Junior Seminar in Mathematical Physics: Proofs from the Book

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Organizational matters

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- Meetings:
  1. Sep 18: Orientation, organization and intro to SM.
  2. Sep 25: SM1 Existence of infinite volume limit by Olivia, SM2 Transfer matrix solution of $d = 1$ by Kashti, SM3 Basic inequalities by Zach
  3. Oct 2: SM4, SM5, SM6
  4. Oct 9: SM7, SM8, SM9
  5. Oct 23: SM10, SM11, SM12
  6. Oct 30: SM13, SM14, intro to QM
  7. Nov 6: QM1, QM2, QM3
  8. Nov 13: QM4, QM5, QM6
  9. Nov 20: QM7, QM8, QM9
  10. Dec 4: QM10, QM11, QM12
  11. Dec 11: QM13, QM14, QM15

Contents

Part I
Statistical Mechanics: the classical $O(N)$ model

The main source of material for mathematical classical statistical mechanics is the Friedli and Velenik book [1]. Another useful source is the Peled Spinka lecture notes [2].

1 The basic model

Let $d, N \in \mathbb{N}_{\geq 1}$ be given (the space dimension and *spin* dimension). Let $\Lambda \equiv \Lambda_L := [-L, L]^d \cap \mathbb{Z}^d$ be a finite box within $\mathbb{Z}^d$. This box can also be considered as a finite *graph* $G = (V, E)$ i.e. a set of vertices $V = \Lambda \subseteq \mathbb{Z}^d$ and a set of edges E which indicate who is neighbor to whom. We have $|V| = (2L+1)^d$ and $|E| = d \times 2L \times (2L+1)^{d-1}$.

Pick some $\beta \in (0, \infty)$. We define the partition function of the d-dimensional classical $O(N)$ model, at inverse temperature β, initially in finite volume L as:

$$Z_{\beta,L}^{d,O(N)} := \int_{\psi:\Lambda\to\mathbb{S}^{N-1}} \exp\left(-\frac{1}{2}\beta\langle\psi, -\Delta\psi\rangle\right) \mathrm{d}\mu(\psi).$$

Here

$$\mu = \prod_{x\in\Lambda} \mu_0$$

i.e., the $|\Lambda|$-fold product measure all of the same copy of the measure μ_0. This is the a-priori measure. Naturally we choose the (normalized) volume measure on $\mathbb{S}^{N-1}$: in the case of $N = 1$ this is the (normalized) counting measure on $\{\pm 1\}$ but for $N \geq 2$ it is the natural measure which measures (normalized) area on the unit sphere.

Moreover, the symbol $\langle\psi, -\Delta\psi\rangle$ should be understood as the number

$$\langle\psi, -\Delta\psi\rangle \equiv \sum_{x,y\in\Lambda} (-\Delta)_{xy} \langle\psi_x, \psi_y\rangle_{\mathbb{R}^N}$$

and $-\Delta$ is an $|\Lambda|\times|\Lambda|$ matrix to be specified shortly.

The finite volume is a handy technical tool to avoid talking about probability measures of infinite processes. Ultimately our aim is to derive any result (read: estimate) uniformly in L so that conclusions are made about the $L\to\infty$ limit (that limit exists, but let us avoid this question for a minute).

Let us study $-\Delta$. The number of points in our box is $k:=|\Lambda|=(2L+1)^d$. Hence $\mathbb{S}^{N-1}$ is the $(N-1)$-dimensional sphere within $\mathbb{R}^N$, and we should view ψ as a map from Λ into $\mathbb{R}^N$. I.e., for any $x\in\Lambda$, $\psi_x\in\mathbb{R}^N$ and $\|\psi_x\|_2^2=1$. Thus, with some abuse of notation, if $A\in\mathrm{Mat}_{k\times k}(\mathbb{R})$ then

$$\langle\psi,A\psi\rangle\equiv\sum_{x,y\in\Lambda}\sum_{i=1}^{N}(\psi_x)_i\,A_{xy}\,(\psi_y)_i\,.$$

(Truly we should have written $A\otimes\mathbb{1}_N$ instead of A....).

The symbol $-\Delta$ is the discrete Laplacian. For every choice of L, it is a $k\times k$ matrix, with $k=(2L+1)^d$, given as follows:

$$(-\Delta v)_x:=\sum_{y\sim x}v_x-v_y\qquad(v:\Lambda\to\mathbb{R},x\in\Lambda)$$

where $y\in\Lambda$ obeys $y\sim x$ if and only if y is "adjacent" to x in Λ, that is, a nearest neighbor. In terms of matrices,

$$-\Delta=D-A$$

where A is the *adjacency* matrix of the graph Λ (i.e. it equals 1 if there is an edge between two vertices and 0 otherwise) and D is the diagonal *degree* matrix of the graph (specifying the number of edges going into any vertex). Here is the point where the discussion of boundary conditions enters: we may decide that for those vertices of Λ at the boundary, they have less neighbors than those in the bulk (*free boundary conditions*), or we may decide to wrap Λ around itself, i.e., to make a torus, to form *periodic boundary conditions.* Simultaneously, we may also consider other custom options, e.g., that the boundary is pinned to a certain range of values. By the way, the values of the boundary need not necessarily be on the sphere. In principle these choices need to be specified when $-\Delta$ is discussed.

For example let us illustrate this with the choice $d=1$ and then, say, $L=4$, (so $k=9$). We get

$$-\Delta_{\text{free}}=\begin{bmatrix}1&-1&&&&&&&\\-1&2&-1&&&&&&\\&-1&2&-1&&&&&\\&&-1&2&-1&&&&\\&&&-1&2&-1&&&\\&&&&-1&2&-1&&\\&&&&&-1&2&-1&\\&&&&&&-1&2&-1\\&&&&&&&-1&1\end{bmatrix}$$

and

$$-\Delta_{\text{periodic}}=\begin{bmatrix}2&-1&&&&&&&-1\\-1&2&-1&&&&&&\\&-1&2&-1&&&&&\\&&-1&2&-1&&&&\\&&&-1&2&-1&&&\\&&&&-1&2&-1&&\\&&&&&-1&2&-1&\\&&&&&&-1&2&-1\\-1&&&&&&&-1&2\end{bmatrix}.$$

It should be noted that we could also rewrite the bilinear form as follows. Assume $-\Delta_{\text{periodic}}$ is used for the moment (then $2d|\Lambda|=2|E|$ where E is the set of edges). Then

$$\begin{aligned}\langle\psi,-\Delta\psi\rangle&=\sum_{x\in\Lambda}\psi_x\cdot\sum_{y\sim x}\psi_x-\psi_y\\&=\sum_{x\in\Lambda}\underbrace{\psi_x\cdot\psi_x}_{=1}\underbrace{\left(\sum_{y\sim x}\right)}_{\deg(x)}-\sum_{x\in\Lambda,y\sim x}\psi_x\cdot\psi_y\\&=2|E|-2\sum_{\{x,y\}\in\Lambda:x\sim y}\psi_x\cdot\psi_y\,.\end{aligned}$$

However,

$$\left\|\psi_x-\psi_y\right\|_{\mathbb{R}^N}^2=\left\|\psi_x\right\|_{\mathbb{R}^N}^2+\left\|\psi_y\right\|_{\mathbb{R}^N}^2-2\psi_x\cdot\psi_y=2\left(1-\psi_x\cdot\psi_y\right)$$

so

$$\langle\psi,-\Delta\psi\rangle=\sum_{\{\,x,y\,\}\in\Lambda:x\sim y}\left\|\psi_x-\psi_y\right\|_{\mathbb{R}^N}^2=2\,|E|-2\sum_{\{\,x,y\,\}\in\Lambda:x\sim y}\psi_x\cdot\psi_y\,.$$

We emphasize that constant (ψ-independent) terms in the bilinear form are irrelevant since we are only interested in ratios. Hence we understand

$$\langle\psi,-\Delta\psi\rangle$$

as measuring the total amount of (squared) disagreement throughout the grid Λ: Moreover, since these are unit vectors, $\psi_x\cdot\psi_y$ gives the cosine of the angle between the two vectors as measured using the geodesic length on the sphere. Full agreement is when $\psi_x\cdot\psi_y=1$, so maximal agreement is the *minimum* value of $\langle\psi,-\Delta\psi\rangle$, which we call *the energy* usually denoted by H and also called *the Hamiltonian* or *the interaction*. We can also consider more general energy functions

$$H:\left(\mathbb{S}^{N-1}\right)^\Lambda\to\mathbb{R}\,.$$

Hence generally

$$\exp\left(-\frac{1}{2}\beta\,\langle\psi,-\Delta\psi\rangle\right)=\exp\left(-\beta H\left(\psi\right)\right)\,.$$

Since $\beta>0$, those configurations $\psi:\Lambda\to\mathbb{S}^{N-1}$ which minimize the energy functional H are those which maximize agreement throughout. For this reason these models are called *ferromagnetic*. Anti-ferromagnetic models maximize disagreement and may be obtained by $H\mapsto -H$.

Note that the matrix $-\Delta$ may be diagonalized easily (it is symmetric). E.g. for $-\Delta_{\text{periodic}}$ one may use the Fourier series (or its discrete version on Λ). The k eigenvalues lie within the interval

$$[0,4d]\;.$$

Zero is always an eigenvalue and it corresponds to the eigenvector (assuming we *avoid* pinning the field, described right below) which is a constant configuration throughout: that is the energy minimizing configuration.

Pinned boundary conditions are implemented as follows. One takes $-\Delta_{\text{free}}$ or $-\Delta_{\text{periodic}}$, but also picks some fixed $B\subseteq\Lambda$ (the boundary set, though in principle it can be any subset of Λ, also just one vertex in the middle) and a "boundary values field" $\varphi:B\to\mathbb{S}^{N-1}$ (actually the co-domain is allowed to even be a general $\mathbb{R}^N$ vector) and then instead of take the integral over all configurations $\Lambda\to\mathbb{S}^{N-1}$, restrict to the integral over the set

$$\Omega_\varphi:=\left\{\,\psi:\Lambda\to\mathbb{R}^N\;\middle|\;\psi|_B=\varphi\wedge\left\|\psi_x\right\|=1\forall x\in\Lambda\setminus B\,\right\}\,.$$

I.e., really, it is actually an integral over

$$|\Lambda|-|B|$$

spheres.

Finally, we also want to allow for an external magnetic field $h:\Lambda\to\mathbb{R}^N$. It enters into the Hamiltonian as

$$H\left(\psi\right)=\frac{1}{2}\,\langle\psi,-\Delta\psi\rangle-\langle h,\psi\rangle\;.$$

If we take h to be non-zero only along $\partial\Lambda$ (those vertices with less than $2d$ degree) then achieve a similar effect to setting the values of ψ on the boundary of a slightly large graph $\overline{\Lambda}$ to h.

The distinction from $-\Delta_{\text{periodic}}$ to $-\Delta_{\text{free}}$ is not terribly important for us now so going forward, unless otherwise noted, we shall use $-\Delta_{\text{free}}$ together with some given field $\varphi:B\to\mathbb{S}^{N-1}$ (the object φ carries the specification of its domain automatically).

In principle the measure depends on the boundary condition φ also, so we should really denote

$$Z_{d,N,\beta,L,\varphi,h}:=\int_{\psi\in\Omega_\varphi}\exp\left(-\frac{1}{2}\beta\,\langle\psi,-\Delta\psi\rangle+\beta\,\langle h,\psi\rangle\right)\prod_{x\in\Lambda}\mathrm{d}\mu_0\left(\psi_x\right).$$

1.1 Terminology

We list some terminology from statistical mechanics:

1. The quantity $Z_{d,N,\beta,L,\varphi}$ is called *the partition function*. The summand within it is called *the Gibbs factor* and the associated probability measure $\mathbb{P}_{d,N,\beta,L,\varphi}$ on Ω_φ is called *the Gibbs measure*.

2. If $N=1$ we have the *Ising model*. If $N=2$ we have the XY or $O(2)$ model. If $N=3$ we have the *classical (isotropic) Heisenberg* or $O(3)$ model. The $N\to\infty$ is sometimes referred to as the *spherical* limit.

3. The $L\to\infty$ limit (you cannot take that limit at the level of $Z_{d,N,\beta,L,\varphi}$, you must only take it for *ratios* such as $\mathbb{P}_{d,N,\beta,L,\varphi}$ or $\mathbb{E}_{d,N,\beta,L,\varphi}$) is referred to as *the thermodynamic* or *infinite volume* limit. Hence for now let us take for granted that there is some measure $\mathbb{P}_{d,N,\beta,\varphi}$ which is to be understood as a measure on the space of functions $\mathbb{Z}^d\to\mathbb{S}^{N-1}$ and is obtained as the limit of sequence of measures $\{\mathbb{P}_{d,N,\beta,L,\varphi}\}_L$. We will study the existence and nature of this limit very soon. Note that some care has to be taken with the specification of the boundary conditions here because we let $\varphi:B\to\mathbb{S}^{N-1}$ and $B\subseteq\Lambda_L$, so if L varies so does B and hence φ, in principle. Then it remains to be seen if the infinite volume object $\mathbb{P}_{d,N,\beta,\varphi}$ has any "memory" of φ or not and what sense does it make to keep carrying φ in the notation. We will study this question too below.

4. *Uniqueness of the Gibbs measure* is the general statement that $\mathbb{P}_{d,N,\beta,\varphi}$ does *not* depend on φ, i.e., there is a-posteriori only *one* infinite-volume Gibbs measure.

5. *The two-point function* is the map
$$\mathbb{Z}^d\times\mathbb{Z}^d\ni(x,y)\mapsto\mathbb{E}_{d,N,\beta,\varphi}\left[\langle\psi_x,\psi_y\rangle_{\mathbb{R}^N}\right]\in[-1,1]\,.$$
The *truncated-two-point-function* is
$$\mathbb{Z}^d\times\mathbb{Z}^d\ni(x,y)\mapsto\left(\mathbb{E}_{d,N,\beta,\varphi}\left[\langle\psi_x,\psi_y\rangle_{\mathbb{R}^N}\right]-\langle\mathbb{E}_{d,N,\beta,\varphi}[\psi_x],\mathbb{E}_{d,N,\beta,\varphi}[\psi_y]\rangle\right)\in[-1,1]\,.$$
Measures how far away spins are correlated with each other.

6. Magnetization at a site x is $\mathbb{E}_{d,N,\beta,\varphi}[\psi_x]$ and total magnetization is
$$m_{d,N,\beta,\varphi}:=\lim_{L\to\infty}\mathbb{E}_{d,N,\beta,\varphi,\Lambda}\left[\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\psi_x\right]\in\mathbb{R}^N\,.$$

7. The quantity $F_{d,N,\beta,L,\varphi}:=-\frac{1}{\beta}\log\left(Z_{d,N,\beta,L,\varphi}\right)$ is called *the free energy*. Its volume-density is
$$f_{d,N,\beta,\varphi}:=\lim_{L\to\infty}\frac{1}{|\Lambda_L|}F_{d,N,\beta,L,\varphi}$$
should this limit exist.

8. There are two main phenomena we are interested in when studying this model:
   (a) *Long range order* (low temperatures, high β): the system has intrinsic global, collective magnetization, $m_{d,N,\beta,\varphi}\neq 0$, $\mathbb{E}_{d,N,\beta,\varphi}\left[\langle\psi_x,\psi_y\rangle_{\mathbb{R}^N}\right]$ does not decay as $\|x-y\|\to\infty$.
   (b) *Disordered phase* (high temperatures, low β): the system is does not show preference to any particular direction, $m_{d,N,\beta,\varphi}=0$, $\mathbb{E}_{d,N,\beta,\varphi}\left[\langle\psi_x,\psi_y\rangle_{\mathbb{R}^N}\right]$ decays as $\|x-y\|\to\infty$ (however slowly). *Correlation length* is the rate of exponential decay.
   (c) *Phase transition*: the shift of the system from one type of the above behavior to another type of the above behavior as a continuous parameter (usually the inverse temperature) is varied.
   (d) *Criticality* or *critical point*: The set of parameters of the system on the boundary between two phases, i.e. the point (or line, or manifold) of phase transition, $\mathbb{E}_{d,N,\beta,\varphi}\left[\langle\psi_x,\psi_y\rangle_{\mathbb{R}^N}\right]$ decays but *polynomially*.

9. *Universality* refers to a type of behavior of certain quantities, usually asymptotically, usually near the critical point.

10. *Gaussianity*, *free-field*, or *spin-wave* behavior is the phenomenon that the random field $\psi:\Lambda\to\mathbb{R}^N$ behaves as if it had a Gaussian measure (it does not). Gaussian upper bounds are upper bounds (e.g., on the two point function) in terms of the two-point function of the *Gaussian field* or the *Gaussian free field*.

11. By *symmetry* we refer to the operation of rotating a vector in $\mathbb{R}^N$ from one direction to another, and observing that *something* remains the same. For example, the inner product

$$\langle \psi_x, \psi_y \rangle$$

is invariant if we apply an orthogonal (rotation) matrix on both vectors simultaneously. The group of $N \times N$ orthogonal matrices, $O(N)$, is the main group of symmetries of our model, since $Z_{d,N,\beta,L,\varphi=0}$ possesses a global $O(N)$ symmetry, in the sense that the probability density (or the push forward of the probability measure if you wish) remains the same if we apply a global $M \in O(N)$ matrix to the magentization vector on all vertices of Λ. There is a slight issue here with boundary conditions which would spoil that, so in principle if $\varphi \neq 0$ then we have to rotate the boundary conditions too.

    (a) *Continuous symmetry* means that the group of symmetries is a smooth manifold as opposed to a discrete finite group. Compare the discontinuous case $N = 1$ (whence $O(1) = \{ \pm 1 \} \cong \mathbb{Z}_2$) with the continuous case $N \geq 2$ ($O(2) \cong \mathbb{S}^1$). It turns out that discrete versus continuous symmetry plays a role.

    (b) *Symmetry breaking or spontaneous symmetry breaking* is the situation where the finite volume Gibbs *factor* is invariant under some symmetry (for any given finite volume) yet *the infinite volume measure is not.* This phenomenon is of utmost interest to us and will have parallels in quantum mechanics as well. Long-range order from above is an example of such symmetry breaking whereas the disordered phase is the absence of symmetry breaking.

    (c) *Mermin–Wagner* (sometimes Mermin–Wagner–Hohenberg) is the general result that there is no continuous symmetry breaking if $d \leq 2$.

12. *Kosterlitz–Thouless* or *Berezinskii-Kosterlitz–Thouless* is a phenomenon of a whole critical interval of temperatures, usually $[\beta_c, \infty)$.

13. *Mass gap* or *massive* field is a field whose two-point function decays exponentially in $\|x - y\|$. This is called in this way because in a Gaussian free field, if we replace the Laplacian

$$-\Delta$$

with a massive Laplacian

$$-\Delta + m^2 \mathbb{1}$$

then we indeed get exponentially decay of the two point function with rate m.

2 The GFF

The *Gaussian free field* is defined in almost identical way to the $O(N)$ model, with the exception of replacing the a-priori measure μ_0 from the volume measure on $\mathbb{S}^{N-1}$ by the Lebesgue measure on $\mathbb{R}^N$. This can be risky because now we run the risk of integrals not converging. This danger can be mitigated in various different ways, either through boundary conditions or the addition of a mass to the Laplacian.

Let us study the model whose partition function is

$$Z^{\mathrm{GFF}}_{d,N,\beta,L} := \int_{\psi:\Lambda\to\mathbb{R}^N} \exp\left(-\frac{1}{2}\beta \left\langle \psi, \left(-\Delta + m^2 \mathbb{1}\right) \psi \right\rangle\right) \mathrm{d}\psi$$

where by $\mathrm{d}\psi$ we mean the Lebesgue measure on $\left(\mathbb{R}^N\right)^\Lambda$. Because $m \neq 0$ then all Gaussian integrals converge regardless of the spectrum of $-\Delta$ (whether it has eigenvalue zero or not; with Dirichlet boundary conditions for example $-\Delta$ anyway has no zero mode and then the integral converges even with $m = 0$). We then have

$$Z^{\mathrm{GFF}}_{d,N,\beta,L} = \int_{\psi:\Lambda\to\mathbb{R}^N} \exp\left(-\frac{1}{2}\beta \left\langle \psi, \left(-\Delta + m^2 \mathbb{1}\right) \psi \right\rangle\right) \mathrm{d}\psi = \frac{(2\pi)^{N\frac{|\Lambda|}{2}} \beta^{-\frac{N|\Lambda|}{2}}}{\sqrt{\det_\Lambda \left(-\Delta + m^2 \mathbb{1}\right)^N}}$$

and

$$\int_{\psi:\Lambda\to\mathbb{R}^N} \exp\left(-\frac{1}{2}\beta \left\langle \psi, \left(-\Delta + m^2 \mathbb{1}\right) \psi \right\rangle + \langle J, \psi \rangle\right) \mathrm{d}\psi = Z^{\mathrm{GFF}}_{d,N,\beta,L} \exp\left(\frac{1}{2\beta} \left\langle J, \left(-\Delta + m^2 \mathbb{1}\right)^{-1} J \right\rangle\right)$$

so that

$$\mathbb{E}\left[\psi_x \cdot \psi_y\right] = \frac{N}{\beta} \left[\left(-\Delta + m^2 \mathbb{1}\right)^{-1}\right]_{xy} \qquad (x, y \in \Lambda) .$$

First note that beyond the overall constant outside, $\left[\left(-\Delta+m^2\mathbb{1}\right)^{-1}\right]_{xy}$ is independent of β. In particular the fate of exponential decay or not of the two-point function is independent of β and hence the GFF has no phase transitions. This is also clear by looking at the integral and making a change of variable $\psi_x \mapsto \sqrt{\beta}\psi_x$.

Note that for $\|x\| \gg \frac{1}{m}$, we have

$$\left[\left(-\Delta+m^2\mathbb{1}\right)^{-1}\right]_{0,x} \sim c_d\left(m\right)\|x\|^{-\frac{d-1}{2}}\exp\left(-\frac{1}{m}\|x\|\right).$$

However, if we're really interested in the $m=0$ case, then we have the following behavior ($m\to 0^+$ asymptotics at fixed large $\|x\|$):

$$\left[\left(-\Delta+m^2\mathbb{1}\right)^{-1}\right]_{0,x} = \begin{cases} \frac{1}{2m}-\frac{\|x\|}{2}+O\left(m\right) & d=1 \\ \frac{1}{2\pi}\log\left(\frac{1}{m}\right)-\frac{2}{\pi}\log\left(\|x\|\right)+C+o\left(1\right) & d=2 \\ \frac{\Gamma\left(\frac{d}{2}-1\right)}{4\pi^{\frac{d}{2}}}\|x\|^{2-d} & d\geq 3 \end{cases}.$$

We see that in $d\leq 2$ this limit $m\to 0^+$ does not exist. One way to cure this is to always consider differences:

$$\left[\left(-\Delta+m^2\mathbb{1}\right)^{-1}\right]_{0,x}-\left[\left(-\Delta+m^2\mathbb{1}\right)^{-1}\right]_{0,1} = \begin{cases} -\frac{\|x\|}{2}+\frac{1}{2} & d=1 \\ -\frac{2}{\pi}\log\left(\|x\|\right)+o\left(1\right) & d=2 \\ \frac{\Gamma\left(\frac{d}{2}-1\right)}{4\pi^{\frac{d}{2}}}\left(\|x\|^{2-d}-1\right) & d\geq 3 \end{cases}.$$

This suggests that objects like

$$\mathbb{E}\left[\|\psi_x-\psi_0\|^2\right]$$

are more appropriate than

$$\mathbb{E}\left[\psi_x\cdot\psi_0\right]$$

where studying $d\leq 2$ massless Gaussian fields.

3 Olivia: The existence of the infinite volume Gibbs measure and the DLR conditions

4 Zach: Inequalities: Ginibre, FKG, GHS

5 Kashti: Full solution in $d = 1$ via transfer matrices

6 The Aizenman-Simon proof of the disordered phase for all small β and other cluster expansions

7 The Peierls solution for $N = 1$, $d \geq 2$

8 Mermin-Wagner (Dobrushin-Shlosman, Pfister)

9 The McBryan-Spencer upper bound for $N \geq 2$, $d \leq 2$

10 Reflection positivity and chessboard inequalities

11 The Fröhlich-Simon-Spencer proof of LRO for $N \geq 2$, $d \geq 3$

12 The Lee-Yang theorem

Part II
Quantum mechanics: Anderson localization and topological insulators

References

[1] Sandro Friedli and Yvan Velenik. *Statistical Mechanics of Lattice Systems: A Concrete Mathematical Introduction.* Cambridge University Press, Cambridge, 2017.

[2] Ron Peled and Yinon Spinka. Lectures on the spin and loop $O(n)$ models. *arXiv preprint*, arXiv:1708.00058, 2017. Version v3, revised 3 Jul 2019.