\right) = \pm\frac{\pi}{2}$$

$$\left| \int_{y=0}^{\pi/2} \exp(i\lambda \cosh(R+iy))\, i\,dy \right| \leq$$

$$\leq \int_{y=0}^{\pi/2} \exp(-\lambda \sin(y) \sinh(R))\, dy$$

$$\lesssim \frac{1}{\sinh(R)} \quad \text{if} \quad R>0.$$

So we must go up if $R>0$
down $R<0$.



So we get by Thm 10.7,

$$I(\lambda) \approx \sqrt{\frac{2\pi}{\lambda(-i)}}\, \exp(i\lambda).$$