Microlocal Morse theory of wrapped Fukaya categories

Sheel Ganatra, John Pardon, and Vivek Shende

24 September 2018

Abstract

Consider on the one hand the partially wrapped Fukaya category of a cotangent bundle stopped at an appropriately stratifiable singular isotropic. Consider on the other hand the derived category of sheaves with microsupport along the singular isotropic. We show here that the former is equivalent to the compact objects of the latter. Applications include a sheaf-theoretic description of the wrapped Fukaya categories of plumbings and of codimension one Weinstein hypersurfaces of cosphere bundles. While inspired by the Nadler–Zaslow correspondence, our results do not depend on it.

1 Introduction

Let $X$ be a Liouville manifold (or sector) and $\Lambda \subseteq \partial_\infty X$ a closed subset at infinity. We write $\mathcal{W}(X, \Lambda)$ for the Fukaya category in which objects are Lagrangians disjoint at infinity from $\Lambda$ and in which morphisms are defined by wrapping in the complement of $\Lambda$. This category is defined and studied in [7, 52, 19, 20], and one has particularly good control over it when $X$ is Weinstein and $\Lambda$ is a (possibly singular) isotropic.

Our interest here is in the case of cotangent bundles, $X = T^*M$. In this case, we can also study the sheaves on $M$, microsupported inside $\Lambda$. We write $\text{Sh}(M)$ for the (dg-)category of unbounded complexes of sheaves of $\mathbb{Z}$-modules on $M$, localized along the stalkwise quasi-isomorphisms. The data of an object $F \in \text{Sh}(M)$ is the assignment of some complex of $\mathbb{Z}$-modules $F(U)$ (called the sections of $F$ over $U$) to every open set $U \subseteq M$, along with restriction maps $F(U) \to F(V)$ for inclusions $V \subseteq U$.

The microsupport of a sheaf $F$ is a closed conical locus $\text{ss}(F) \subseteq T^*M$. Its role is to encode when restriction maps are quasi-isomorphisms. The basic idea is illustrated by the fact [26, Corollary 5.4.19] that for a smooth function $\phi : M \to \mathbb{R}$ and real numbers $a < b$, one has $F(\phi^{-1}(-\infty, b)) \xrightarrow{\sim} F(\phi^{-1}(-\infty, a))$ whenever the graph of $d\phi$ is disjoint from $\text{ss}(F)$ over $\phi^{-1}([a, b])$. For example, armed with the further knowledge that the microsupport of the constant sheaf is the zero section, one may conclude from this that the cohomology of sublevel sets of $\phi$ changes only at levels where $d\phi = 0$.

We prove here the following result:

**Theorem 1.1.** Let $M$ be a real analytic manifold, and let $\Lambda \subseteq T^*M$ be a subanalytic closed isotropic subset. Then $\text{Perf} \mathcal{W}(T^*M, \Lambda)^{op}$ is equivalent to the category of compact objects in $\text{Sh}_\Lambda(M)$. Moreover, this equivalence carries the linking disk at any smooth Legendrian point $p \in \Lambda$ to a co-representative of the microstalk functor at $p \in \Lambda$. 


Rather than writing $\mathcal{W}(T^*M, \Lambda)^{\text{op}}$, one could equivalently negate the Liouville form and write $\mathcal{W}(-T^*M, \Lambda)$, or pull back by the antipodal map and write $\mathcal{W}(T^*M, -\Lambda)$. We also remind the reader that the notation $\text{Perf}$ refers to the idempotent-completed pre-triangulated closure of an $A_\infty$ or dg category (see §A.5).

The reader is cautioned that the compact objects of $\text{Sh}_\Lambda(M)$ do not necessarily have perfect stalks or bounded homological degree. That is, they need not be constructible sheaves in the usual sense. The necessity of considering such objects on the sheaf side was pointed out in [35], where the above result was implicitly conjectured by the title.

The main idea of the proof of Theorem 1.1 is to first establish the result by direct calculation when $\Lambda = N^*_\infty S$ is the union of conormals of strata of a triangulation $S$ of $M$, and then to show that both sides transform in the same way when $\Lambda$ becomes smaller. In Section 3, we formulate these two steps as axioms for a system of categories $\Lambda \mapsto \mathcal{C}(\Lambda)$ parameterized by closed subanalytic singular isotropics $\Lambda \subseteq S^*M$. The remainder of the paper consists in establishing these axioms on both sides. On the sheaf side, these amount to the (standard) fact that the category of sheaves constructible with respect to an appropriate stratification is the representation category of the poset of the stratification, plus stratified Morse theory [21] as reformulated microlocally with sheaf coefficients in [26]. We describe this in detail in Section 4. On the Fukaya side, the analogue of this microlocal Morse theory is the wrapping exact triangle of [20]. The structural results of [20] are also fundamentally useful in the computation of the Fukaya category of a cotangent bundle, with wrapping stopped by the union of conormals to a triangulation. This is carried out in Section 5.

That there should be a relationship between constructible sheaves on $M$ and the Fukaya category of $T^*M$ was first suggested by Nadler and Zaslow in [37, 33]. The appeal of this statement was that the category of constructible sheaves is essentially combinatorial, leading readily to computations.

By contrast with their work, our theorem concerns wrapped, rather than infinitesimal, Fukaya categories. Wrapped Floer cohomology and the wrapped Fukaya category are more complicated, global, typically infinite rank, invariants than their infinitesimal counterparts studied in [37, 33]. From this point of view, it may seem at first surprising that sheaves can model both infinitesimal and wrapped categories. This possibility was suggested by [35]; as noted there one can expect to recover the infinitesimal category from the wrapped category, but not vice versa. A discussion from our point of view, recovering a version of the original Nadler–Zaslow correspondence from Theorem 1.1, appears in Section 6.3.

Because we treat the wrapped category, our result has broader implications than the original [37, 33]. In particular, it is necessary to use the wrapped category on the $A$-model side of mirror symmetry to match categories of coherent sheaves when the $B$-model side is not both smooth and compact. Additionally, the connection between sheaves and Legendrian contact homology is best understood from the wrapped perspective. We detail in Section 6 the extent to which various sheaf calculations [17, 50, 49, 48, 27, 34, 35, 18] can now be understood as computations of wrapped Fukaya categories. We also give a new computation of the wrapped Fukaya categories of cotangent bundles (in particular generalizing [4, 6] to the case where $M$ may be non-compact or have boundary), and we compute the wrapped Fukaya categories of plumbings.
In the proof of Theorem 1.1, we do not rely on the results [37, 33] of Nadler and Zaslow.\footnote{We emphasize this because some symplectic geometers do not view [37] as fully rigorous; the worry involves the construction of the infinitesimal Fukaya category given in [37]. We do not wish to express any opinion on this matter, beyond noting that we are aware of it and that it does not affect the present results.} In fact it would not have helped to do so: beyond the foundations we set up in [19, 20], the main new ideas and difficulties in the proof of Theorem 1.1 concern the geometry of wrapping (as underlies the results of [20]), as opposed to the calculation of Floer cohomology. In fact, the only Floer cohomology calculations which need to be made in this entire article are between Lagrangians which intersect in at most one point.

In particular, we expect the proofs of the results in this paper, in particular Theorem 1.1, would apply in the case of more general (e.g. sphere spectrum) coefficients, provided one has access to the definitions of the sheaf and Fukaya categories, respectively, in these settings.

**Convention.** Throughout this document, we work in the setting of dg- and, equivalently, $A_\infty$-categories over $\mathbb{Z}$ (or more generally any commutative ring). We only ever consider “derived” functors, we only ever mean “homotopy” limits or colimits, and we systematically omit the word “quasi”. By modules, we mean dg- or $A_\infty$-modules, e.g. by $\mathbb{Z}$-modules we mean the category of chain complexes of abelian $\mathbb{Z}$-modules, localized at quasi-isomorphisms, except when, as in this sentence, we qualify it with the word ‘abelian’. In §A we detail our assumptions about these categories and collect relevant categorical notions which will appear throughout the paper.

### 1.1 Acknowledgements

We thank Mohammed Abouzaid, Roger Casals, Tobias Ekholm, Benjamin Gammage, David Nadler, Amnon Neeman, and Lenhard Ng for helpful discussions, some of which took place during visits to the American Institute of Mathematics.

S.G. would like to thank the Mathematical Sciences Research Institute for its hospitality during a visit in Spring 2018 (supported by NSF grant DMS–1440140) during which some of this work was completed. This research was conducted during the period J.P. served as a Clay Research Fellow and was partially supported by a Packard Fellowship and by the National Science Foundation under the Alan T. Waterman Award, Grant No. 1747553. V.S. was supported by the NSF CAREER grant DMS–1654545.

### 2 Stratifications

Let $X$ be a topological space. By a stratification $S$ of $X$, we mean a locally finite decomposition into disjoint locally closed subsets $\{X_\alpha\}_{\alpha \in S}$, called strata, such that each boundary $\overline{X_\alpha} \setminus X_\alpha$ is a union of other strata $X_\beta$. The collection of strata $S$ is naturally a poset, in which there is a map $\beta \to \alpha$ iff $X_\alpha \subseteq \overline{X_\beta}$. When $X$ is a manifold, we implicitly require the strata to be as well.

**Remark 2.1.** The poset $S$ does not generally reflect the topology of the space $X$. Conditions under which it does (contractibility of various strata/stars) are well known and recalled below.
We will say a subset $Y \subseteq X$ is $S$-constructible when it is a union of strata of the stratification $S$ of $X$. We say that a stratification $T$ refines a stratification $S$ when the strata of $S$ are $T$-constructible.

We recall that an abstract simplicial complex on a vertex set $V$ is a collection $\Sigma$ of nonempty finite subsets of $V$, containing all singletons and all subsets of elements of $\Sigma$. By a simplicial complex, we mean the geometric realization $|\Sigma|$ of an abstract simplicial complex $\Sigma$; it comes with a stratification by the ‘open simplices’ (which, of course, are locally closed, not necessarily open, subsets of $|\Sigma|$). We say a stratification $S$ on $X$ is a triangulation when there is a homeomorphism $|\Sigma| \cong X$ identifying stratifications.\(^{2}\) Note the following are not triangulations: a stratification of a circle into single point and its complement, or into two points and their complement. The stratification into three points and their complement is a triangulation.

The open star of a stratum is the union of strata whose closures contain it. Taking stars reverses the inclusion: we have $X_\alpha \subseteq \overline{X_\beta} \iff \text{star}(X_\beta) \subseteq \text{star}(X_\alpha)$. Note that $\text{star}(X_\alpha) \cap \text{star}(X_\beta) = \bigcup_{\alpha, \beta \to \gamma} \text{star}(X_\gamma)$. For triangulations, we can do better: $\text{star}(X_\alpha) \cap \text{star}(X_\beta) = \text{star}(X_\gamma)$ where $\gamma$ is the simplex spanned by the vertices of $\alpha$ union the vertices of $\beta$ (if this simplex is present), and otherwise $\text{star}(X_\alpha) \cap \text{star}(X_\beta) = \emptyset$.

Given a $C^1$ stratification $S$ of a $C^1$ manifold $M$, we write $N^*S \subseteq T^*M$ for the union of conormals to the strata. The stratification is said to be Whitney (a) iff $N^*S$ is closed; or, as it is usually formulated, if limits of tangent spaces to strata contain the tangent spaces of their boundary strata. In any setting where we consider cotangent bundles, we will only ever consider Whitney (a) stratifications.

Eventually we will restrict to the setting of analytic manifolds and subanalytic stratifications. We recall that a set is defined to be subanalytic when locally (i.e. in a neighborhood of every point of its closure) it is the analytic image of a relatively compact semianalytic set. The canonical modern reference is [11].\(^{3}\)

By a subanalytic stratification, we mean a stratification in which all strata are subanalytic. By a subanalytic triangulation, we mean a triangulation $f : |\Sigma| \cong X$, where the restriction of $f$ to every closed simplex is analytic and the restriction of $f$ to every open simplex is an immersion. It is a fundamental result that for any locally finite subanalytic partition $M = \bigsqcup M_\alpha$, there is a subanalytic triangulation in which all $M_\alpha$ are constructible. See [11] for proofs of the above results.

In [26] there is a notion of $\mu$-stratification, this being a certain strengthening of the Whitney conditions; it is shown in [26, Thm. 8.3.20] that any subanalytic stratification can be refined to a $\mu$-stratification.

\(^{2}\)Strictly speaking, it would be better to define a triangulation as a simplicial complex $(V, \Sigma)$ together with a map $|\Sigma| \cong X$ identifying stratifications. This distinction, however, will not concern us.

\(^{3}\)Wherever we have written ‘subanalytic’, one could substitute any fixed analytic-geometric category, so long as one is willing (we are) to have strata that are $C^p$ for arbitrarily large $p$ but not necessarily $C^\infty$ [54].
3 Microlocal Morse categories

3.1 Strata poset categories and refinement functors

Let $S$ be a stratification. We fix the following notation for the Yoneda embedding:

$$S \to \text{Fun}(S^{\text{op}}, \text{Set}),$$

$$\alpha \mapsto \text{Hom}(\cdot, \alpha) =: 1_{\star(\alpha)}. \quad (3.1)$$

Note that

$$\text{Hom}(1_{\star(\alpha)}, 1_{\star(\beta)}) = 1_{\star(\beta)}(\alpha) = \begin{cases} \{1\} & \text{star}(\alpha) \subseteq \text{star}(\beta) \\ \emptyset & \text{otherwise}. \end{cases} \quad (3.2)$$

For any $S$-constructible open set $U$, we introduce the functor $1_U \in \text{Fun}(S^{\text{op}}, \text{Set})$ defined by the analogous formula

$$\text{Hom}(1_{\star(\alpha)}, 1_U) = 1_U(\alpha) := \begin{cases} \{1\} & \text{star}(\alpha) \subseteq U \\ \emptyset & \text{otherwise}. \end{cases} \quad (3.3)$$

Note that $\text{star}(\alpha) \subseteq U$ iff $\alpha \subseteq U$.

Now let $S'$ be a stratification refining $S$. There is a natural map $r : S' \to S$, sending a stratum in $S'$ to the unique stratum in $S$ containing it. We write

$$r^* : \text{Fun}(S'^{\text{op}}, \text{Set}) \to \text{Fun}(S^{\text{op}}, \text{Set})$$

for the pullback of functors along this map $r$. For $\tau' \in S'$ and an $S$-constructible open set $U$, we have

$$\text{Hom}(1_{\star(\tau')}, r^*1_U) = (r^*1_U)(\tau') = 1_U(r(\tau')) = \begin{cases} \{1\} & \text{star}(r(\tau')) \subseteq U \\ \emptyset & \text{otherwise}. \end{cases} \quad (3.5)$$

Since $U$ is open and $S$-constructible, we have $\text{star}(r(\tau')) \subseteq U$ iff $\text{star}(\tau') \subseteq U$, so we conclude that $r^*1_U = 1_U$.

We now linearize. We write $Z[S]$ for the linearization of a poset $S$. We write $\text{Mod} S$ for the category of modules $\text{Fun}(S^{\text{op}}, \text{Mod} Z) = \text{Fun}(Z[S]^{\text{op}}, \text{Mod} Z)$, and we use $r^* : \text{Mod} S \to \text{Mod} S'$ for pullback of modules as above. As with any pullback of modules, this functor has a left adjoint given by extension of scalars, which by abuse of notation we write as $r : \text{Mod} S' \to \text{Mod} S$ due to the commuting diagram

$$\begin{array}{ccc}
S' & \xrightarrow{s+1_{\star(s)}} & \text{Mod} S' \\
\downarrow r & & \downarrow r \\
S & \xrightarrow{s+1_{\star(s)}} & \text{Mod} S. \end{array} \quad (3.6)
$$

Restriction of scalars $r^*$ is co-continuous, so its left adjoint $r$ extension of scalars preserves compact objects, giving a map $r : \text{Perf} S' \to \text{Perf} S$ (which can also be viewed as the canonical extension of $r : S' \to S$ to the idempotent completed pre-triangulated hulls).
3.2 A category for any $\Lambda$

We now wish to define a \textit{microlocal Morse category} $\mathcal{C}(\Lambda)$ for any subanalytic singular isotropic $\Lambda \subseteq S^*M$, together with functors $\mathcal{C}(\Lambda') \to \mathcal{C}(\Lambda)$ for inclusions $\Lambda' \supseteq \Lambda$. We define this system of categories $\Lambda \mapsto \mathcal{C}(\Lambda)$, the \textit{microlocal Morse theatre}, by formulating axioms which characterize it uniquely.

The previous subsection defined categories $\Perf S$ together with functors $r: \Perf S' \to \Perf S$ whenever $S'$ is a refinement of $S$. For our current purpose, these categories do not have the correct significance for general stratifications $S$ (compare Remark 2.1). As such, we will consider these categories only for triangulations $S$.\footnote{In fact, there are weaker conditions on a stratification $S$ (which are satisfied if $S$ is a triangulation) implying that $\Perf S$ is the correct category to associate to $S$.} The microlocal Morse theatre is an extension of this functor $S \mapsto \Perf S$ on triangulations in the following sense:

**Definition 3.3.** A \textit{microlocal Morse pre-theatre} $\Lambda \mapsto \mathcal{C}(\Lambda)$ is a functor from the category of subanalytic singular isotropics inside $S^*M$ to the category of dg-categories over $\mathbb{Z}$, together with an isomorphism of functors $(S \mapsto \mathcal{C}(N^*S)) = (S \mapsto \Perf S)$ on $\mu$-triangulations $S$.

**Remark 3.2.** Any isomorphism of functors $(S \mapsto H^*(\mathcal{C}(N^*_S))) = (S \mapsto H^*(\Perf S))$ automatically lifts to an isomorphism $(S \mapsto \mathcal{C}(N^*_S)) = (S \mapsto \Perf S)$ by Lemma 5.30. This will be crucial when discussing Fukaya categories.

We will characterize the microlocal Morse theatre in terms of microlocal Morse theory.\footnote{More conventionally [21], this is called stratified Morse theory. We find the term ‘microlocal’ more descriptive, and also the word stratified would otherwise take on too many meanings in this article.} Let $f: M \to \mathbb{R}$ be a function and $S$ a stratification. An intersection of $\Gamma_d$ with $N^*S$ is called an $S$-critical point, which is said to be Morse if it is a transverse intersection at a smooth point of $N^*S$. The function $f$ is said to be $S$-Morse when all its $S$-critical points are Morse. When $S$ is subanalytic, such functions are plentiful, and can be chosen analytic. (See [21, Thm. 2.2.1] for this assertion, which is collected there from various results in the literature.) More generally, for any singular isotropic $\Lambda \subseteq S^*M$, a $\Lambda$-critical point of $f$ is an intersection of $\Gamma_d$ with the union of the zero section and the cone over $\Lambda$, it is said to be Morse if transverse and at a smooth point, and any $f$ whose $\Lambda$-critical points are all Morse is called $\Lambda$-Morse.

**Definition 3.3.** In any microlocal Morse pre-theatre $\Lambda \mapsto \mathcal{C}(\Lambda)$, the \textit{Morse characters} $X_{\Lambda,p}(f,\epsilon, S) \in \mathcal{C}(\Lambda)$ are defined as follows for smooth Legendrian points $p \in \Lambda$.

Let $f: M \to \mathbb{R}$ be an analytic function with a Morse $\Lambda$-critical point at $p$ with critical value 0, no other $\Lambda$-critical points with critical values in the interval $[-\epsilon, \epsilon]$, and with relatively compact sublevel set $f^{-1}(-\infty, \epsilon)$. Let $S$ be a triangulation for which $\Lambda \subseteq N^*_S$ and for which both $f^{-1}(-\infty, -\epsilon)$ and $f^{-1}(-\infty, \epsilon)$ are $S$-constructible.

The Morse character $X_{\Lambda,p}(f,\epsilon, S)$ is then defined as the image of

$$\text{cone}(1_{f^{-1}(-\infty, -\epsilon)} \to 1_{f^{-1}(-\infty, \epsilon)}) \in \Perf S = \mathcal{C}(N^*_S).$$

(3.7)

under the map $\mathcal{C}(N^*_S) \to \mathcal{C}(\Lambda)$. 
The Morse character $\mathcal{X}_{\Lambda,p}(f,\epsilon,S) \in \mathcal{C}(\Lambda)$ depends \textit{a priori} on the ‘casting directors’ $(f,\epsilon,S)$. Casting directors $(f,\epsilon)$ exist at any smooth Legendrian point $p \in \Lambda$ by general position, and $S$ exists by the following result:

**Proposition 3.4** ([26, 8.3.10]). For any closed subanalytic singular isotropic $\Lambda \subseteq S^* M$, there exists a subanalytic stratification $S$ of $M$ such that $\Lambda \subseteq N^*_\Lambda S$. □

**Definition 3.5.** A \textit{microlocal Morse theatre} is a microlocal Morse pre-theatre $\Lambda \mapsto \mathcal{C}(\Lambda)$ such that for any inclusion $\Lambda \subseteq \Lambda'$ and any collection of Morse characters $\mathcal{X}_{\Lambda',p}(f,\epsilon,S) \in \mathcal{C}(\Lambda')$ at smooth Legendrian points $p \in \Lambda' \setminus \Lambda$ with at least one in every component of the smooth Legendrian locus of $\Lambda' \setminus \Lambda$, the functor $\mathcal{C}(\Lambda') \to \mathcal{C}(\Lambda)$ is (the idempotent completion of) the quotient by these Morse characters.

The definition of a microlocal Morse theatre allows one to readily compute any particular microlocal Morse category $\mathcal{C}(\Lambda)$: embed $\Lambda$ into some $N^*_\Lambda S$ using Proposition 3.4, cast Morse characters in $\mathcal{C}(N^*_\Lambda S) = \text{Perf} S$ for all Legendrian components of $N^*_\Lambda S \setminus \Lambda$, and take the quotient of $\text{Perf} S$ by these characters and idempotent complete. It follows that:

**Proposition 3.6.** Any two microlocal Morse theatres $\Lambda \mapsto \mathcal{C}(\Lambda)$ are uniquely isomorphic. □

A \textit{dramatic realization} is a particular construction of the microlocal Morse theatre $\Lambda \mapsto \mathcal{C}(\Lambda)$. We give two dramatic realizations, namely via sheaves and via Lagrangians in Sections 4 and 5, respectively. Both these dramatic realizations cast the Morse characters as certain familiar objects. They moreover show that the Morse characters in fact depend only on $p$ and are independent of the casting directors.

**Theorem 3.7.** The microlocal Morse theatre $\Lambda \mapsto \mathcal{C}(\Lambda)$ exists, and the Morse characters $\mathcal{X}_{\Lambda,p} \in \mathcal{C}(\Lambda)$ are independent of the casting directors and form a local system over the smooth Legendrian locus of $\Lambda$.

**Proof.** This follows from either Theorem 4.21 or Theorem 5.36. □

**Proof of Theorem 1.1.** Combine Theorems 4.21 and Theorem 5.36. □

In fact, both dramatic realizations show that $\mathcal{C}(\Lambda)$ is invariant under contact isotopy of $S^* M$, something which is not apparent from the present combinatorial prescription. This is immediate on the Fukaya side, and on the sheaf side it is ‘sheaf quantization’ [23]. In fact, there are even stronger invariance statements: it is shown in [20] that in fact $\mathcal{C}(\Lambda)$ is invariant under isotopy of $S^* M \setminus \Lambda$ inside $S^* M$; meanwhile, it is shown in [36] that $\mathcal{C}(\Lambda)$ is invariant under “non-characteristic deformations” of $\Lambda$.

**Remark 3.8.** The construction of this subsection makes sense in any stable setting, e.g. over the sphere spectrum. To show existence of the microlocal Morse theatre in such a more general setting, one could set up either microlocal sheaf theory or the Fukaya category over the sphere spectrum. In principle, one could also show existence directly from the stratified Morse theory of [21], as it already establishes results about homotopy types of spaces (not just their cohomologies). A more interesting question is whether any symplectically invariant statement can be made beyond the stable setting.
4 Sheaf categories

We recall the general formalism of sheaves, and properties of stratifications. We then recall from [26] the notion of microsupport, and the category $\text{Sh}_\Lambda(M)$ of sheaves on $M$ whose microsupport at infinity is contained in $\Lambda$. We show that the assignment $\Lambda \mapsto \text{Sh}_\Lambda(M)$ is a microlocal Morse theatre in the sense of Definition 3.5.

4.1 Categories of sheaves and functors between them

Here we give a brief review of the general formalism of sheaves. Our presentation is somewhat modern in that we never discuss sheaves of abelian groups, rather we work at the dg level and with unbounded complexes from the beginning, but it is essentially the same as any standard account such as [25, 26, 43], complemented by [51] in order to work with unbounded complexes, and in particular for the proper base change theorem in this setting.

Given a topological space $T$, we write $\text{Op}(T)$ for the category whose objects are open sets and morphisms are inclusions. A ($\mathbb{Z}$-module valued) presheaf on $T$ is by definition a functor $\text{Op}(T)^{\text{op}} \to \text{Mod}$. In particular, a presheaf $\mathcal{F}$ takes a value $\mathcal{F}(U) \in \text{Mod}$ on an open set $U \subseteq T$, termed its sections; given open sets $U \subseteq V$ it gives a morphism $\mathcal{F}(V) \to \mathcal{F}(U)$, termed the restriction, etcetera. Given any subset $X \subseteq T$, we write $\mathcal{F}(X) = \varprojlim_{X \subseteq U} \mathcal{F}(U)$; when $X$ is a point, this is termed the stalk and is written $\mathcal{F}_x$.

The category of sheaves is the full subcategory of presheaves on objects $\mathcal{F}$ taking covers to limits:

$$\mathcal{F} \left( \bigcup_{i \in I} U_i \right) \xrightarrow{\sim} \varprojlim_{\emptyset \neq J \subseteq I} \mathcal{F} \left( \bigcap_{j \in J} U_j \right)$$

It is also the localization of the category of presheaves along the stalkwise quasi-isomorphisms. The composition of these two operations is termed “sheafification”, giving, for any presheaf $\mathcal{F}^{\text{pre}}$: a sheaf $\mathcal{F}$ such that any map from $\mathcal{F}^{\text{pre}}$ to a sheaf factors uniquely through $\mathcal{F}$.

We write $\text{Sh}(T)$ for the (dg) category of sheaves of (dg) $\mathbb{Z}$-modules on $T$. It is complete and cocomplete. It has very few compact objects, but is well generated [39, 40]. Its homotopy category is what was classically called the unbounded derived category of sheaves on $T$.

For any continuous map $f : S \to T$, there is an adjoint pair $f^* : \text{Sh}(T) \leftrightarrow \text{Sh}(S) : f_*$. The pushforward $f_*$ is given by the formula $(f_*\mathcal{F})(U) = \mathcal{F}(f^{-1}(U))$, while the pullback $f^*$ is the sheafification of the presheaf given by $(f^*\mathcal{G})(V) = \mathcal{G}(f(V))$.

Example 4.1. Consider $f : S \to$ point, and the constant sheaf $\mathbb{Z}_S := f^*\mathbb{Z}$. Note that in our conventions, $\mathbb{Z}_S(U)$ is a chain complex computing the cohomology of $U$. This should illustrate where, in this account of sheaf theory, is hiding the usual homological algebra of resolutions: it is in the sheafification.

Being a left adjoint, $f^*$ is cocontinuous (preserves colimits, in particular, sums). When $j : U \to T$ is the inclusion of an open set, $j^*$ is given by the simpler formula $(j^*\mathcal{F})(V) = \mathcal{F}(V)$.

\[\text{Well generation is a significantly weaker version of compact generation, which is nevertheless sufficient to appeal to Brown representability, i.e. the assertion that co-continuous functors are representable; that consequently co-continuous functors are left adjoints and continuous functors are right adjoints, etc.}\]
no sheafification required, and hence preserves limits as well. In particular it must also be a right adjoint. The corresponding left adjoint is easy to describe:

\[ j_! F(V) = \begin{cases} F(V) & V \subseteq U \\ 0 & \text{otherwise} \end{cases} \]

The sheaf \( j_! F \) is termed the extension by zero, since its stalks in \( U \) are isomorphic to the corresponding stalks of \( F \), and its stalks outside of \( U \) are zero. For a sheaf \( F \) on \( T \), we write \( F_U := j_! j^* F \). By adjunction there is a canonical morphism \( F_U \to F \). The object \( Z_U \) co-represents the functor of sections over \( U \), i.e. \( \text{Hom}(Z_U, F) = F(U) \).

Being a right adjoint, \( f_* \) is continuous. When \( f \) is proper, it is in addition co-continuous.

More generally, for a morphism of locally compact spaces \( f : S \to T \), one defines\(^7\)

\[ f! : \text{Sh}(S) \to \text{Sh}(T) \]

\[ F \mapsto \lim_{U \subset \subset T} f_* F_U \]

Here the notation \( U \subset \subset T \) means that the closure of \( U \) is compact. When \( S \) is an open subset, this recovers the original definition. When \( f \) is proper, then \( f_! = f_* \). When \( f \) is the map to a point, then \( f_! f^* Z \) is the compactly supported cohomology.

As \( f_! \) is built from colimits, left adjoints, and pushforwards from compact sets, it is co-continuous. As such it has a right adjoint, denoted \( f^! \). When \( f \) is the inclusion of an open subset, we already had the right adjoint \( f_! \), so in this case \( f^* = f^! \).

For any locally closed subset \( v : V \subseteq T \), we extend the notation \( F_V := v_! v^* F \). This sheaf has the same stalks as \( F \) at points in \( V \), and has vanishing stalks outside.

For an open-closed decomposition \( U \subseteq T \), we have the following criterion for when (pullbacks of) these functors are fully faithful:

**Lemma 4.2.** Let \( \Pi \) be a poset with a map to \( \text{Op}(M) \), and let \( Z[\Pi] \) denote its dg linearization. The following are equivalent:

- \( H^*(U) \cong \mathbb{Z} \) for all \( U \in \Pi \) and \( H^*(U) \cong H^*(U \setminus V) \) whenever \( U \not\subseteq V \).

- The composition \( Z[\Pi] \to \text{Op}(M) \xrightarrow{1} \text{Sh}(M) \) is fully faithful.

\(^7\)This particular way of defining the \( ! \) pushforward is taken from [43]. It has the virtue of making the co-continuity of \( f_! \) obvious.
The composition $\mathbb{Z}[\Pi]^{\text{op}} \to \text{Op}(M)^{\text{op}} \xrightarrow{\sim} \text{Sh}(M)$ is fully faithful.

Proof. Let us show that the first condition is equivalent to the second. We have

$$\text{Hom}_M(Z_U, Z_V) = \text{Hom}_M(u_! Z, v_! Z) = \text{Hom}_U(Z, u^! v_! Z) = H^*(U, Z_{V \cap U}) = \text{cone}(H^*(U) \to H^*(U \setminus V)), \quad (4.3)$$

where we have used the second triangle in (4.1). The second condition asks that this be $Z$ when $U \subseteq V$ and zero otherwise, which is exactly what is asserted in the first condition.

A similar calculation, or taking Verdier duals and noting $\text{Hom}(\mathcal{F}, \mathcal{S}) = \text{Hom}(\mathcal{D}\mathcal{S}, \mathcal{D}\mathcal{F})$, shows the first is equivalent to the third.

Lemma 4.3. Let $\Pi$ be a poset with a map to $\text{Op}(M)$ satisfying the equivalent conditions in Lemma 4.2, and suppose that $W \subseteq M$ is an open set such that $H^*(U) \xrightarrow{\sim} H^*(U \setminus W)$ is an isomorphism whenever $U \subseteq W$. Then the pullback of the module $\text{Hom}(-, Z_W)$ along $\mathbb{Z}[\Pi] \xrightarrow{1} \text{Sh}(M)$ is the indicator functor

$$1_W : U \mapsto \begin{cases} Z & U \subseteq W, \\ 0 & \text{otherwise}. \end{cases} \quad (4.4)$$

Proof. This is true by the same calculation as above. \qed

4.2 Constructible sheaves

Let $T$ be a topological space and $S : T = \bigsqcup T_\alpha$ a stratification. Write $T_S$ for the topological space with underlying set $T$ and base given by the stars of strata in $S$ (note that the intersection of any two stars is expressible a union of stars). Note the continuous map $T \to T_S$.

Remark 4.4. Let $\pi : T \to T'$ be any map weakening a topology. For any open set $U$ of $T'$, and any sheaf $\mathcal{F}$ on $T$, one has by definition $\pi_* \mathcal{F}(U) = \mathcal{F}(U)$. It follows that $\pi^* Z_U = Z_U$, as this sheaf co-represents the functor of sections over $U$.

Lemma 4.5. Pulling back sheaves under $S \xrightarrow{\text{star}} \text{Op}(T_S)$ defines an equivalence

$$\text{Sh}(T_S) \xrightarrow{\sim} \text{Fun}(S^{\text{op}}, \text{Mod} \mathbb{Z}) = \text{Mod} S \quad (4.5)$$

$$\mathcal{F} \mapsto (s \mapsto \mathcal{F}(\text{star}(s))) = \text{Hom}_{T_S}(Z_{\text{star}(s)}, \mathcal{F}) \quad (4.6)$$

which sends $Z_{\text{star}(s)}$ to $\text{Hom}_S(\cdot, s) = 1_{\text{star}(s)}$.

Proof. The functor in question is simply restricting a sheaf on $T_S$ to the base consisting of stars of strata. This functor is fully faithful because a map of sheaves is determined by its restriction to a base for the topology. It is essentially surjective because there are no nontrivial covers of stars of strata by stars of strata. The behavior on objects is as asserted because $Z_{\text{star}(s)}$ and $s$ are the co-representatives of the functors of sections over $s$ and the value of the module at $s$, respectively. \qed
Lemma 4.6. If $S'$ refines $S$, then the following diagram commutes:

\[
\begin{array}{ccc}
\text{Sh}(T_{S'}) & \longrightarrow & \text{Mod } S' \\
\pi^* & & \uparrow r^* \\
\text{Sh}(T_S) & \longrightarrow & \text{Mod } S
\end{array}
\]

(4.7)

where $\pi^*$ denotes pullback of sheaves under the continuous map $\pi : T_{S'} \rightarrow T_S$ and $r^* : \text{Mod } S \rightarrow \text{Mod } S'$ denotes the pullback along the natural map $r : S' \rightarrow S$.

Proof. By Remark 4.4 and the characterization of the horizontal functors as $Z_U \mapsto \mathbb{1}_U$. □

A sheaf is said to be constant when it is isomorphic to the star pullback of a sheaf on a point, and locally constant when this is true after restriction to an open cover. For a stratification $S$ of $M$, we say a sheaf is $S$-constructible if it is locally constant when star restricted to each stratum of $S$.

Note that the image of the pullback map $\text{Sh}(T_{S}) \rightarrow \text{Sh}(T)$ is contained in $\text{Sh}_S(T)$ (i.e. consists of $S$-constructible sheaves).

Lemma 4.7. For a triangulation $S$, the map $\text{Sh}(T_S) \rightarrow \text{Sh}_S(T)$ is an equivalence.

Proof. To show full faithfulness, in view of the equivalence of Lemma 4.5 it is enough to check that $\text{Hom}_{T_S}(Z_{\text{star}(s)}, Z_{\text{star}(t)}) = \text{Hom}_T(Z_{\text{star}(s)}, Z_{\text{star}(t)})$. The former is the indicator of $\text{star}(s) \subseteq \text{star}(t)$ again by Lemma 4.5. To show that $\text{Hom}_T(Z_{\text{star}(s)}, Z_{\text{star}(t)})$ is as well, by Lemma 4.2 it is enough to show that $\mathbb{H}^*(\text{star}(s)) \rightarrow \mathbb{H}^*(\text{star}(s) \setminus \text{star}(t))$ is an isomorphism for $\text{star}(s) \not\subseteq \text{star}(t)$. If $\text{star}(s) \not\subseteq \text{star}(t)$, then $\text{star}(s) \setminus \text{star}(t)$ is the join of something with $s$, and is hence contractible.

Regarding essential surjectivity, note that the rightmost exact triangle of (4.1) serves to decompose any sheaf into an iterated extension of (extensions by zero of) sheaves on the strata; hence any constructible sheaf into (extensions by zero of) locally constant sheaves on the strata. Since the strata are all contractible, these sheaves are in fact constant. This shows that the $Z_s$ generate. To conclude that the $Z_{\text{star}(s)}$ generate, use the exact triangle $Z_{\text{star}(s)} \rightarrow Z_{\text{star}(s)} \rightarrow Z_s \xrightarrow{[1]}$ and induction on dimension of strata (noting that the first term is in the span of $Z_t$ for $\dim(t) < \dim(s)$). □

4.3 Microsupport

The notion of microsupport is developed in [26].\(^9\) We recall some basic facts here.

For what follows, let $M$ denote an analytic manifold. Given a sheaf $\mathcal{F}$ and a smooth function $\phi : M \rightarrow \mathbb{R}$, consider a point $m$ in a level set $\phi^{-1}(t)$. We say that $m \in M$ is a cohomological $\mathcal{F}$-critical point of $\phi$ if, for inclusion of the superlevelset $i : \phi^{-1}(\mathbb{R}_{\geq t}) \hookrightarrow M$, one has $(i^*\mathcal{F})_m \neq 0$.

\(^8\)Some sources, such as [26], also ask that the word constructible should mean that sheaves should have perfect stalks and bounded cohomological degree. We do not.

\(^9\)In [26], the authors work in the bounded derived category. As noted in [42], the only real dependence on this was in the proof of one lemma, which is extended to the unbounded setting in that reference.
The microsupport $\text{ss}(F) \subseteq T^*M$ is by definition the closure of the locus of differentials of functions at their cohomological $\mathcal{F}$-critical points [26]. It is conical.

If $\mathcal{F}$ is locally constant, then a cohomological $\mathcal{F}$-critical point can only occur where the function in question has zero derivative. Thus the microsupport of a locally constant sheaf is contained in the zero section (and is equal to it where the sheaf is not locally zero). If $U \subseteq M$ is an open set and $m$ is a point in the smooth locus of $\partial U$, then over $m$, the locus $\text{ss}(\mathcal{Z}_U) = \text{ss}(u_*\mathcal{Z})$ is the half-line of outward conormals to $\partial U$. The locus $\text{ss}(u_*\mathcal{Z})$ is the inward conormal.

For a subset $X \subseteq T^*M$, we write $\text{Sh}_X(M)$ for the full subcategory of $\text{Sh}(M)$ spanned by objects with microsupport contained in $X$. Similarly, for $X \subseteq S^*M$, we write $\text{Sh}_X(M)$ for the full subcategory of $\text{Sh}(M)$ with microsupport at infinity contained in $X$. Evidently if $0_M \subseteq X$, then $\text{Sh}_X(M) = \text{Sh}_{0M}(M)$.

If $\mathcal{F}$ is a sheaf and $\phi$ is a smooth function with $d\phi_x = \xi$, then for $i : \phi^{-1}(\mathbb{R}_{\geq 1}) \to M$, if $(i^!F)_x \neq 0$, we have $(x, \xi) \in \text{ss}(\mathcal{F})$. Given this, one wants to assign the complex $(i^!F)_x$ itself as an invariant of $\mathcal{F}$ at $(x, \xi)$. This is not generally possible, but it can be done when $\xi$ is a point in the smooth Lagrangian locus of $\text{ss}(\mathcal{F})$ [26, Proposition 7.5.3]. Namely, at any smooth Lagrangian point $(x, \xi) \in X \subseteq S^*M$, there is a 'microstalk' functor

$$\mu_{(x,\xi)} : \text{Sh}_X(M) \to \text{Sh}(pt).$$

(4.8)

It is given by a shift of $\mathcal{F} \mapsto (i^!F)_x$ for any $\phi$ with $d_x\phi = \xi$ with the graph of $d\phi$ transverse to $X$. The shift can be fixed using the index of the three transverse Lagrangians $(\text{ss}(\mathcal{F}), T^*_xM, \Gamma_{d\phi})$. When $\xi = 0$, the microstalk functor is simply the stalk functor.

**Theorem 4.8** ([26, 5.4.19, 5.4.20, 7.5.3] or [21, 44]). Let $X \subseteq T^*M$ be a closed conical subset, let $\phi : M \to \mathbb{R}$ be a proper function, and assume that over $\phi^{-1}((a,b))$, one has $\Gamma_{d\phi} \cap \text{ss}(\mathcal{F}) = (x, \xi)$, where $(x, \xi)$ is a smooth Lagrangian point of $X$.

Let $A : \phi^{-1}((-\infty, a)) \to M$, $A' : \phi^{-1}((a, \infty)) \to M$, $B : \phi^{-1}((\infty, b)) \to M$, and $B' : \phi^{-1}((b, \infty)) \to M$ be the inclusions. Then (up to a shift), the following functors $\text{Sh}_X(M) \to \text{Sh}(pt)$ are isomorphic:

- **The microstalk functor** $\mu_{(x,\xi)}$.
- $\text{Hom}(\text{cone}(A_*\mathcal{Z} \to B_*\mathcal{Z}), -)$.
- $\text{Hom}(\text{cone}(A'_*\mathcal{Z} \to B'_*\mathcal{Z}), -)$.

Here the maps are the canonical ones coming from restriction of sections.

We do not say that $\text{cone}(A_*\mathcal{Z} \to B_*\mathcal{Z})$ co-represents the microstalk because it is not an element of $\text{Sh}_X(M)$. As observed in [35], such co-representatives do exist, for categorical reasons.

Indeed, the microsupport of a sum or product is contained in the union of the microsupports. It follows that the subcategory $\text{Sh}_X(M) \subseteq \text{Sh}(M)$ is closed under sums and products. In particular, $\text{Sh}_X(M)$ is complete and co-complete, and the inclusion $\text{Sh}_X(M) \to \text{Sh}(M)$ is continuous and co-continuous. More generally, if $X \subseteq X'$, then the inclusion $\iota : \text{Sh}_X(M) \to \text{Sh}_{X'}(M)$ is continuous and co-continuous. Thus it has both adjoints: $(\iota^*, \iota, \iota^!)$.
For example, if $V \subseteq M$ is a closed subset, then taking $X = T^*M|_V$ and $X' = T^*M$ recovers the adjoint triple for the pushforward along $V \to M$, because $\Sh_{T^*M|_V}(M) = \Sh(V)$.

Using the left adjoint and Theorem 4.8, we can obtain a co-representative for the microstalk as follows. Take any $X' \supseteq \ss(cone(A_iZ \to B_iZ))$, e.g. $X' = T^*M$. Then $\iota^*\cone(A_iZ \to B_iZ) \in \Sh_X(M)$ co-represents the microstalk.

We do not generally have a good understanding of $(\iota^*, \iota, \iota^!)$, but when $X' \setminus X$ is isotropic we have the following:

**Theorem 4.9.** Let $X \subseteq T^*M$ be closed and conical, and let $\Lambda \subseteq T^*M \setminus X$ be a closed conical subanalytic isotropic. Then $\Sh_X(M) \subseteq \Sh_{X \cup \Lambda}(M)$ is the kernel of all microstalks at Lagrangian points of $\Lambda$. Thus the left adjoint $\iota^*$ to this inclusion $\iota$ realizes the quotient

$$\Sh_{X \cup \Lambda}(M)/\mathcal{D} \xrightarrow{\sim} \Sh_X(M),$$

(4.9)

where $\mathcal{D}$ denotes co-representing objects for the microstalks at Lagrangian points of $\Lambda$.

**Proof.** If $\ss(\mathcal{F}) \subseteq X$, then the microstalks of $\mathcal{F}$ at Lagrangian points of $\Lambda$ vanish by definition of microsupport. To prove the converse, suppose that $\ss(\mathcal{F}) \subseteq X \cup \Lambda$ and that the microstalks of $\mathcal{F}$ vanish at all Lagrangian points of $\Lambda$, and let us show that $\ss(\mathcal{F}) \subseteq X$. By the fundamental result [26, 6.5.4] that the microsupport is co-isotropic, it is enough to show that $p \notin \ss(\mathcal{F})$ for every Lagrangian point $p \in \Lambda$. It is not quite immediate from the definitions that vanishing of the microstalk implies there is no microsupport, since the microsupport is defined in terms of arbitrary test functions, whereas microstalks are defined in terms of microlocally transverse test functions. To see it is true, and that moreover the microstalk is locally constant along $\Lambda$, one can apply a contact transformation so that $\Lambda$ becomes locally the conormal to a smooth hypersurface; for details see [26, Chap. 7].

### 4.4 Compact objects

Here we elaborate upon some assertions of [35].

We write $\Sh_X(M)^c$ for the compact objects in the category $\Sh_X(M)$. Be warned:

**Proposition 4.10 ([39]).** When $M$ is non-compact, $\Sh(M)^c = 0$.

There are not many more compact objects in the compact case. However, for constructible sheaves with respect to a fixed triangulation, the situation is different:

**Lemma 4.11.** For $\mathcal{S}$ a triangulation, the category $\Sh_{\mathcal{S}}(M)$ is compactly generated, and the objects of $\Sh_{\mathcal{S}}(M)^c$ are the sheaves with perfect stalks and compact support.

**Proof.** Under the identification (Lemma 4.7) $\Sh_{\mathcal{S}}(M) = \Mod \mathcal{S}$, the $\Z_{\text{star}(s)}$ go to compact generators. The devissage in the proof of the same Lemma shows that $\Z_s$ also generate, and can be expressed using finitely many $\Z_{\text{star}(s)}$, hence are compact. The $\Z_s$ evidently generate the sheaves with perfect stalks and compact support.

**Remark 4.12.** Note that while a non-compact manifold does not admit a finite triangulation, it can sometimes be a relatively compact constructible subset of a larger manifold.

Let us carry what we can of this to the categories $\Sh_\Lambda(M)$ when $\Lambda$ is subanalytic.
Proposition 4.13. Let $S$ be a subanalytic stratification. The microstalk at a smooth point of $\nu^*S$ is co-representable by a compact object of $\text{Sh}_S(M)$. The same holds for the stalk at any point in the zero section.

Proof. Consider the microstalk at some point $(x, \xi)$. It is possible to choose real analytic $\phi$ as in Theorem 4.8, see [21, Thm. 2.2.1] or [26, Proposition 8.3.12]. We keep the notation of Theorem 4.8. Refine the stratification to some $S'$ so the $A_!\mathbb{Z}$ and $B!\mathbb{Z}$ are constructible.

By Lemma 4.11, cone$(A!\mathbb{Z} \to B!\mathbb{Z})$ is a compact object in $\text{Sh}_{S'}(M)$. The functor $\iota$ is co-continuous, so its left adjoint $\iota^*$ preserves compact objects. Thus the co-representing object $\iota^*\text{cone}(A!\mathbb{Z} \to B!\mathbb{Z}) \in \text{Sh}_S(M)$ is compact.

Regarding stalks, note that for any $x \in M$, the functor of taking stalks at $x$, which is by definition $\mathcal{F}_x := \lim \mathcal{F}(B_!(x))$, is in fact computed by some fixed $\mathcal{F}_x = \mathcal{F}(B_!(x))$. Indeed, further shrinking of the ball will be non-characteristic with respect to $\nu^*S$, as follows from Whitney’s condition B. Now we argue as above, choosing any analytic function with sublevelset $B_!(x)$.

Corollary 4.14. Let $X \subseteq T^*M$ and $\Lambda \subseteq T^*M \setminus X$ be closed conical subanalytic isotropics. Then

$$(\text{Sh}_{X\Lambda}(M)^c/\mathcal{D})^\pi \xrightarrow{\sim} \text{Sh}_X(M)^c,$$

where $\mathcal{D}$ denotes co-representing objects for the microstalks at Lagrangian points of $\Lambda$.

Proof. The microstalks are compact by Proposition 4.13 and the fact (observed in the proof of that result) that the left adjoint to the inclusion $\text{Sh}_{X\Lambda}(M) \hookrightarrow \text{Sh}_S(M)$ preserves compact objects (for $S$ a stratification with $X \cup \Lambda \subseteq \nu^*S$). Now apply Lemma A.5 to Theorem 4.9.

Corollary 4.15. For any subanalytic isotropic $\Lambda$, the category $\text{Sh}_\Lambda(M)$ is compactly generated by co-representatives of the microstalks at smooth points of $\Lambda$ (including the stalks at points of the zero section $M$ whose cotangent sphere is disjoint from $\Lambda$).

Proof. By Theorem 4.9, the quotient by this set of objects is the zero category.

The following result was shown in [35] using arborealization; here is a direct argument.

Corollary 4.16. The Yoneda embedding induces an equivalence between the full subcategory of $\text{Sh}_\Lambda(M)$ of objects with perfect stalks and the category $\text{PropSh}_\Lambda(M)^c$.

Proof. From the argument in Proposition 4.13, we see that the microstalks are calculated by comparing sections over precompact sets; it follows that a sheaf microsupported in $\Lambda$ (thus constructible) with perfect stalks has perfect microstalks. Together the stalk and microstalk functors split-generate $\text{Sh}_\Lambda(M)^c$ by Corollary 4.15, so we see that a sheaf with perfect stalks defines a proper module.

To see the converse, recall from the argument in Proposition 4.13 that the stalk functors can be expressed in terms of sections over open sets constructible with respect to some $S$ satisfying $\nu^*S \supseteq \Lambda$. The left adjoint to $\text{Sh}_\Lambda(M) \hookrightarrow \text{Sh}_S(M)$ preserves compact objects as observed previously, hence proper over $\text{Sh}_\Lambda(M)^c$ implies perfect stalks.
Remark 4.17. Note that if we restricted from the beginning to sheaves constructible with respect to some fixed finite subanalytic stratification, and for intermediate constructions allowed ourselves only fixed finite refinements, then we could in principle argue without appeal to the “well-generated” version of Brown representability. The results of this article, restricted to compact manifolds, can all be deduced in this setting.

For compact $M$, we establish smoothness and/or properness for some of these categories.

**Proposition 4.18.** If $M$ is compact and $S$ is a triangulation, then $\text{Sh}_S(M)^c$ is smooth and proper.

**Proof.** The $Z_{\text{st}}(s)$ give a finite generating exceptional collection which is proper, and this implies smoothness by Lemma A.10. □

More generally,

**Corollary 4.19.** If $M$ is compact and $\Lambda$ is subanalytic isotropic, then the category $\text{Sh}_\Lambda(M)^c$ is smooth, $\text{Prop Sh}_\Lambda(M)^c \subseteq \text{Perf Sh}_\Lambda(M)^c$, and the category $\text{Prop Sh}_\Lambda(M)^c$ is proper.

**Proof.** By Proposition 4.18 and Corollary 4.14, the category $\text{Sh}_\Lambda(M)^c$ is a quotient of a smooth category, hence smooth (Lemma A.8). Smoothness implies proper modules are perfect (Lemma A.7) and that the category of proper modules is proper. □

**Remark 4.20.** When $(M, \Lambda)$ are non-compact but finite-type in a suitable sense, the same result is true. One can prove it by embedding into a compact manifold as in Remark 4.12.

### 4.5 In conclusion

Collecting the results of this section, we have shown:

**Theorem 4.21.** The functor $\Lambda \mapsto \text{Sh}_\Lambda(M)^c$ is a microlocal Morse theatre in the sense of Definition 3.5, which casts the co-representatives of the microstalk functors at smooth points of $\Lambda$ as the Morse characters.

**Proof.** The most obvious functor $\Lambda \mapsto \text{Sh}_\Lambda(M)$ is the one which carries inclusions $\Lambda \subseteq \Lambda'$ to inclusions $\text{Sh}_\Lambda(M) \hookrightarrow \text{Sh}_{\Lambda'}(M)$; note that this is in fact a strict diagram of categories (as all are simply full subcategories of $\text{Sh}(M)$) and takes values in the category whose objects are large dg categories and whose morphisms are continuous and co-continuous. Passing to left adjoints and taking compact objects (a left adjoint of a co-continuous functor preserves compact objects), we obtain a functor $\Lambda \mapsto \text{Sh}_\Lambda(M)^c$.

For triangulations $S$, the functors

$$S \xrightarrow{S \to Z_{\text{st}}(s)} \text{Sh}_S(M) \xrightarrow{\text{F} \to \text{Hom}(Z_{\text{st}}(-), \text{F})} \text{Mod } S$$

(4.11)

define an equivalence $\text{Perf } S = \text{Sh}_S(M)^c$ by Lemmas 4.5 and 4.7. When $S$ is a $\mu$-stratification, we have $\text{Sh}_S(M)^c = \text{Sh}_{N_\mu S}(M)^c$ by [26, Prop. 8.4.1].

Taking the commutative diagram in Lemma 4.6 and passing to the left adjoints of the vertical maps shows that this equivalence respects refinement of triangulations. This shows that $\Lambda \mapsto \text{Sh}_\Lambda(M)^c$ is a microlocal Morse pre-theatre.

By Theorem 4.8, the Morse characters in $\text{Perf } S$ correspond, under this isomorphism, to co-representatives of the microstalks. According to Corollary 4.14, the functor $\text{Sh}_{\Lambda'}(M)^c \to \text{Sh}_\Lambda(M)^c$ is the quotient by co-representatives of the microstalks. □
5 Wrapped Fukaya categories

5.1 Notation

Here we quickly fix notation and review basic facts (see, e.g., [19, §3.3] for more details). Fix a Liouville manifold or sector $X$.

For exact Lagrangians $L, K \subseteq X$, conical at infinity, we write $HF^*(L, K)$ for the Floer cohomology. We write $HF^*(L, L)$ to mean $HF^*(L^+, L)$, where $L^+$ denotes an (unspecified) small positive pushoff of $L$. This group $HF^*(L, L) = HF^*(L^+, L)$ is a unital algebra, and its unit is termed the continuation element. Composition of continuation elements defines a continuation element in $HF^*(L^{++}, L)$ for $L^{++}$ any (not necessarily small) positive wrapping of $L$. If the positive isotopy $L \sim L^{++}$ takes place in the complement of $\partial_\infty K$, then composition with the continuation map gives an isomorphism $HF^*(L, K) \cong HF^*(L^{++}, K)$ (and similarly in the reverse). More generally, if $L \sim L'$ is any isotopy taking place in the complement of $\partial_\infty K$, then there is an induced identification $HF^*(L, K) = HF^*(L', K)$ (see [19, Lem. 3.21]).

The wrapped Floer cohomology $HW^*(L, K)_X$ is equivalently calculated by

$$\lim_{L \sim L^{++}} HF^*(L^{++}, K) = \lim_{L \sim L^{++}, K \sim K^-} HF^*(L^{++}, K^-) = \lim_{K^- \sim K} HF^*(L, K^-).$$

Here, the direct limits are taken using the continuation maps over positive-at-infinity isotopies of $L$ and negative-at-infinity isotopies of $K$. The freedom to wrap in only one factor is extremely useful in practice.

Given any closed subset $\mathfrak{f} \subset \partial_\infty X$, and $L, K$ disjoint at infinity from $\mathfrak{f}$, we similarly define $HW^*(L, K)_{(X, \mathfrak{f})}$ by restricting wrappings to take place in the complement of $\mathfrak{f}$.

One main point of [19] was the construction of a covariant functor $W(X) \rightarrow W(Y)$ for an inclusion of Liouville sectors $X \subseteq Y$. In [20] we remarked that the same construction gives a functor $W(X, \mathfrak{f} \cap (\partial_\infty X)^\circ) \rightarrow W(Y, \mathfrak{f})$. This covariance is a nontrivial result having to do with the fact that holomorphic disks do not cross the boundary of a Liouville sector. By contrast, it is immediate from the definition that if $\mathfrak{g} \subseteq \mathfrak{f}$ then there is a natural map $W(X, \mathfrak{f}) \rightarrow W(X, \mathfrak{g})$: just wrap more. Both covariance statements allow one to calculate in a potentially simpler geometry, and push forward the result.

The following Lemma allows one to explicitly describe cofinal wrapping sequences. Its typical use is the following. To compute $HW^*(L, K)$, one finds a cofinal sequence $L_t$ as in the Lemma, such that the induced maps $HF^*(L_t, K) \rightarrow HF^*(L_{t+1}, K)$ are eventually all isomorphisms. Then $HW^*(L, K) = HF^*(L_t, K)$ for any $L_t$ in this stable range.

**Lemma 5.1** ([20, Lemma 2.1]). Let $Y$ be a contact manifold and $\Lambda_t$ a positive isotopy of Legendrians. If $\Lambda_t$ escapes to infinity as $t \rightarrow \infty$ (i.e. is eventually disjoint from any given compact subset of $Y$), then it is a cofinal wrapping of $\Lambda_0$. □

5.2 Foundations

In [19, 20], for any Liouville sector $X$ and any closed subset $\mathfrak{f} \subseteq (\partial_\infty X)^\circ$, we constructed $A_\infty$ categories $W(X, \mathfrak{f})$ whose objects are exact Lagrangians in $X \setminus \mathfrak{f}$, conical at infinity. The cohomology category is simply wrapped Floer cohomology $H^*W(L, K) = HW^*(L, K)_{(X, \mathfrak{f})}$. 
We wish to consider here categories \( W(T^*M, \Lambda) \) for manifolds \( M \) and closed subsets \( \Lambda \subseteq T^*M = \partial_\infty T^*M \). When \( M \) is non-compact, this does not strictly fit into the framework of \([19, 20]\), so we describe here the construction (which is only a minor variation on \([19, \text{Sec. 3}] \) and \([20, \text{Sec. 2}]\), to which we refer the reader for more details). We do not discuss the case when \( M \) has boundary, since it is not difficult to check that the category one would define for such \( M \) is equivalent to the category we define here for its interior \( M^\circ \).

To define \( W(T^*M, \Lambda) \), we choose the following data:

(i) A countable poset \( \mathcal{O} \) of exact conical at infinity Lagrangians inside \( T^*M \) (equipped with grading/orientation data) disjoint from \( \Lambda \) and with relatively compact image in \( M \) (and \( \mathcal{O} \) must contain at least one in every isotopy class of such Lagrangians). We require \( \mathcal{O} \) to be cofinite, namely \( \{ K \in \mathcal{O} \mid K \leq L \} \) is finite for all \( L \in \mathcal{O} \). We require that every totally ordered subset \( L_0 > \cdots > L_k \in \mathcal{O} \) must be mutually transverse. Finally, we require that for every \( L \in \mathcal{O} \), there exist a cofinal sequence \( L = L_0 < L_1 < \cdots \in \mathcal{O} \) along with positive isotopies \( L_0 = L_0 \leadsto L_1 \leadsto \cdots \) which are cofinal inside the positive wrapping category of \( L \) inside \( T^*M \) away from \( \Lambda \).

(ii) A collection \( C \) of elements of \( HF^*(L, K) \) for pairs \( L > K \in \mathcal{O} \) consisting only of continuation elements for various positive isotopies \( K \leadsto L \) disjoint from \( \Lambda \). This collection \( C \) must be such that for every \( L \in \mathcal{O} \), there exists a sequence \( L = L_0 < L_1 < \cdots \in \mathcal{O} \) cofinal in \( \mathcal{O} \) and positive isotopies \( L = L_0 \leadsto L_1 \leadsto \cdots \) cofinal in the wrapping category of \( L \) (away from \( \Lambda \)) such that every associated continuation element in \( HF^*(L_{i+1}, L_i) \) is in \( C \).

(iii) For every \( L \in \mathcal{O} \), a choice of compact codimension zero submanifold-with-boundary \( M_L \subseteq M \) containing the image of \( L \), such that \( M_L \subseteq (M_K)^0 \) for \( L < K \).

(iv) Floer data for \( \mathcal{O} \), consisting of choices of strip-like coordinates \( \xi \) as in \([19, \text{(3.54)-(3.55)}]\) and almost complex structures

\[
J_{L_0,\ldots,L_k} : \mathcal{S}_{k,1} \to \mathcal{J}(T^*M_{L_0})
\]

for \( L_0 > \cdots > L_k \in \mathcal{O} \). These almost complex structures are required to make a fixed choice of projection from near \( \partial T^*M_{L_0} \) to \( \mathbb{C}_{\mathbb{R} \geq 0} \) holomorphic.

Remark 5.2. The essential difference between the current situation and the setup of \([19, 20]\) is that \( T^*M \), rather than being a Liouville sector itself, is only a filtered ascending union of Liouville sectors \( T^*M_\alpha \subseteq T^*M \) (over compact codimension zero submanifolds-with-boundary \( M_\alpha \subseteq M \)). The present discussion, while phrased in terms of cotangent bundles, would apply without change to any such ‘ind-Liouville sector’, i.e. any filtered ascending union of Liouville sectors.

By counting holomorphic disks with respect to given data \((\mathcal{O}, C, M, \xi, J)\), we define a directed (by the poset) strictly unital \( A_\infty \) category \( \mathcal{O} \) with \( \text{hom}_\mathcal{O}(K, L) = 0 \) unless \( K \geq L \). The definition of \( W(T^*M, \Lambda) \) is as the localization \( \mathcal{W} := \mathcal{O}[C^{-1}] \). This has the correct cohomology category, calculated by wrapping as above (see \([19, \text{Lemma 3.37}]\)). Given \((\mathcal{O}, C)\), it is straightforward to construct \((M, \xi, J)\) by induction to achieve transversality. The construction of \((\mathcal{O}, C)\) proceeds by applying the following Lemma to \((\mathcal{O}_0, C_0 = \emptyset)\) where \( \mathcal{O}_0 \) is any countable set containing at least one Lagrangian in every isotopy class, thought of as a poset with no relations:
Lemma 5.3. Let $\mathcal{O}_0$ be any cofinite countable poset of Lagrangians inside $(T^*M, \Lambda)$ (with every totally ordered subset mutually transverse), and let $C_0$ be a collection of continuation elements. There exists another such pair $(\mathcal{O}_1, C_1)$ together with a downward closed embedding $\mathcal{O}_0 \hookrightarrow \mathcal{O}_1$ with $C_1|_{\mathcal{O}_0} = C_0$, such that $(\mathcal{O}_1, C_1)$ in addition satisfies: for every $L \in \mathcal{O}_1$, there exists a cofinal sequence $L = L_0 < L_1 < \cdots \in \mathcal{O}_1$ and positive isotopies $L_0 \rightsquigarrow L_1 \rightsquigarrow \cdots$ cofinal in the wrapping category of $L$ (away from $\Lambda$) such that the associated continuation elements are in $C_1$.

Proof. Exhaust $\mathcal{O}_0$ by downward closed finite subsets $Z_0 \subseteq Z_1 \subseteq \cdots$. The additional Lagrangians $\mathcal{O}_1 \setminus \mathcal{O}_0$ are indexed by integers $i \geq 0$ (ordered accordingly), and the $i$th such additional Lagrangian lies above precisely the Lagrangians $Z_i$ inside $\mathcal{O}_0$. For each Lagrangian $L \in \mathcal{O}_0$, we choose (generically) cofinal wrappings $L \rightsquigarrow L_1 \rightsquigarrow L_2 \rightsquigarrow \cdots$, and we add the $L_1 < L_2 < \cdots$ to the list of Lagrangians we want to include in $\mathcal{O}_1 \setminus \mathcal{O}_0$.

There are countably many such sequences $(L_1, L_2, \cdots)$, so we can include all of them into $\mathbb{Z}_{\geq 0}$ at the same time, ensuring moreover that $L < L_1$. Namely, we enumerate $\mathcal{O}_0 = \{K_0, K_1, \ldots\}$, and we process these $K_i$ in order as follows: given $L = K_i$, embed $L_1 < L_2 < \cdots$ into $\mathbb{Z}_{\geq 0}$ (in the complement of everything else previously embedded there) such that $L_1$ is put high enough to ensure $L < L_1$, and such that there are still infinitely many ‘slots’ in $\mathbb{Z}_{\geq 0}$ remaining (to be used in the countably many subsequent steps). At the end of the process, there may be ‘unfilled’ slots in $\mathbb{Z}_{\geq 0}$, however this is of no importance. After the completion of the process of embedding all of the countably many sequences $(L_1, L_2, \cdots)$ into $\mathbb{Z}_{\geq 0}$, we further inductively perturb these $L_i$ to $\tilde{L}_i$ (in a manner preserving positivity and cofinality of the isotopies $L \rightsquigarrow L_1 \rightsquigarrow L_2 \rightsquigarrow \cdots$) to ensure mutual transversality of every totally ordered subset of the thusly defined poset $\mathcal{O}_1$.

The continuation elements $C_1$ are simply $C_0$ union those associated to the positive isotopies $L \rightsquigarrow \tilde{L}_1 \rightsquigarrow \tilde{L}_2 \rightsquigarrow \cdots$. \hfill $\square$

Applying Lemma 5.3, we obtain a pair $(\mathcal{O}, C)$ for any $\Lambda \subseteq S^*M$, thus giving rise to a category $\mathcal{W}(T^*M, \Lambda)$. This defines each of the categories $\mathcal{W}(T^*M, \Lambda)$ individually, however we also want pushforward functors $\mathcal{W}(T^*M, \Lambda') \to \mathcal{W}(T^*M, \Lambda)$ for inclusions $\Lambda' \supseteq \Lambda$. That is, denoting by $\Lambda$ the poset of all closed subsets $\Lambda \subseteq S^*M$ (ordered by reverse inclusion), we want a functor

$$\mathcal{W} : \Lambda \to A_{\infty}\text{-}\text{cat} \quad (5.2)$$

$$\Lambda \mapsto \mathcal{W}(T^*M, \Lambda) \quad (5.3)$$

(in the sense that $\mathcal{W}(T^*M, \Lambda') \to \mathcal{W}(T^*M, \Lambda') \to \mathcal{W}(T^*M, \Lambda)$ coincides with $\mathcal{W}(T^*M, \Lambda'') \to \mathcal{W}(T^*M, \Lambda)$ on the nose). To construct this functor $\mathcal{W}$, it suffices to construct it for finite subsets of $\Lambda$, in the following sense. Namely, suppose that for every finite subset $F \subseteq \Lambda$, we have a functor

$$\mathcal{W}_F : F \to A_{\infty}\text{-}\text{cat} \quad (5.4)$$

$$\Lambda \mapsto \mathcal{W}_F(T^*M, \Lambda) \quad (5.5)$$

along with quasi-equivalences $\mathcal{W}_F \to \mathcal{W}_F|_F$ (again, strictly compatible for triples $F \subseteq F' \subseteq F''$) which are all naive inclusions, i.e. are injective on objects and with all higher ($k \geq 2$)
functor operations vanishing. Then we may define

$$\mathcal{W}(\Lambda) := \lim_{\Lambda \in F \subseteq \Lambda} \mathcal{W}_F(\Lambda)$$ \hfill (5.6)

(which makes sense since the transition functors are naive inclusions). Now for \( \Lambda \supseteq \Lambda' \), and for any \( F \ni \Lambda \), there is a (naive) inclusion \( \mathcal{W}_F(\Lambda) \to \mathcal{W}_{F \cup \{\Lambda\}}(\Lambda) \to \mathcal{W}_{F \cup \{\Lambda\}}(\Lambda') \) living over the map of directed systems \( \{\Lambda \in F \subseteq \Lambda\} \to \{\Lambda' \in F' \subseteq \Lambda\} \) which sends \( \Lambda \in F \) to \( \Lambda' \in F \cup \{\Lambda\} \), compatibly with maps in the system. This defines \( \mathcal{W}(\Lambda) \to \mathcal{W}(\Lambda') \) as desired, and one can check that these maps indeed compose as desired.

It thus suffices to construct the compatible systems of categories \( \mathcal{W}_F \) for finite subsets \( F \subseteq \Lambda \). To construct these \( \mathcal{W}_F \), it is enough to define the corresponding \( (\mathcal{O}_F, C_F) \) (the subsequent inductive construction of Floer data is straightforward, and hence will not be discussed further).

We would thus like to define, for every finite \( F \subseteq \Lambda \), a functor \( \Lambda \mapsto (\mathcal{O}_F(\Lambda), C_F(\Lambda)) \) such that each map \( (\mathcal{O}_F(\Lambda'), C_F(\Lambda')) \hookrightarrow (\mathcal{O}_F(\Lambda), C_F(\Lambda)) \) is as in the conclusion of Lemma 5.3, namely downward closed and satisfying \( C_F(\Lambda)|_{\mathcal{O}_F(\Lambda')} = C_F(\Lambda') \). Furthermore, we would like to have the same sort of maps \( (\mathcal{O}_F, C_F) \hookrightarrow (\mathcal{O}_{F'}, C_{F'}) \) for \( F \subseteq F' \). The construction of \( (\mathcal{O}_F, C_F) \) is by induction on finite subsets \( F \subseteq \Lambda \) (ordered by inclusion). For a fixed \( F \), the construction of \( (\mathcal{O}_F(\Lambda), C_F(\Lambda)) \) is by induction on \( \Lambda \in F \) (ordered by reverse inclusion). To define \( (\mathcal{O}_F(\Lambda), C_F(\Lambda)) \), we begin with \( (\mathcal{O}_0, C_0) \) equal to the colimit of everything which must map into \( (\mathcal{O}_F(\Lambda), C_F(\Lambda)) \), namely \( (\mathcal{O}_0, C_0) \) is the colimit of \( (\mathcal{O}_{F'}(\Lambda'), C_{F'}(\Lambda')) \) over pairs \( (\Lambda', F') \) with \( \Lambda \subseteq \Lambda' \subseteq F' \subseteq F \) (with its natural poset structure) with at least one of the inclusions strict. If necessary, add on to \( \mathcal{O}_0 \) additional Lagrangians in order to represent all isotopy classes (with no relations to the rest of \( \mathcal{O}_0 \), and no additions to \( C_0 \)). Applying Lemma 5.3 to the result defines \( (\mathcal{O}_F(\Lambda), C_F(\Lambda)) \).

## 5.3 Gradings and orientations

We briefly review the setup for defining gradings and orientations in Floer theory; for more details see Seidel [45, 46].

The Grassmannian of Lagrangian subspaces of a given symplectic vector space of dimension \( 2n \) is homotopy equivalent to \( U(n)/O(n) \), and its fundamental group has a preferred isomorphism with \( \mathbb{Z} \) given by \( \det^2 : U(n)/O(n) \to U(1) \). A \( \mathbb{Z} \)-grading on a symplectic manifold \( X \) is a choice of fiberwise universal cover \( \mathcal{L}\text{Gr}X \to \mathcal{L}\text{Gr}X \). A \( \mathbb{Z} \)-grading on a Lagrangian \( L \subseteq X \) with respect to a given \( \mathbb{Z} \)-grading on \( X \) is a lift of the tautological section \( L \to (\mathcal{L}\text{Gr}X)|_L \) to \( (\mathcal{L}\text{Gr}X)|_L \). The Lagrangian Grassmannian of the symplectic manifold \( T^*M \) has a canonical section (over the zero section \( M \)) namely \( [x \mapsto T_x^*M] \). Using this section as the fiberwise basepoint, we obtain a canonical \( \mathbb{Z} \)-grading \( \mathcal{L}\text{Gr}T^*M \to \mathcal{L}\text{Gr}T^*M \).

We consider exclusively this \( \mathbb{Z} \)-grading on \( T^*M \) for defining the Fukaya category of a cotangent bundle \( T^*M \). The Lagrangian fibers \( T_x^*M \subseteq T^*M \) have tautological \( \mathbb{Z} \)-gradings (in particular, varying continuously in \( x \)) relative to this canonical \( \mathbb{Z} \)-grading on \( T^*M \).

A Pin-structure on a Lagrangian \( L \subseteq X \) relative to a \( K(\mathbb{Z}/2, 1) \)-bundle \( b \) over \( X \) is an isomorphism of \( K(\mathbb{Z}/2, 1) \)-bundles over \( L \) between \( b|_L \) and \( \lim \mathbb{P}(TL \oplus \mathbb{R}^n) \). We consider exclusively the \( K(\mathbb{Z}/2, 1) \)-bundle \( b := \lim \mathbb{P}(T^*M \oplus \mathbb{R}^n) \) over \( M \) (pulled back to \( T^*M \)) for
defining the Fukaya category of a cotangent bundle $T^*M$. The Lagrangian fibers $T^*_xM \subseteq T^*M$ are tautologically Pin relative to $b$ (varying continuously in $x$).

5.4 Wrapping exact triangle, stop removal, generation

The fundamental ingredients underlying our work in this section are the wrapping exact triangle and its consequence stop removal, both proved in [20]. The wrapping exact triangle can be thought of as quantifying the price of wrapping through a stop. It should be compared with Theorem 4.8.

**Theorem 5.4** (Wrapping exact triangle [20, Theorem 1.7]). Let $(X, \Lambda)$ be a stopped Liouville sector, and let $p \in \Lambda$ be a point near which $\Lambda$ is a Legendrian submanifold. If $L \subseteq X$ is an exact Lagrangian submanifold and $L^w \subseteq X$ is obtained from $L$ by passing $\partial^\infty L$ through $\Lambda$ transversally at $p$ in the positive direction, then there is an exact triangle

$$L^w \to L \to D_p \xrightarrow{[1]}$$

in $\mathcal{W}(X, \Lambda)$, where $D_p \subseteq X$ denotes the small Lagrangian disk linking $\Lambda$ at $p$ and the map $L^w \to L$ is the continuation map.

The following result about wrapped Fukaya categories is a consequence of the wrapping exact triangle, and can be compared with Theorem 4.9.

**Theorem 5.5** (Stop removal [20, Theorem 1.13]). Let $(X, f)$ be a stopped Liouville manifold (or sector), and let $\Lambda \subseteq (\partial^\infty X)^o \setminus f$ be an isotropic submanifold. Then pushforward induces a quasi-equivalence

$$\mathcal{W}(X, f \cup \Lambda)/\mathcal{D} \xrightarrow{\sim} \mathcal{W}(X, f),$$

where $\mathcal{D}$ denotes the collection of small Lagrangian disks linking (Legendrian points of) $\Lambda$.

We will also need to know that fibers generate:

**Theorem 5.6.** The cotangent fibers split-generate $\mathcal{W}(T^*M)$.

**Proof.** When $M$ is compact (including the case with boundary), this is [20, Theorem 1.9 and Example 1.10]. For a general possibly non-compact $M$, we observe that any Lagrangian $L \in \mathcal{W}(T^*M)$ is in the essential image of the pushforward functor $\mathcal{W}(T^*M_L) \to \mathcal{W}(T^*M)$, for some compact codimension zero submanifold-with-boundary $M_L \subseteq M$. Now push forward the fact that $L$ is split-generated by a fiber in $\mathcal{W}(T^*M_L)$.

**Remark 5.7.** In fact, [20, Theorem 1.9] and the argument above shows the fibers generate $\mathcal{W}(T^*M)$, however we only need split-generation.

Another ingredient which proves useful in our computations is the Künneth theorem for Floer cohomology and wrapped Fukaya categories, also proved in [20].
5.5 Conormals

For a relatively compact open set $U \subseteq M$ with smooth boundary, we write $L_U \subseteq T^*_M M$ for (a smoothing of) the union of $U \subseteq T^*_M M$ with its outward conormal. We write $-$ to mean the antipodal map on $T^*_M M$, hence $-L_U$ is a smoothing of the union of $U$ with its inward conormal.

For a Lagrangian $L$, we write $L^+$ for an unspecified small positive Reeb pushoff of $L$, and $L^-$ for a negative pushoff. If $U$ is an relatively compact open set with smooth boundary, we write $U^\epsilon$ for an $\epsilon$ neighborhood of $U$ in some metric, and $U^-\epsilon$ for the open set such that $(U^-\epsilon)^\epsilon = U$. When $\epsilon$ is unimportant and unchanging, we write these as $U^+$ and $U^-$. Our conormal conventions are chosen to ensure that the conormal to $U^+$ is a positive pushoff of the conormal to $U$; in other words

$$L_{U^+} = L_U^+ \quad L_{U^-} = L_U^-$$

That is, positive Reeb flow pushes outward conormals out.

More generally, if $U$ is a manifold with corners, then we write we write $L_{\tilde{U}}$ where $\tilde{U}$ is obtained from $U$ by smoothing out the boundary.

In all of the above cases, we could also equivalently say that $L_U$ is a rounding of $ss(Z_U)$ (compare Section 4.3).

Each $L_U$ is exact and possesses a canonical relative Pin structure and grading: the codimension zero inclusion $U^- \subset L_U$ is a homotopy equivalence, and $U^-$ is a codimension 0 submanifold of, and thereby inherits all of this data from, the zero section.

5.6 Floer cohomology between conormals of balls and stable balls

By a ball, we mean an open set with smooth boundary whose closure is diffeomorphic to the unit ball.

**Lemma 5.8.** Let $U, V \subseteq M$ be balls with $\overline{U} \subseteq V$. Then $HF^*(L_V, L_U) = \mathbb{Z}$, and is canonically generated by the continuation element.

**Proof.** In this case there is a positive isotopy from $L_U^+$ to $L_V$ in the complement of $\partial_{\infty} L_U$. ■

**Lemma 5.9.** Let $V$ be an open set with smooth boundary, and let $U$ be a $\epsilon$-ball centered a point on $\partial V$. Then $HF^*(L_U, L_V) = 0 = HF^*(L_V, L_U)$.

**Proof.** During the obvious isotopy of $U$ outward to become disjoint from $V$, their conormals never intersect at infinity. ■

By a stable ball, we mean a contractible open set with smooth boundary. The reason we study stable balls is that we do not know how to prove that for a subanalytic triangulation, the inward perturbation (in the sense of Section 5.7) of an open star is a ball; it is, however, obviously a stable ball.

To compute Floer cohomology between conormals of stable balls, we reduce to the case of conormals to balls by stabilizing (i.e. taking their product with conormals to standard balls in $\mathbb{R}^k$) and appealing to the Künneth theorem for Floer cohomology. We begin by showing that the stabilization of a stable ball is indeed a ball (thus justifying the name). This uses the following famous corollary of the $h$-cobordism theorem:
Theorem 5.10. A stable ball of dimension \( \geq 6 \) with simply connected boundary is a ball. \( \square \)

Corollary 5.11. Let \( M \) be a stable ball. Then \( M \times I^k \) is a ball provided \( \dim M + k \geq 6 \) and \( k \geq 1 \).

Proof. We just need to check that the boundary of \( M \times I^k \) is simply connected. It suffices to show that for any stable ball \( N \) of dimension \( \geq 2 \), the boundary of \( N \times I \) is simply connected. The boundary of \( N \times I \) is (up to homotopy) two copies of \( N \) glued along their common boundary. Since \( N \) is contractible, the fundamental group of this gluing vanishes provided \( \partial N \) is connected. If \( \partial N \) were disconnected, then by Poincaré duality, the cohomology group \( H_\dim N - 1(N) \) would be nonzero, which contradicts contractibility as \( \dim N \geq 2 \). \( \square \)

Proposition 5.12. Let \( U, V \subseteq M \) be stable balls with \( \overline{U} \subseteq V \). Then \( HF^*(L_V, L_U) = \mathbb{Z} \), and it is equipped with a canonical generator \( 1_{VU} \) (which is just the continuation map when \( U \) and \( V \) are balls). These generators behave well under composition: for any triple of stable balls \( U, V, W \subseteq M \) with \( \overline{U} \subseteq V \) and \( \overline{V} \subseteq W \), we have \( 1_{WV1_{VU}} = 1_{WU} \).

Proof. We multiply by \( L_U, L_V \) by \( L_{B1(0)}, L_{B2(0)} \subseteq T^*\mathbb{R}^k \) where \( k \) is sufficiently large to guarantee that \( U \times B1(0) \) and \( V \times B2(0) \) are balls by Corollary 5.11. By the Künneth formula for Floer cohomology (see e.g. [20, Lemma 6.3]), we have

\[
HF^*(L_V \times L_{B2(0)}, L_U \times L_{B1(0)}) = HF^*(L_V, L_U) \otimes HF^*(L_{B2(0)}, L_{B1(0)}) = HF^*(L_V, L_U).
\]

On the other hand, by the result for balls Lemma 5.8, we have

\[
HF^*(L_V \times L_{B2(0)}, L_U \times L_{B1(0)}) = HF^*(L_{V \times B2(0)}, L_{U \times B1(0)}) = \mathbb{Z}.
\]

After arguing that the above identification is compatible with rounding of corners, this defines the canonical generator \( 1_{VU} \in HF^*(L_V, L_U) \). The proof that \( 1_{WV1_{VU}} = 1_{WU} \) is the same: stabilize to reduce to the corresponding fact for honest continuation maps. \( \square \)

Corollary 5.13. Let \( U \subseteq M \) be any stable ball. Then the map \( L_U \to T^*_p M \) from Proposition 5.12 is an isomorphism in \( \mathcal{W}(T^*M) \) for any point \( p \in U \).

Proof. By pushing forward, it suffices to treat the case \( M = U^+ \). It further suffices to show the result after applying the Künneth embedding \( \mathcal{W}(T^*(U^+)) \to \mathcal{W}(T^*(U^+ \times I^k)) \) (see [20, Theorem 1.5]). The canonical map \( L_U \to [\text{fiber}] \) from Proposition 5.12 is, by definition, sent by the Künneth functor to the continuation map \( L_{U \times I^k} \to [\text{fiber}] \) (which is defined since the stabilized stable ball \( U^+ \times I^k \) is a ball). The continuation map is an isomorphism in the wrapped Fukaya category of the ball, so we are done by full faithfulness of Künneth. \( \square \)

There is similarly an improved version of Lemma 5.9:

Lemma 5.14. Let \( V \) be an open set with smooth boundary, and let \( U \) be stable ball such that \( U \cap \partial V \) is also a stable ball. Then \( HF^*(L_U, L_V) = 0 = HF^*(L_V, L_U) \).

Proof. Stabilization reduces to Lemma 5.9 (note that \( U \cap \partial V \) necessarily divides \( U \) into two stable balls). \( \square \)
A more subtle result about stable balls is the following, which will be important later:

**Proposition 5.15.** Let $X^m \subseteq Y^n$ be an inclusion of stable balls, with $\partial X \subseteq \partial Y$. Assume there exists another stable ball (with corners) $Z^{m+1} \subseteq Y^n$ such that $\partial Z$ is the union of $X$ with a smooth submanifold of $\partial Y$. Then the map $L_Y \rightarrow L_{B_{\epsilon}(x)}$ from Proposition 5.12 is an isomorphism in $\mathcal{W}(T^*Y, N^*_\infty X)$ for any $x \in X$.

**Proof.** By stabilization, we reduce to the case that $X$, $Y$, and $Z$ are all balls. This implies that, up to diffeomorphism, everything is standard: $Y$ is the unit ball, $X$ is the intersection of $Y$ with a linear subspace, and $Z$ is the intersection of $Y$ with a linear halfspace. Indeed, since $X$ and $Z$ are balls, we can use $Z$ to push $X$ to $Z \cap \partial Y$, thus showing that $X$ is simply a slight inward pushoff of the ball $Z \cap \partial Y \subseteq \partial Y$.

By definition, the map from Proposition 5.12 becomes the continuation map under stabilization. Once everything is standard, it is obvious that the continuation map $L_Y \rightarrow L_{B_{\epsilon}(x)}$ is an isomorphism, since the positive isotopy $L_{B_{\epsilon}(x)} \rightarrow L_Y$ is disjoint from $N^*_\infty X$ at infinity. □

**Remark 5.16.** We will apply Proposition 5.15 when (before rounding) $X$ is a simplex in a triangulation, $Y$ is its star, and $Z$ is any simplex containing $X$ of dimension one larger.

### 5.7 Fukaya categories of conormals to stars

Let $S$ be a stratification of $M$ by locally closed smooth submanifolds. Whitney’s conditions on $S$ are:

(a) For strata $X \subseteq \overline{Y}$ and points $x \in X$ and $y \in Y$, as $y \rightarrow x$ the tangent spaces $T_yY$ become arbitrarily close to containing $T_xX$ (uniformly over compact subsets of $X$).

(b) For strata $X \subseteq \overline{Y}$ and points $x \in X$ and $y \in Y$, as $y \rightarrow x$ the secant lines between $x$ and $y$ become arbitrarily close to being contained in $T_yY$ (uniformly over compact subsets of $X$).

We assume $S$ satisfies Whitney’s conditions (a) and (b).

Consider an $S$-constructible relatively compact open set $U \subseteq M$. Fixing a Riemannian metric on $M$, let $N_\varepsilon$ denote the $\varepsilon$-neighborhood. For $\underline{\varepsilon} = (\varepsilon_0, \ldots, \varepsilon_{\dim M})$ some positive real numbers, we define

$$U^{-\underline{\varepsilon}} := U \setminus \bigcup_{\text{strata } X \subseteq M \setminus U} N_{\varepsilon_{\dim X}} X \quad (5.9)$$

When $\varepsilon_i$ sufficiently small in terms of $\varepsilon_0, \ldots, \varepsilon_{i-1}$ (a condition we indicate simply by “$\underline{\varepsilon}$ sufficiently small”, and other authors [21, 37] express in terms of “fringed sets”), $U^{-\underline{\varepsilon}}$ is (the interior of) a manifold with corners; taking $\underline{\varepsilon} \rightarrow 0$ we see that $U^{-\underline{\varepsilon}}$ is diffeomorphic to $U$ (rel any fixed compact subset of $U$). When the choice of sufficiently small $\underline{\varepsilon}$ is unimportant and not varying, we write $U^{-}$ for $U^{-\underline{\varepsilon}}$.

**Definition 5.17.** For a constructible open subset $U$ of $S$, we refer to the $U^{-\underline{\varepsilon}}$ above as the **inward perturbation** of $U$ with respect to $S$.
Remark 5.18. More generally, one can define, for every stratum $X_{\alpha}$ of $S$, the set $X'_{\alpha} := N_{\dim X_{\alpha}} X_{\alpha} \setminus \bigcup_{\beta \leq \alpha} N_{\dim X_{\beta}} X_{\beta}$, and to any (not necessarily open) $S$-constructible subset $\bigcup_{\alpha \in A} X'_{\alpha} \subseteq M$, we can associate the manifold-with-corners $\bigcup_{\alpha \in A} X'_{\alpha}$.

Proposition 5.19. For sufficiently small $\varepsilon > 0$, the conormals $\pm L_{U^{-}},$ are disjoint at infinity from $N^* S$, and as $\varepsilon \to 0$, they converge to (i.e. become contained in arbitrarily small neighborhoods of) $N^* S$.

Proof. Let $x \in \partial U^{-}$, and let $S_x \subseteq S$ denote the collection of strata $X$ for which $x \in \partial N_{\dim X} X$. Note that $S_x$ has a unique smallest stratum $X_x$ for sufficiently small $\varepsilon$ (proof: construct $\varepsilon$, by induction).

Let $\xi$ be a conormal direction to $\partial U^{-}$ at $x$, that is $\xi$ is a linear combination of conormal directions to $\partial N_{\dim X} X$ at $x$ for various strata $X \in S_x$. A conormal direction to $\partial N_{\dim X} X$ at $x$ is, for sufficiently small $\varepsilon$, arbitrarily close to the conormal sphere of $X$ near $x$. By Whitney’s condition (a), the conormal sphere of $X$ near $x$ is, in turn, for sufficiently small $\varepsilon$, arbitrarily close to the conormal sphere of the minimal stratum $X_x$ near $x$. It follows that the conormal direction $\xi$ is arbitrarily close to the conormal of the minimal stratum $X_x$ near $x$. This shows that the conormals $\pm L_{U^{-}}$ converge to $N^* S$ as $\varepsilon \to 0$.

To show that the conormals to $U^{-}$ are disjoint at infinity from $N^* S$, we use Whitney’s condition (b). The only strata whose conormals could possibly be hit are those strata inside $U$. Given a point $x \in \partial U^{-}$, taking $\varepsilon > 0$ sufficiently small ensures that the secant lines between $x$ and the nearby strata on the boundary of $U$ have span transverse to $\partial U^{-}$. Such secant lines are approximately tangent to the stratum containing $x$, so we conclude that this stratum is also transverse to $\partial U^{-}$.

Remark 5.20. Any subanalytic family of Legendrians inside $S^* M$ whose projections converge to $\partial U$ converges to the conormals of some refinement of $S$. In contrast, the above proposition does not require refining the stratification.

Corollary 5.21. Let $L$ be any Lagrangian, disjoint at infinity from $N^* S$. Then for all $\varepsilon$ sufficiently small, $\CS(L_{U^{-}}, L) \xrightarrow{\varepsilon} \CW(L_{U^{-}}, L_{N^* S})$.

Proof. Proposition 5.19 and Lemma 5.1 imply that taking $\varepsilon \to 0$ constitutes a cofinal wrapping of $L_{U^{-}}$ in the complement of $N^* S$.

Let $S$ be a triangulation of $M$. To each stratum $s \in S$, we associate its open star $\star(s)$ and the conormal $L_{\star(s)}$ to its inward perturbation $\star(s)^{-}$. By Proposition 5.19, for $\varepsilon$ sufficiently small, the conormal to $\star(s)^{-}$ is disjoint at infinity from $N^* S$, so it defines an object of $\CS(T^* M, N^* S)$. Since $\star(s)$ is contractible, $L_{\star(s)}$ is the conormal to a stable ball, and hence the results of the previous subsection apply, allowing us to deduce the following:

Proposition 5.22. We have

$$\HW(L_{\star(s)}, L_{\star(t)}) = \begin{cases} \mathbb{Z} & t \to s \\ 0 & \text{otherwise} \end{cases} \quad (5.10)$$

generated in the former case by the map from Proposition 5.12.
Proof. Fix a small \( \varepsilon > 0 \) and let \( \delta \to 0 \). By Proposition 5.19, the wrapped Floer cohomology \( HW^*(L_{s\text{star}}(s), L_{\text{star}(t)}) \) is calculated by \( HF^*(L_{s\text{star}(s)} - \varepsilon, L_{\text{star}(t)} - \varepsilon) \).

Now if \( t \to s \), then \( \text{star}(t)^{-\varepsilon} \subseteq \text{star}(s)^{-\varepsilon} \) is an inclusion of stable balls, so Proposition 5.12 produces a canonical generator of \( HF^*(L_{s\text{star}(s)} - \varepsilon, L_{\text{star}(t)} - \varepsilon) = \mathbb{Z} \).

Now suppose that \( t \to s \). If \( \text{star}(s) \cap \text{star}(t) = \emptyset \), then the desired vanishing is trivial. Otherwise, we have \( \text{star}(s) \cap \text{star}(t) = \text{star}(r) \) where \( r \) is the simplex spanned by the union of the vertices of \( s \) and \( t \). To show the desired vanishing, it suffices by Proposition 5.14 to show that \( \text{star}(t)^{-\varepsilon} \cap \partial \text{star}(s)^{-\varepsilon} \) is a stable ball. This space \( \text{star}(t)^{-\varepsilon} \cap \partial \text{star}(s)^{-\varepsilon} \) is homotopy equivalent to the star of \( t \) inside the link of \( s \), and so is contractible. \( \square \)

It will be convenient to have another perspective on the objects \( L_{s\text{star}} \). Let \( L_s \) denote the conormal to a small ball centered at any point on the stratum \( s \) (this conormal is disjoint from \( N_\infty^* S \) at infinity by Whitney’s condition (b)). One reason the \( L_s \) are nice to consider is the following calculation:

**Lemma 5.23.** For any \( S \)-constructible open set \( U \), we have

\[
HW^*(L_U, L_s)_{N_\infty^* S} = \begin{cases} 
\mathbb{Z} & \text{star}(s) \subseteq U \\
0 & \text{otherwise}
\end{cases} \tag{5.11}
\]

Proof. We calculate using Corollary 5.21. If \( s \) is a stratum in the interior of \( U \), then the ball centered at \( s \) is contained in \( U \), so there is a single intersection point. If \( s \) is a stratum not contained in the closure of \( U \), then the morphism space obviously vanishes since the two Lagrangians are disjoint.

Finally, we claim that if \( s \) is a stratum on the boundary of \( U \), the morphism space still vanishes. To see this, start with a small \( \varepsilon > 0 \), and choose \( L_s \) to be the conormal of a small ball disjoint from \( U^{-\varepsilon} \). Now we take \( \varepsilon \to 0 \) with this \( L_s \) fixed, and we claim that the outward conormal to \( U^{-\varepsilon} \) never passes through the outward conormal of this small ball. Indeed, the portion of \( \partial U^{-\varepsilon} \) coming from \( \partial N_\varepsilon(s) \) will be tangent to the small ball, however with the opposite coorientation. The remaining nearby parts of \( \partial U^{-\varepsilon} \), namely coming from \( \partial N_\varepsilon(t) \) for strata \( t \) whose boundaries contain \( s \), will be transverse to the boundary of the small ball by Whitney’s condition (b): any secant line from the center of the ball to a point on its boundary intersected with \( t \) is, by Whitney’s condition (b) approximately tangent to \( t \) (hence to \( \partial N_\varepsilon(t) \)). \( \square \)

Another reason that the \( L_s \) are nice to consider is that we can show using the wrapping exact triangle and stop removal that they (split-)generate:

**Proposition 5.24.** For any stratification \( S \), the objects \( L_s \) for strata \( s \) split-generate \( \mathcal{W}(T^*M, N_\infty^* S) \).

Proof. Denote by \( N_\infty^* S_{\leq k} \) the stratification where we keep all strata of dimension \( \leq k \) and combine all other strata into a single top stratum. We consider the sequence of categories

\[
\mathcal{W}(T^*M, N_\infty^* S) = \mathcal{W}(T^*M, N_\infty^* S_{\leq k}) \to \mathcal{W}(T^*M, N_\infty^* S_{\leq k-2}) \to \cdots \\
\cdots \to \mathcal{W}(T^*M, N_\infty^* S_{\leq 0}) \to \mathcal{W}(T^*M). \tag{5.12}
\]
Each of these functors removes a locally closed Legendrian submanifold \( N^*_\infty S_{\leq k} \setminus N^*_\infty S_{\leq k-1} \), and thus by stop removal Theorem 5.5, is the quotient by the corresponding linking disks.

The linking disk at a point on \( N^*_\infty S_{\leq k} \setminus N^*_\infty S_{\leq k-1} \) can be described as follows. A point on \( N^*_\infty S_{\leq k} \setminus N^*_\infty S_{\leq k-1} \) is simply a point \( x \) on a \( k \)-dimensional stratum together with a covector \( \xi \) at \( x \) conormal to the stratum. Consider a small ball \( B_a \) centered at \( x \), and consider a smaller ball \( B_b \subseteq B_a \) disjoint from the stratum containing \( x \). There is a family of balls starting at \( B_b \) and shrinking down to \( B_b \) whose boundaries are tangent to the stratum containing \( x \) only at \( (x, \xi) \). It follows from the wrapping exact triangle Theorem 5.4 that the cone on the resulting continuation map \( L_{B_b} \rightarrow L_{B_b} \) is precisely the linking disk at \( (x, \xi) \).

We have thus shown that the linking disks to each locally closed Legendrian \( N^*_\infty S_{\leq k} \setminus N^*_\infty S_{\leq k-1} \) are generated by the objects \( L_s \). By Theorem 5.6 above, these \( L_s \) also split-generate the final category \( \mathcal{W}(T^*M) \). We conclude that the \( L_s \) split-generate \( \mathcal{W}(T^*M, N^*_\infty S) \), as the quotient by all of them vanishes.

**Remark 5.25.** A small variation on the above proof and an appeal to [20, Theorem 1.9] shows that the objects \( L_s \) in fact generate \( \mathcal{W}(T^*M, N^*_\infty S) \). We give the weaker argument above to minimize the results we need to appeal to.

**Proposition 5.26.** The map \( L_{\text{star}(s)} \rightarrow L_s \) from Proposition 5.12 is an isomorphism in \( \mathcal{W}(T^*M, N^*_\infty S) \).

**Proof.** We proceed by induction on the codimension of \( s \). When \( s \) has codimension zero, the desired statement follows from Corollary 5.13.

Now suppose that \( s \) has positive codimension. For any \( t \) of strictly smaller codimension than \( s \), we have \( \text{Hom}(L_{\text{star}(t)}, L_{\text{star}(s)}) = 0 \) by Proposition 5.22 and \( \text{Hom}(L_{\text{star}(t)}, L_s) = 0 \) by Lemma 5.23.

Now by the discussion in the proof of Proposition 5.24, the functor

\[
\mathcal{W}(T^*M, N^*_\infty S) \rightarrow \mathcal{W}(T^*M, N^*_\infty S_{\leq \dim s})
\]

quotients by cones of \( L_t \) for \( t \) of strictly smaller codimension than \( s \). By the induction hypothesis and the calculations of the previous paragraph, such cones are left-orthogonal to \( L_s \) and \( L_{\text{star}(s)} \). Hence it suffices to check that \( L_{\text{star}(s)} \rightarrow L_s \) is an isomorphism in \( \mathcal{W}(T^*M, N^*_\infty S_{\leq \dim s}) \).

Finally, we observe that \( L_{\text{star}(s)} \rightarrow L_s \) is an isomorphism in \( \mathcal{W}(T^*M, N^*_\infty S_{\leq \dim s}) \) by Proposition 5.15. Namely, we take \( Y = \text{star}(s)^-, X = s \cap \text{star}(s)^- \), and \( Z = t \cap \text{star}(s)^- \) for any simplex \( t \) containing \( s \) and of one higher dimension.

**Remark 5.27.** For a smooth triangulation \( S \), there is an obvious positive isotopy from \( L_s \) to \( L_{\text{star}(s)} \) disjoint from \( N^*_\infty S \) (thus proving Proposition 5.26 in this case), obtained by expanding a small ball centered at a point on \( s \) to \( \text{star}(s) \), keeping the boundary transverse to the strata of \( S \). We do not know whether this proof can be generalized from smooth triangulations to subanalytic triangulations.
5.8 Functors from poset categories to Fukaya categories

**Definition 5.28.** Let $M$ be a manifold with stratification $S$, and let $U : \Pi \to \text{Op}_S(M)$ be a map from a poset $\Pi$ to the poset of $S$-constructible open subsets of $M$. Suppose further that each $U(\pi,)$ (from Section 5.7) is a stable ball. Define a functor on cohomology categories

$$H^*F_U : \mathbb{Z}[\Pi] \to H^*\mathcal{W}(T^*M, N^*_\infty S)^{\text{op}} \quad (5.14)$$

by $H^*F_U(\pi) := L_{U(\pi)}$ and $H^*F_U(1_{\pi,\pi'}) = 1_{U(\pi),U(\pi')} \in HW^*(L_{U(\pi')}, L_{U(\pi)})$ is the canonical generator from Proposition 5.12.

**Remark 5.29.** Note that this definition of $H^*F_U$ depends on having chosen the correct $K(\mathbb{Z}/2,1)$-bundle over $T^*M$ to twist by (compare §5.3). Having chosen the wrong such bundle would show up in the functor respecting composition only up to a sign. The resulting 2-cocycle, or rather its class in $H^2(\Pi, \mathbb{Z}/2)$, would represent (the pullback to $\Pi$ from $M$ of) the obstruction to choosing continuously varying relative $\text{Pin}$-structures on all cotangent fibers.

**Proposition 5.30.** Let $H^*F : \mathbb{Z}[\Pi] \to H^*\mathcal{C}$ be any functor on cohomology categories such that $H^*\mathcal{C}(F(x), F(y))$ is free and concentrated in degree zero for every pair $x \leq y \in \Pi$. Then there exists an $A_\infty$ functor $F$ lifting $H^*F$, and moreover the space of natural isomorphisms between any two such lifts is contractible.

**Proof.** We show existence of a lift $F$ by induction. Take $F^1$ to be any map with the correct action on cohomology. Having chosen $F^1, \ldots, F^{k-1}$, the obstruction to the existence of an $F^k$ satisfying the $A_\infty$ functor equations of order $k$ is a degree $2-k$ cohomology class in (the cohomology of)

$$\prod_{\pi_0 \leq \cdots \leq \pi_k \in \Pi} \text{Hom}(\mathcal{C}(F(\pi_0), F(\pi_1)) \otimes \cdots \otimes \mathcal{C}(F(\pi_{k-1}), F(\pi_k)), \mathcal{C}(F(\pi_0), F(\pi_k))). \quad (5.15)$$

Appealing to cofibrancy of $\mathcal{C}(-,-)$ and the fact that $H^*\mathcal{C}(F(x), F(y))$ is free and concentrated in degree zero, we conclude that the obstruction class must vanish when $k \geq 3$ for degree reasons. When $k = 2$, the obstruction class measures the failure of $H^*F$ to respect composition, so by hypothesis this obstruction also vanishes. Hence in either case, there exists an $F^k$ compatible with the previously chosen $F^1, \ldots, F^{k-1}$. (Compare [46, Lemma 1.9], where this obstruction theory argument is explained in more detail.)

To analyze the space of natural quasi-isomorphisms, we again argue inductively to show that all obstructions vanish (again using the $\mathbb{Z}$-grading).

**Corollary 5.31.** There is a unique up to contractible choice $A_\infty$ functor

$$F_U : \mathbb{Z}[\Pi] \to \mathcal{W}(T^*M, N^*_\infty S)^{\text{op}} \quad (5.16)$$

lifting the functor on cohomology categories from Definition 5.28.

**Proof.** By Corollary 5.21, the wrapped Floer cohomology group $HW^*(L_{U(\pi)}$, $L_{U(\pi')})$ is simply the Floer cohomology of two nested stable balls, which is $\mathbb{Z}$ by Proposition 5.12. Thus Proposition 5.30 is applicable.
Remark 5.32. To extend Corollary 5.31 to the Fukaya category with a \( \mathbb{Z}/N \)-grading, we would need to add to the requirement that \( F \) (and natural transformations \( F_1 \to F_2 \)) must lift to \( \mathbb{Z} \)-graded categories locally (the \( \mathbb{Z} \)-grading is only defined locally, over any contractible open subset of \( M \)).

**Definition 5.33.** For a triangulation \( S \), let

\[
F_S : \mathbb{Z}[S] \to W(T^*M, N_{\infty}^*S)^{\text{op}}
\]

(5.17)

denote the functor induced from Definition 5.28 and Corollary 5.31 by the map associating to each simplex of \( S \) its open star.

**Theorem 5.34.** The functor \( F_S \) is a Morita equivalence.

**Proof.** Proposition 5.22 shows is full faithfulness of \( F_S \), and Propositions 5.24 and 5.26 together show essential surjectivity of \( F_S \) (after passing to \( \text{Perf} \)). \( \square \)

We now show that \( F_S \) is compatible with refinement (compare Lemma 4.6):

**Theorem 5.35.** For any refinement of triangulations \( S' \) refining \( S \), the following diagram commutes:

\[
\begin{array}{ccc}
\mathbb{Z}[S'] & \xrightarrow{F_{S'}} & W(T^*M, N_{\infty}^*S')^{\text{op}} \\
\downarrow r & & \downarrow \rho \\
\mathbb{Z}[S] & \xrightarrow{F_S} & W(T^*M, N_{\infty}^*S)^{\text{op}}
\end{array}
\]

(5.18)

up to contractible choice.

**Proof.** There are two functors \( \rho \circ F_{S'} \) and \( F_S \circ r \) from \( \mathbb{Z}[S'] \) to \( W(T^*M, N_{\infty}^*S) \). By Corollary 5.30, it suffices to define a canonical natural isomorphism between the induced functors on cohomology categories. It is most natural to define this canonical natural isomorphism in the direction \( F_S \circ r \Rightarrow \rho \circ F_{S'} \).

To a stratum \( s \) of \( S' \), the composition \( F_S \circ r \) associates the conormal of \( \text{star}_S(r(s)) \), and the composition \( \rho \circ F_{S'} \) associates the conormal of \( \text{star}_{S'}(s) \). Both are stable balls, and there is an inclusion \( \text{star}_S(r(s)) \supseteq \text{star}_{S'}(s) \), so by Proposition 5.12 there is a canonical map from one to the other. Using the composition property of Proposition 5.12, it is easy to check that this defines a natural transformation \( H^*(F_S \circ r) \Rightarrow H^*(\rho \circ F_{S'}) \).

This natural transformation is in fact a natural isomorphism since the natural maps from both \( L_{\text{star}_S(r(s))} \) and \( L_{\text{star}_{S'}(s)} \) to \( L_s = L_{r(s)} \) are isomorphisms by Proposition 5.26. \( \square \)

**5.9 In conclusion**

**Theorem 5.36.** The functor \( \Lambda \mapsto W(T^*M, \Lambda)^{\text{op}} \) is a microlocal Morse theater in the sense of Definition 3.5, which casts the linking disks at smooth points of \( \Lambda \) as the Morse characters.
**Proof.** Definition 5.33 and Theorems 5.34 and 5.35 give the identification between \( S \mapsto \text{Perf} \; S \) and \( S \mapsto \mathcal{W}(T^*M, N^*_\infty S)^{op} \).

Stop removal Theorem 5.5 says that \( \mathcal{W}(T^*M, \Lambda') \to \mathcal{W}(T^*M, \Lambda) \) is the quotient by the linking disks at the smooth points of \( \Lambda' \setminus \Lambda \). It therefore suffices to show that the Morse characters are precisely these linking disks.

Recall from Definition 3.3 that a Morse character at a smooth point \( p \in \Lambda \) is defined as follows. We choose a function \( f : M \to \mathbb{R} \) and an \( \epsilon > 0 \) such that \( f \) has no critical values in \([-\epsilon, \epsilon]\) and \( df \) is transverse to \( \Lambda \) over \( f^{-1}[-\epsilon, \epsilon] \), intersecting it only at \( p \) (where \( f \) vanishes). We also choose a triangulation \( S \) such that \( \Lambda \subseteq N^*_\infty S \) and \( f^{-1}(-\infty, -\epsilon) \) and \( f^{-1}(-\infty, \epsilon) \) are constructible. The Morse character associated to these choices is then defined as the image in \( \mathcal{W}(T^*M, \Lambda) \) of

\[
\text{cone}(1_{f^{-1}(-\infty, -\epsilon)} \to 1_{f^{-1}(-\infty, \epsilon)}) \in \text{Perf} \; S = \mathcal{W}(T^*M, N^*_\infty S)^{op}.
\]

To show that this cone is indeed the linking disk at \( p \) in \( \mathcal{W}(T^*M, \Lambda) \), we appeal to the wrapping exact triangle Theorem 5.4. It thus suffices to show that \( 1_{f^{-1}(-\infty, \pm \epsilon)} \in \mathcal{W}(T^*M, N^*_\infty S) \) is identified with the conormal of \( f^{-1}(-\infty, \pm \epsilon) \), and the map \( 1_{f^{-1}(-\infty, -\epsilon)} \to 1_{f^{-1}(-\infty, \epsilon)} \) is identified with the continuation map associated to the obvious positive isotopy from the conormal of \( f^{-1}(-\infty, -\epsilon) \) to the conormal of \( f^{-1}(-\infty, \epsilon) \) coming from the fact that \( f \) has no critical values in the interval \([-\epsilon, \epsilon]\).

We first note that because the level sets \( f^{-1}(\pm \epsilon) \) are smooth, it is straightforward to produce an isotopy between the above inward conormals and the **inward perturbation with respect to** \( S \) of \( f^{-1}(-\infty, \pm \epsilon) \), in the sense of Definition 5.17. Henceforth we work with the latter, and denote them as \( L_{f^{-1}(-\infty, \pm \epsilon)} \).

It follows from Lemma 5.23 that

\[
HW^*(L_{f^{-1}(-\infty, \pm \epsilon)}, L_s) N^*_\infty S = \begin{cases} \mathbb{Z} & \text{star}(s) \subseteq f^{-1}(-\infty, \pm \epsilon) \\ 0 & \text{otherwise} \end{cases}
\]

As \( L_{\text{star}(s)} = L_s \), this identifies the pullback under \( F_S \) (Definition 5.33) of the Yoneda module of \( L_{f^{-1}(-\infty, \pm \epsilon)} \) with the indicator \( 1_{f^{-1}(-\infty, \pm \epsilon)} \).

As \( \partial_{\infty} L_{f^{-1}(-\infty, \epsilon)} \) falls immediately into the stop \( N^*_\infty S \), we have

\[
HW^*(L_{f^{-1}(-\infty, \epsilon)}, L_{f^{-1}(-\infty, -\epsilon)}) N^*_\infty S = HF^*(L_{f^{-1}(-\infty, \epsilon)}, L_{f^{-1}(-\infty, -\epsilon)}).
\]

By Yoneda, the map \( 1_{f^{-1}(-\infty, -\epsilon)} \to 1_{f^{-1}(-\infty, \epsilon)} \) corresponds to an element of the above group. It remains to show that this element is the continuation map.

We test both against the generators \( L_s \) for \( \mathcal{W}(M, N^*_\infty S) \); for \( 1_{f^{-1}(-\infty, -\epsilon)} \to 1_{f^{-1}(-\infty, \epsilon)} \) the answer is determined by Theorem 5.34. Both tests give zero unless the stratum \( s \) is contained in \( f^{-1}(-\infty, -\epsilon) \); in this case we may arrange that \( L_s \) is the conormal to a subset of \( f^{-1}(-\infty, -\epsilon) \) as well. For these \( L_s \), the evident isotopy \( L_{f^{-1}(-\infty, -\epsilon)} \prec L_{f^{-1}(-\infty, \epsilon)} \) is disjoint at infinity from \( L_s \), so the desired assertion follows from [19, Lemma 3.26].

\( \square \)
6 Examples

6.1 Cotangent bundles

Let $M$ be a smooth manifold. The cotangent fibers $F_q \in \mathcal{W}(T^*M)$ generate by Abouzaid [3, 4] when $M$ is closed and by [20, Theorem 1.9] in general.

When $M$ is closed, Abbondandolo–Schwarz [1] and Abouzaid [6] calculated the endomorphism algebra of the fiber as $CW^*(F_q, F_q) = C_{-\bullet}(\Omega_q M)$ (using relative Pin structures as in Section 5.3). The present Theorem 1.1 (which does not depend on any of [1, 6, 4, 3]) gives a proof of this fact for all (not necessarily closed) $M$:

**Corollary 6.1.** There is a quasi-isomorphism $CW^*(F_q, F_q) = C_{-\bullet}(\Omega_q M)$. Moreover if $M \subseteq N$ is a codimension zero inclusion, there is a commutative diagram

$$
\begin{array}{ccc}
CW^*(F_q, F_q)_{T^*M} & \longrightarrow & C_{-\bullet}(\Omega_q M) \\
\downarrow & & \downarrow \\
CW^*(F_q, F_q)_{T^*N} & \longrightarrow & C_{-\bullet}(\Omega_q N)
\end{array}
$$

(6.1)

where the left hand vertical arrow is covariant inclusion and the right hand vertical arrow is induced by pushforward of loops.

**Proof.** Note that there exists a real analytic structure on $M$ whose induced smooth structure agrees with the given one. Taking $\Lambda = \emptyset$ in Theorem 1.1 gives $\text{Perf} \mathcal{W}(T^*M) = \text{Sh}_\emptyset(M)$. It is well known the latter is the category $\text{Perf} C_{-\bullet}(\Omega_q M)$, e.g. because both are the global sections of the constant cosheaf of linear categories with costalk $\text{Mod} \mathbb{Z}$.

We may derive the more precise assertion that $C_{-\bullet}(\Omega_q M)$ is endomorphisms of the cotangent fiber by following a fiber through the equivalence, e.g. by considering the inclusion of the cotangent bundle of a disk, or equivalently by introducing a stop along the conormal of the boundary of a disk and then removing it. \hfill \square

6.2 Plumbings

Many authors have studied Fukaya categories of plumbings [5, 8, 16] and their sheaf counterparts [10]. Here we compute the wrapped category of a plumbing.

Let $\Pi_{2n}$ be the Liouville pair $(\mathbb{C}^n, \partial_{\infty}(\mathbb{R}^n \cup i\mathbb{R}^n))$; we term it the plumbing sector. Plumbings are formed by taking a manifold $M$ (usually disconnected) with spherical boundary $\partial M = \bigsqcup S^{n-1}$, and gluing the Liouville pair $(T^*M, \partial M)$ to some number of plumbing sectors along the spheres.

One can model the wrapped Fukaya category of the plumbing sector directly in sheaf theory: we can view it as the pair $(T^*\mathbb{R}^n, N^*_\infty \{0\})$, and the category $\text{Sh}_{N^*_\infty \{0\}} \mathbb{R}^n$ has a well-known description in terms of the Fourier transform as described in [10]. This category is equivalent to $\mathcal{W}(\Pi_{2n})$ by Theorem 1.1. To apply the gluing results of [20], however, we need to know how the wrapped Fukaya categories of the two boundary sectors include, which is slightly more than what Theorem 1.1 tells us. Hence we give a direct computation of
the wrapped Fukaya category of the plumbing sector. Take a positive Reeb pushoff of the boundary of a cotangent fiber in $T^*\mathbb{R}^n$, so it is now the outward conormal of a small ball. Deleting the original cotangent fiber, we obtain the Liouville sector $T^*S^{n-1} \times \mathcal{A}_2$ where $\mathcal{A}_2$ denotes the Liouville sector $(\mathbb{C}, \{e^{2\pi ik/3}\}_{k=0,1,2})$. We can get back to the plumbing sector $\Pi_2n$ by adding back the missing fiber, which amounts to attaching a Weinstein handle along one of the boundary sectors $T^*(S^{n-1} \times I)$. We may thus deduce from [20, Thm. 1.20, Thm. 1.5, and Cor. 1.11] that:

**Lemma 6.2.**

$$\text{Perf}(\Pi_{2n}) = \text{Perf}(\text{colim}(\text{Perf}(-) \leftarrow \text{Perf}(\Omega S^{n-1}) \to \text{Perf}(- \to -) \otimes \text{Perf}(\Omega S^{n-1})))$$

Gluing in the remaining manifolds, we conclude:

**Corollary 6.3.** The wrapped Fukaya category of a plumbing is calculated by (Perf applied to) the colimit of the diagram:

$$\bigcup \text{Perf}(-) \xrightarrow{\text{colim}} \bigcup \text{Perf}(\Omega S^{n-1}) \xrightarrow{\text{Perf}(- \to -) \otimes \text{Perf}(\Omega S^{n-1})} \bigcup \text{Perf}(\Omega M_i)$$

where $M_i$ are the components of $M$.

### 6.3 Proper modules and infinitesimal Fukaya categories

Recall that for a dg or $A_\infty$ category $\mathcal{C}$, we write $\text{Prop}\mathcal{C} := \text{Fun}(\mathcal{C}, \text{Perf}\mathbb{Z})$ for the category of proper (aka pseudo-perfect) modules. It is immediate from our main result that $\text{PropSh}_\Lambda(M)^c = \text{PropW}(T^*M, \Lambda)^{op}$.

Recall from Corollary 4.16 that any proper $\text{Sh}_\Lambda(M)^c$-module is representable by an object of $\text{Sh}_\Lambda(M)$ with perfect stalks, i.e. a constructible sheaf in the classical sense. Let us describe some objects in the Fukaya category $\mathcal{W}(T^*M, \Lambda)$ which necessarily give rise to proper modules (and thus to sheaves on $M$ with perfect stalks, microsupported inside $\Lambda$).

**Definition 6.4.** For any stopped Liouville manifold $(X, f)$, we define the forward stopped subcategory $\mathcal{W}^*(X, f)$ to be the full subcategory of $\mathcal{W}(X, f)$ generated by Lagrangians which admit a positive wrapping into $f$, meaning $\partial_\infty L$ becomes contained in arbitrarily small neighborhoods of $f$. By Lemma 5.1, such a wrapping is necessarily cofinal.

**Example 6.5.** If $f$ admits a ribbon $F$ (or, alternatively, is itself equal to a Liouville hypersurface $F$) then $\mathcal{W}^*(X, f)$ contains all Lagrangians whose boundary at infinity is contained in a neighborhood of a small negative Reeb pushoff of $f$ (or $F$).
Example 6.6. All compact (exact) Lagrangians are contained in $\mathcal{W}^c(X, f)$, as their boundary at infinity $\emptyset$ is wrapped into $f$ by the trivial wrapping.

Proposition 6.7. All objects of $\mathcal{W}^c(X, f)$ co-represent proper modules over $\mathcal{W}(X, f)$; that is, the restriction of the Yoneda embedding $\mathcal{W}(X, f) \rightarrow \text{Mod} \mathcal{W}(X, f)^{\text{op}}$ to $\mathcal{W}^c(X, f)$ has image contained in $\text{Prop} \mathcal{W}(X, f)^{\text{op}}$.

Proof. Morphisms in the wrapped category can be computed by cofinally positively wrapping the first factor. Any $L \in \mathcal{W}^c(X, f)$ admits such a wrapping $\{L_t\}_{t \geq 0}$ which converges at infinity to $f$. It follows that after some time $t$, its boundary at infinity stays disjoint at infinity from $K$, and hence $CW^*(L, K) = CF^*(L_t, K)$ for sufficiently large $t$.

Corollary 6.8. The equivalence $\text{Perf} \mathcal{W}(T^*M, \Lambda)^{\text{op}} = \mathcal{S}h_{\mathcal{A}}(M)^c$ sends $\mathcal{W}^c(T^*M, \Lambda)$ fully faithfully into $\text{Prop} \mathcal{S}h_{\mathcal{A}}(M)^c$.

Recall that for a triangulation $S$, the category $\mathcal{W}(T^*M, N^*_\infty S)^{\text{op}}$ is Morita equivalent to $\mathbb{Z}[S]$, hence smooth and proper. The generators $L_{\text{stat}}(s)$ of $\mathcal{W}(T^*M, N^*_\infty S)$ used to prove this equivalence were shown in that proof to lie in $\mathcal{W}^c(T^*M, N^*_\infty S)$, so we have:

Proposition 6.9. For a triangulation $S$, the inclusion $\mathcal{W}^c(T^*M, N^*_\infty S) \subseteq \mathcal{W}(T^*M, N^*_\infty S)$ is a Morita equivalence.

Remark 6.10. Corollary 6.8 is very similar to the original Nadler–Zaslow correspondence [37], restricted to Lagrangians with fixed asymptotics. To be more precise, recall that Nadler–Zaslow wish to consider an infinitesimially wrapped Fukaya category $\mathcal{W}^\inf_f(T^*M)$ of Lagrangians ‘asymptotic at infinity to $f$’ and then show it is equivalent to a category of sheaves on $M$ with microsupport inside $f$.

If $f$ is a smooth Legendrian and $\mathcal{W}^\inf_f(T^*M)$ is defined to consist of Lagrangians which are conical at infinity, ending inside $f$, then there is a fully faithful embedding $\mathcal{W}^\inf_f(T^*M) \hookrightarrow \mathcal{W}^c(T^*M, f)$, sending a Lagrangian ending inside $f$ to its small negative pushoff (which then tautologically wraps positively back into $f$). Hence Corollary 6.8 recovers a version of [37] when $f$ is a smooth Legendrian. One can certainly imagine constructing such an embedding $\mathcal{W}^\inf_f(T^*M) \hookrightarrow \mathcal{W}^c(T^*M, f)$ for more general (e.g. subanalytic isotropic) $f$ (for some particular definition of $\mathcal{W}^\inf_f(T^*M)$).

Remark 6.11. We do not know when $\mathcal{W}^c(T^*M, \Lambda)^{\text{op}} \hookrightarrow \text{Prop} \mathcal{S}h_{\mathcal{A}}(M)^c$ is a Morita equivalence. Note that the assertion of such an equivalence (for $\mathcal{W}^\inf_N(T^*M)^{\text{op}}$) is not made in [33], although that work is occasionally misquoted to suggest that it is. What is actually said is that one can get all objects of $\text{Prop} \mathcal{S}h_{\mathcal{A}}(M)^c$ from twisted complexes of objects of $\mathcal{W}^\inf_N(T^*M)^{\text{op}}$ for a possibly larger $\Lambda'$ which, as twisted complexes, pair trivially with all Lagrangians contained in a neighborhood of $\Lambda' \setminus \Lambda$ (and thus could be said to be “Floer-theoretically supported away from $\Lambda' \setminus \Lambda$”).

To make a precise statement along the lines of Remark 6.11, realizing a version of the Nadler–Zaslow equivalence, we have:

Proposition 6.12. If $S$ is any triangulation of compact $M$ with $\Lambda \subseteq N^*_\infty S$, and $\mathcal{D}$ denotes the collection of linking disks to smooth points of $N^*_\infty S \setminus \Lambda$, then

$$\text{Prop} \mathcal{S}h_{\mathcal{A}}(M)^c = \text{Prop} \mathcal{W}(T^*M, \Lambda)^{\text{op}} = (\text{Tw} \mathcal{W}^c(T^*M, N^*_\infty S)^{\text{op}})_{\text{Ann}(\mathcal{D})} \quad (6.2)$$
where Tw denotes twisted complexes (i.e. any model for the the pre-triangulated, non idempotent-completed, hull), and the subscript Ann(Ď) indicates taking the full subcategory of objects annihilated by CW∗(−, D) = 0 for all D ∈ Ď.

Proof. For such an S, the functor j : W(T∗M, N∗∞S) → W(T∗M, Λ) is the quotient by Ď by Theorem 5.5. Pullback of modules under any localization is a fully faithful embedding, identifying the category of modules over the localized category with the full subcategory of modules over the original category which annihilate the objects quotiented by (see §A.3 and [19, Lemmas 3.12 and 3.13]). Properness of a module is also clearly equivalent to properness of its pullback. We thus conclude that

\[ j^* : \text{Prop } W(T^*M, \Lambda)^{\text{op}} \hookrightarrow \text{Prop } W(T^*M, N^*_\infty S)^{\text{op}} \]

embeds the former as the full subcategory of the latter annihilating Ď.

Now W(T∗M, N∗∞S) (Morita equivalent to Perf S^{\text{op}} by Proposition 5.34) is smooth and proper by Lemma A.10 (since M is compact and thus there are finitely many simplices). Hence Prop W(T∗M, N∗∞S)^{\text{op}} = Perf W(T∗M, N∗∞S)^{\text{op}} = Perf W^{\text{op}}(T∗M, N∗∞S)^{\text{op}} (by Proposition 6.9). Finally, we observe that idempotent completion is unnecessary by Lemma A.9, as Perf S has a generating exceptional collection. \qed

Remark 6.13. For non-compact M, the same proof implies that

\[ \text{Prop Sh}_A(M)^c = (\text{Prop } W^c(T^*M, N^*_\infty S)^{\text{op}})_{\text{Ann}(Ď)} \supseteq (\text{Perf } W^c(T^*M, N^*_\infty S)^{\text{op}})_{\text{Ann}(Ď)} \]

but the inclusion is not generally an equality.

Example 6.14. Let us explain how our ‘stopped’ setup can be used to make ordinary (not wrapped) Floer cohomology calculations using sheaves. Suppose given two Lagrangians L, K ⊆ T∗M for which Λ := ∂∞L ∪ ∂∞K is subanalytic. We are interested in computing HF∗(L+, K). Thus consider the wrapped category W(T∗M, Λ) and small negative pushoffs L−, K− ∈ W(T∗M, ∂L ∪ ∂K), and observe that

\[ HF^*(L^+, K) = HW^*(L^-, K^-)_Λ. \]

By our main result, the right hand side can be computed as Hom(ℱK, ℱL) in the sheaf category Sh_Λ(M), provided we can determine the sheaves ℱL and ℱK to which L− and K− are sent by our Theorem 1.1.

Here we make only a few observations regarding how to determine these sheaves. Because linking disks go to microstalks and L−, K− are forward stopped, we can see immediately that ℱL, ℱK have microstalks Z along the respective loci ∂∞L, ∂∞K ⊆ Λ. For the same reason, for p away from the front projection of Λ = ∂∞L ∪ ∂∞K, we have

\[ ℱ_L|_p \cong CF^*(L, T^*_p M) \quad \text{and} \quad ℱ_K|_p \cong CF^*(K, T^*_p M). \]

In some cases, e.g. in case that L intersects every cotangent fiber either once or not at all, this data already suffices to determine ℱL. In particular, this situation occurs in [49], where sheaf calculations are made exhibiting cluster transformations arising from comparing different fillings of Legendrian knots. The present discussion suffices to translate those calculations into calculations in Lagrangian Floer theory.
6.4 Legendrians and constructible sheaves

**Corollary 6.15.** Let \( \Lambda \subseteq J^1\mathbb{R}^n \subseteq S^*\mathbb{R}^{n+1} \) be a smooth compact Legendrian. Let \( D \) be its linking disk, and consider the algebra

\[
\mathcal{A}_\Lambda := CW^*(D, D)_{T^*\mathbb{R}^{n+1}, \Lambda}.
\]

Then \( \text{Mod} \mathcal{A}_\Lambda^{\text{op}} \) is equivalent to the category \( \text{Sh}_\Lambda(T^*\mathbb{R}^{n+1})_0 \) of sheaves microsupported inside \( \Lambda \) and with vanishing stalk at infinity. This equivalence identifies the forgetful functor \( \text{Mod} \mathcal{A}_\Lambda^{\text{op}} \to \text{Mod} \mathbb{Z} \) with the microstalk along \( \Lambda \). Hence \( \text{Prop} \mathcal{A}_\Lambda^{\text{op}} \) is equivalent to the subcategory of \( \text{Sh}_\Lambda(T^*\mathbb{R}^{n+1})_0 \) of objects with perfect microstalk along \( \Lambda \) (or equivalently, with perfect stalks).

**Proof.** Our generation results [20, Theorem 1.9] imply that \( \mathcal{W}(T^*\mathbb{R}^{n+1}, \Lambda) \) is generated by \( D \) and a cotangent fiber near infinity. Because we assume that \( \Lambda \subseteq J^1\mathbb{R}^n \), the cotangent fiber at negative (in the last coordinate) infinity can be cofinally positively wrapped without intersecting \( \Lambda \), and likewise the (isomorphic) cotangent fiber at positive infinity can be cofinally negatively wrapped without intersecting \( \Lambda \). These large wrappings are conormals to large disks in \( \mathbb{R}^{n+1} \) containing the projection of \( \Lambda \); they thus have vanishing wrapped Floer cohomology (in both directions) with the linking disk \( D \) to \( \Lambda \). Thus \( \mathcal{W}(T^*\mathbb{R}^{n+1}, \Lambda) \) is generated by the two orthogonal objects \( D \) and the fiber at infinity.

Denote by \( \mu, \sigma \in \text{Sh}_\Lambda(T^*\mathbb{R}^{n+1})^c \) the objects corresponding to \( D \) and the fiber at infinity, respectively. They are orthogonal, and have endomorphism algebras \( \mathcal{A}_\Lambda^{\text{op}} \) and \( \mathbb{Z} \), respectively.

We have \( \text{Sh}_\Lambda(T^*\mathbb{R}^{n+1}) = \text{Mod} \text{Sh}_\Lambda(T^*\mathbb{R}^{n+1})^c = \text{Mod} \mathcal{W}(T^*\mathbb{R}^{n+1}, \Lambda)^{\text{op}} = \text{Mod} \mathcal{A}_\Lambda^{\text{op}} \oplus \text{Mod} \mathbb{Z} \), and this equivalence is given concretely by \( \mathcal{F} \mapsto \text{Hom}(\mu, \mathcal{F}) \oplus \text{Hom}(\sigma, \mathcal{F}) \). By Theorem 1.1, \( \text{Hom}(\mu, \mathcal{F}) \) is the microstalk along \( \Lambda \) and \( \text{Hom}(\sigma, \mathcal{F}) \) is the stalk at infinity.

To see that perfect stalks is equivalent to perfect microstalks along \( \Lambda \) for objects of \( \text{Sh}_\Lambda(T^*\mathbb{R}^{n+1})_0 \), argue as follows. Suppose microstalks are perfect. Stalks are computed by \( \text{Hom}(\mathbb{Z}_{B_\epsilon(x)}, \mathcal{F}) \) for some sufficiently small \( \epsilon > 0 \) (in terms of \( \Lambda \)), since changing \( \epsilon \) is non-characteristic by Whitney’s condition B for the image of \( \Lambda \). Now moving \( B_\epsilon(x) \) generically to infinity picks up some number of microstalks when its conormal passes through \( \Lambda \) (transversally), and eventually gives zero since the stalk of \( \mathcal{F} \) near infinity vanishes. Thus perfect microstalks implies perfect stalks. To see that perfect stalks implies perfect microstalks, recall that \( \text{Sh}_\Lambda(T^*\mathbb{R}^{n+1}) \) is a full subcategory of \( \text{Sh}_S(T^*\mathbb{R}^{n+1}) \) for a triangulation \( S \) whose conormal contains \( \Lambda \). Perfect stalk and compact support objects of \( \text{Sh}_S(T^*\mathbb{R}^{n+1}) \) are generated by \( \mathbb{Z}_{\text{star}(a)} \), which all have perfect microstalks.

Let us comment on the relation of the above result to the ‘augmentations are sheaves’ statement in [50, 41]. There is an evident similarity: both relate augmentations of an algebra associated to a Legendrian to categories of sheaves microsupported in that Legendrian. But they are not exactly the same: the algebra \( \mathcal{A}_\Lambda \) is not by definition the Chekanov–Eliashberg dga, and moreover in [41] the category of augmentations is defined by a somewhat complicated procedure, not just as proper modules over a dga. Also in [41], the authors restrict attention to augmentations, i.e. 1-dimensional representations of the dga, whereas the above result concerns the entire representation category (the underlying \( \mathbb{Z} \)-module of the representation being the microstalk), specializing to a comparison of rank \( k \) representations with rank \( k \) microstalk sheaves for every \( k \).
In fact, $\mathcal{A}_\Lambda$ was conjectured by Sylvan to be a version of the Chekanov–Eliashberg dga with enhanced $C_*(\Omega\Lambda)$ coefficients. A precise statement comparing $\mathcal{A}_\Lambda$ to such a generalized “loop space dga” can be found in [14, Conj. 3], where it is explained that the comparison should follow from a slight variant of the surgery techniques of [12] (the complete proof of which has yet to appear). The relation between the multiple copy construction of [41] and the loop space dga can also be extracted from [14].

Finally we note that a version of the above discussion serves to translate between the arguments of [47] and [15].

6.5 Fukaya-Seidel categories of cotangent bundles

Let $W : T^*M \to \mathbb{C}$ be an exact symplectic fibration with singularities. The associated Fukaya–Seidel category is by definition $\mathcal{W}(T^*M, W^{-1}(-\infty))$. According to [20], retracting the stop to its core does not affect the category: $\mathcal{W}(T^*M, W^{-1}(-\infty)) = \mathcal{W}(T^*M, c_{W^{-1}(-\infty)})$. Thus if the fiber is Weinstein, then we may calculate the corresponding Fukaya–Seidel category using Theorem 1.1 (provided the core can be made subanalytic).

In particular, the sheaf-theoretic work on mirror symmetry for toric varieties may now be translated into assertions regarding the wrapped Fukaya category. Recall that [17] introduced for any $n$-dimensional toric variety $T$ a certain Lagrangian $\Lambda_T \subset T^*(S^1)^n$. They conjectured, and [27] proved, that $\text{Sh}_{\partial_\infty \Lambda_T}((S^1)^n)^c = \text{Coh}(T)$, where we use Coh to denote the dg category of coherent complexes. By Theorem 1.1, we may conclude:

**Corollary 6.16.** $\text{Perf} \mathcal{W}(T^*(S^1)^n, \partial_\infty \Lambda_T)^{\text{op}} = \text{Coh}(T)$

We note that this is a much stronger statement than [2], in that it includes the case of non-Fano, non-compact, singular, and stacky $T$.

When $T$ is smooth and Fano, it was expected that the $\text{Coh}(T)$ should be equivalent to the Fukaya–Seidel category of the mirror Hori–Vafa superpotential [24]. To compare this expectation with Corollary 6.16, it suffices to show that $\partial_\infty \Lambda_T$ is in fact the core of the fiber of said superpotential in the Fano case. This is shown under certain hypotheses in [18] and in general in [55].

For more general $T$ this equivalence is known to be false, though $\text{Coh}(T)$ conjecturally still embeds into the Fukaya–Seidel category of the Hori–Vafa mirror restricted to an open subset in $\mathbb{C}$ containing some of the critical values of the superpotential, see [9, §5] or [2]. It may be interesting to explore this conjecture using the present methods.

6.6 Weinstein hypersurfaces in cosphere bundles

Let $X \subset S^*M$ be a Weinstein hypersurface (i.e. codimension one), and let $X^c$ be its positive pushoff. We showed in [20] that the covariant pushforward $\mathcal{W}(X) \to \mathcal{W}(T^*M, X \sqcup X^c)$ is fully faithful. Retracting the stops and appealing to generation by cores [20, Theorem 1.9], we see that $\mathcal{W}(X)$ is equivalent to the full subcategory of $\mathcal{W}(T^*M, c_X \sqcup c_X^c)$ generated by the linking disks to $c_X$. By Theorem 1.1, we may conclude that $\mathcal{W}(X)$ is Morita equivalent to the full subcategory of $\text{Sh}_{c_X \sqcup c_X^c}(M)$ generated by the co-representatives of the microstalks at smooth points of $c_X$. This provides a “sheaf theoretic prescription” for $\mathcal{W}(X)$.

---

10Strictly speaking, they conjectured the proper module version of this statement.
Remark 6.17. We do not know how to tell whether a given Weinstein manifold can be embedded in some cosphere bundle as a hypersurface.

The above sheaf theoretic prescription for $W(X)$ does not yet match the category typically associated by sheaf theorists to $c_X$. This latter category is defined as follows. One forms the “Kashiwara–Schapira stack” by sheafifying the presheaf of categories on $T^*M$ given by the formula $\mu \text{Sh}^{\text{pre}}(\Omega) := \text{Sh}(M)/\text{Sh}_{T^*\Lambda}(M)$. The presheaf $\mu \text{Sh}^{\text{pre}}$ is already discussed in [26]; working with its sheafification is a more modern phenomenon, see e.g. [22, 35].

The notion of microsupport makes sense for a section of this sheaf, and we write $\mu \text{Sh}_{\Lambda}$ for the subsheaf of full subcategories of objects with microsupport inside $\Lambda$. The category typically associated to $c_X$ in the sheaf theory literature is $\mu \text{Sh}_{c_X}(c_X)$.

The remainder of the present discussion depends on the following “anti-microlocalization lemma”, proved in [36]:

Lemma 6.18 ([36]). For any compact singular Legendrian $\Lambda \subseteq S^*M$, the natural map $\mu \text{Sh}_{\Lambda}(\Lambda)^e \rightarrow \mu \text{Sh}_{\Lambda}(\Lambda)$; the map above is its left adjoint restricted to compact objects as in [35]. The fact that the image is generated by microstalks is the usual fact that sheaves have no microsupport along a smooth Lagrangian iff the microstalk is zero. The content of Lemma 6.18 is in the full faithfulness. Similar results can be found in [22].

The anti-microlocalization embedding from Lemma 6.18 parallels the embedding $W(X) \hookrightarrow W(T^*M, c_X \sqcup c_X)$ discussed just above. As the equivalence $\text{Sh}_{c_X \sqcup c_X}(X)^c = \text{Perf}(W(T^*M, c_X \sqcup c_X)^{\text{op}})$ from Theorem 1.1 identifies microstalks and linking disks, and these generate the images of $\mu \text{Sh}_{c_X}(c_X)^c$ and $W(X)$ respectively, we conclude that by restriction Theorem 1.1 defines an equivalence $\text{Perf}(W(X)^{\text{op}}) = \mu \text{Sh}_{c_X}(c_X)^c$.

Example 6.19 (Attachment to cotangent bundles). It is known that if $\Lambda \subseteq S^*M$ is smooth, then $\mu \text{Sh}_{\Lambda}$ is locally the stack of local systems; hence $\mu \text{Sh}_{\Lambda}(\Lambda)$ is a category of twisted local systems, the twist coming from Maslov obstructions (see e.g. [22]). Of course, we learn the same thing from the identification $\mu \text{Sh}_{\Lambda}(\Lambda) = \text{Perf}(W(T^*\Lambda)^{\text{op}})$ above. Note that we have to define $\text{Perf}(W(T^*\Lambda)$ using not the twisting data intrinsic to $T^*\Lambda$ as discussed in §5.3, but rather from the data restricted from $T^*M$, hence the appearence of the Maslov obstructions.

Now suppose we have some other manifold $N$ with some component of $\partial N$ diffeomorphic to $\Lambda$ (and that we choose Maslov data consistent across this identification). Then we can study the category of $T^*M \#_{T^*\Lambda} T^*N$. We know that its wrapped Fukaya category is (Perf of) the pushout of $W(T^*N) \leftarrow W(T^*\Lambda) \rightarrow W(T^*M, \Lambda)$ by [20, Theorem 1.20], and the above discussion allows us to describe these functors on the sheaf side. Thus we may compute $W(T^*M \#_{T^*\Lambda} T^*N)$ using sheaves. We can also allow stops in $T^*M$ and $T^*N$ disjoint from $\Lambda$, and we can also treat the case where $\Lambda$ is embedded into $S^*N$ instead of being identified with one of its boundary components.

The simplest case is when $N$ is just a disjoint union of disks, so we are attaching handles. For an example of sheaf calculations in this setting, see [48].

Example 6.20 (Mirror symmetry for very affine hypersurfaces). As in Corollary 6.16, let $\Lambda_T$ be the Lagrangian of [17]. Consider $\text{Sh}_{\partial \infty \Lambda_T}(T^*(S^1)^n) \rightarrow \mu \text{Sh}_{\partial \infty \Lambda_T}(\partial \infty \Lambda_T)$. As mentioned
above, in [27] the source category was computed, and shown to be equivalent to the category of coherent sheaves on an appropriate toric variety. In [18] the map and the target were computed, and identified with the restriction of coherent sheaves to the toric boundary. As we have already mentioned, \( \partial_\infty \Lambda_T \) was also shown there (under a hypothesis later removed in [55]) to be the core of the fiber of the Hori–Vafa mirror \( W_T : (\mathbb{C}^*)^n \to \mathbb{C} \). ([35] had previously treated the case of pants, i.e. when \( W \) is the sum of the coordinates.) Thus we may translate the sheaf theoretic results of [18] to the following:

\[
\begin{align*}
\text{Perf } W(W^{-1}_T(-\infty)) & \sim \text{Coh}(\partial T) \\
\downarrow & \\
\text{Perf } W((\mathbb{C}^*)^n, W^{-1}_T(-\infty)) & \sim \text{Coh}(T)
\end{align*}
\] (6.3)

A Review of categorical notions

We will assume the reader is familiar with the basic definitions of differential graded (dg) and/or \( A_\infty \) categories, functors between them, modules, and bimodules, for which there are many references. In this section we review notation, assumptions, and relevant notions/results.

All of our dg or \( A_\infty \) categories \( \mathcal{C} \) have morphism co-chain complexes linear over a fixed commutative ring (which we take for simplicity of notation to be \( \mathbb{Z} \)), which are \( \mathbb{Z} \)-graded and cofibrant in the sense of [19, §3.1] (an assumption which is vacuous if working over a field). We further assume that all such \( \mathcal{C} \) are at least cohomologically unital, meaning that the underlying cohomology-level category \( H^*(\mathcal{C}) \) has identity morphisms (this follows if \( \mathcal{C} \) itself is strictly unital, as is the case in the dg setting). We say objects in \( \mathcal{C} \) are isomorphic if they are isomorphic in \( H^*(\mathcal{C}) \).

A.1 Functors, modules, and bimodules

For two (\( A_\infty \) or dg) categories \( \mathcal{C} \) and \( \mathcal{D} \), we use the notation

\[
\text{Fun}(\mathcal{C}, \mathcal{D})
\]

(A.1)

to refer to the (\( A_\infty \)) category of \( A_\infty \) functors from \( \mathcal{C} \) to \( \mathcal{D} \) (compare [46, §(1d)], noting that we consider here homologically unital functors). Note that \( \text{Fun}(\mathcal{C}, \mathcal{D}) \) is in fact a dg category whenever \( \mathcal{D} \) is. The morphism space between \( f, g \in \text{Fun}(\mathcal{C}, \mathcal{D}) \) is the derived space of natural transformations (not to be confused with the space of strict natural transformations which can be defined in the dg setting, but not in the more general \( A_\infty \) setting).

An \( A_\infty \) functor functor \( f : \mathcal{C} \to \mathcal{D} \) is called fully faithful (essentially surjective, an equivalence) if the induced functor on cohomology categories \( H^*(f) : H^*(\mathcal{C}) \to H^*(\mathcal{D}) \) is. We use freely the similar notion of a bilinear \( A_\infty \) functor \( \mathcal{C} \times \mathcal{D} \to \mathcal{E} \) (see [31]), which are themselves objects of an \( A_\infty \) category which is dg if \( \mathcal{E} \) is.

Denote by \( \text{Mod } \mathbb{Z} \) the dg category of (implicitly \( \mathbb{Z} \)-linear, \( \mathbb{Z} \)-graded, and cofibrant) chain complexes (i.e., dg \( \mathbb{Z} \)-modules). A left (respectively right) module over a category \( \mathcal{C} \) is, by
definition a functor from \( C^{\text{op}} \) (respectively \( C \)) to \( \text{Mod} \mathbb{Z} \). More generally, a \( (C, D) \) bimodule is a bilinear functor \( C^{\text{op}} \times D \to \text{Mod} \mathbb{Z} \); this notion specializes to the previous two notions by taking \( C \) or \( D = \mathbb{Z} \) (meaning the category with one object \( * \) and endomorphism algebra \( Z \)), see [19, §3.1]. By the above discussion, left modules, right modules, and bimodules are each objects of a dg categories, denoted

\[
\text{Mod} \mathcal{C} = \text{Fun}(C^{\text{op}}, \text{Mod} \mathbb{Z}) \tag{A.2}
\]

\[
\text{Mod} C^{\text{op}} = \text{Fun}(\mathcal{C}, \text{Mod} \mathbb{Z}) \tag{A.3}
\]

\[
[C, D] = \text{Fun}(C^{\text{op}} \times D, \text{Mod} \mathbb{Z}) \tag{A.4}
\]

respectively. We will most frequently discuss left modules, which we simply call modules.

There are canonical fully faithful Yoneda embeddings (see e.g., [46, (1l)] for a more detailed description on morphism spaces):

\[
\mathcal{C} \hookrightarrow \text{Mod} \mathcal{C} \quad \quad X \mapsto \text{hom}_\mathcal{C}(-, X) \tag{A.5}
\]

\[
\mathcal{C}^{\text{op}} \hookrightarrow \text{Mod} \mathcal{C}^{\text{op}} \quad \quad Y \mapsto \text{hom}_\mathcal{C}(Y, -) \tag{A.6}
\]

\[
\mathcal{C} \times D^{\text{op}} \hookrightarrow [C, D] \quad \quad (X, Y) \mapsto \text{hom}_\mathcal{C}(-, X) \otimes_{\mathbb{Z}} \text{hom}_D(Y, -) \tag{A.7}
\]

and we call any (bi)module in the essential image of these embeddings representable. Recall that any \( C \) possesses a canonical (not necessarily representable) \( (C, C) \) bimodule, the diagonal bimodule \( C \Delta \) (defined on the level of objects by \( C \Delta(-, -) = \text{hom}_C(-, -) \)).

A \( (D, C) \) bimodule \( B \) induces, via convolution (aka tensor product), a functor

\[
B \otimes_c - : \text{Mod} \mathcal{C} \to \text{Mod} D \tag{A.8}
\]

\[
M \mapsto B(-, -) \otimes_c M(-) \tag{A.9}
\]

(note that this is a version of the derived tensor product), and more generally a functor \( [\mathcal{C}, \mathcal{E}] \to [D, \mathcal{E}] \) for any category \( \mathcal{E} \). This functor always has a right adjoint, given by \( N \mapsto \text{hom}_{\text{Mod} D}(B, N) \). As one might expect, convolving with the diagonal bimodule is (isomorphic to) the identity. Not every functor \( \text{Mod} \mathcal{C} \to \text{Mod} D \) comes from a bimodule, however there is a characterization of those that do:

**Theorem A.1** (compare [53], Theorem 1.4). The convolution map \( [D, \mathcal{C}] \to \text{Fun}(\text{Mod} \mathcal{C}, \text{Mod} D) \) is fully faithful, and its essential image is precisely the co-continuous functors, i.e. those that preserve small direct sums.

(By ‘\( F \) preserves small direct sums’ we mean ‘the natural map \( \bigoplus F(X_\alpha) \to F(\bigoplus X_\alpha) \) is an isomorphism’.)

**Proof Sketch.** If \( \text{Fun}_{\text{co-cont}}(\text{Mod} \mathcal{C}, \text{Mod} D) \) denotes the co-continuous functors, observe that restriction to (the Yoneda image of) \( \mathcal{C} \) induces tautologically a map (which is an equivalence) \( \text{Fun}_{\text{co-cont}}(\text{Mod} \mathcal{C}, \text{Mod} D) \to \text{Fun}(\mathcal{C}, \text{Mod} D) = \text{Fun}(\mathcal{C}, \text{Fun}(D^{\text{op}}, \text{Mod} \mathbb{Z})) = [D, \mathcal{C}] \); in other words co-continuous functors from \( \text{Mod} \mathcal{C} \) are determined by what they do on \( \mathcal{C} \). One checks that this is a two-sided inverse to the convolution map, up to homotopy.

\[\text{We say } f : \mathcal{C} \to D \text{ has right adjoint (or is the left adjoint of) } g : D \to \mathcal{C} \text{ if there is in isomorphism in } [\mathcal{C}, D] \text{ between } \text{hom}_D(f(-), -) \text{ and } \text{hom}_\mathcal{C}(-, g(-)).\]
Given an $A_{\infty}$ functor $f: \mathcal{C} \to \mathcal{D}$, there is a pair of (adjoint) induced functors on module categories: first, there is an induced restriction map

$$f^*: \text{Mod } \mathcal{D} \to \text{Mod } \mathcal{C}$$

(A.10)

given by pre-composing with $f^{\text{op}}$; one can show this is quasi-isomorphic to tensoring with the graph $(\mathcal{C}, \mathcal{D})$ bimodule $(f^{\text{op}}, \text{id})^* \mathcal{D}_{\Delta} = \mathcal{D}_{\Delta}(f(-), -)$ (see [19, Lemma 3.7]). In particular, there is a natural functor $\mathcal{D} \to \text{Mod } \mathcal{C}$ given by composing (A.10) with the Yoneda embedding for $\mathcal{D}$. There is also (left adjoint to $f^*$) an induction map

$$f_*: \text{Mod } \mathcal{C} \to \text{Mod } \mathcal{D}$$

(A.11)

given by tensoring with the graph $(\mathcal{D}, \mathcal{C})$ bimodule $(\text{id}, f)^* \mathcal{D}_{\Delta} = \mathcal{D}_{\Delta}(-, f(-))$. One can directly compute that $f_*$ sends a representable over $X \in \mathcal{C}$ to an object isomorphic to the representable over $f(X)$. Conversely, we have:

**Lemma A.2.** If a $(\mathcal{D}, \mathcal{C})$ bimodule $\mathcal{B}$ has the property that $\mathcal{B}(-, c)$ is representable by an object $f(c) \in \mathcal{D}$ each $c \in \mathcal{C}$, then convolving with $\mathcal{B}$ is isomorphic in $\text{Fun}(\text{Mod } \mathcal{C}, \text{Mod } \mathcal{D})$ to the induction of a (unique up to isomorphism) $A_{\infty}$ functor $f: \mathcal{C} \to \mathcal{D}$ sending $c$ to $f(c)$. In particular, $f_* = \mathcal{B} \otimes_\mathcal{C} -$ admits a right adjoint, namely $f^*$.

Note that $f^*$ also admits a right adjoint $f_!$, called co-induction, induced by taking hom from $(f^{\text{op}}, \text{id})^* \mathcal{D}_{\Delta}$, by the earlier discussion.

### A.2 Large categories and compact objects

Any category of modules $\text{Mod } \mathcal{C}$ (or more generally bimodules, etc.), inherits from $\text{Mod } \mathbb{Z}$ the following properties: it is pre-triangulated, meaning it possess all mapping cones; in particular $H^0(\text{Mod } \mathcal{C})$ is triangulated in the usual sense. The category $\text{Mod } \mathcal{C}$ also admits arbitrary (small) direct sums, which in conjunction with the previous fact implies that it is co-complete, meaning it admits all (small) colimits (compare [30, Prop. 4.4.26]), and in particular is idempotent complete, meaning in $H^0(\text{Mod } \mathcal{C})$ every idempotent morphism splits (i.e., $H^0(\text{Mod } \mathcal{C})$ is closed under retracts) (compare [40, Prop. 1.6.8]).

In light of the above facts, we now adopt the perspective of large categories $\mathcal{C}$ (such as $\text{Mod } \mathcal{C}$, but without requiring a particular such presentation), which are by definition co-complete hence pre-triangulated, idempotent complete, and admitting arbirary direct sums. We say an object $X \in \mathcal{C}$ is compact if $\text{hom}(X, -)$ commutes with arbitrary direct sums (i.e. is co-continuous). Denoting by $\mathcal{C}^c \subseteq \mathcal{C}$ the full subcategory of compact objects, we say that a co-complete category $\mathcal{C}$ is compactly generated if there is a small collection (i.e., a set) of compact objects $\mathcal{T} \subseteq \mathcal{C}^c$ satisfying the following equivalent conditions:

- An object $X \in \mathcal{C}$ is zero if and only if it is right-orthogonal to $\mathcal{T}$ (meaning $\text{hom}_\mathcal{C}(-, X)$ annihilates $\mathcal{T}$).

- The natural map $\mathcal{C} \to \text{Mod } \mathcal{T}^c$ sending $Y \mapsto \text{hom}_\mathcal{C}(-, Y)$ is an equivalence.

On the level of large (i.e., cocomplete) dg categories, a version of Brown representability gives effective criteria for deducing the existence of adjoints to functors, and/or identifying when functors preserve compact objects.
Theorem A.3 (Compare [40, Thm. 8.4.4] or [30, Cor. 5.5.2.9]). Let \( \mathsf{C} \) and \( \mathsf{D} \) be large categories with \( \mathsf{C} \) compactly generated. If an \( A_\infty \) functor \( f : \mathsf{C} \to \mathsf{D} \) is co-continuous, then \( f \) admits a right adjoint.

Proof Sketch. We suppose that \( \mathsf{D} \) is also compactly generated, so one can write \( \mathsf{C} = \text{Mod} \mathsf{C} \), \( \mathsf{D} = \text{Mod} \mathsf{D} \) with \( \mathsf{C} = (\mathsf{C})^c \) and \( \mathsf{D} = (\mathsf{D})^c \). Then we observe that if \( f \) is co-continuous, it comes (by Theorem A.1) from convolving with a bimodule, which always has a right adjoint as described above. □

Theorem A.3 also holds under the weaker hypothesis that \( \mathsf{C} \) is well generated rather than compactly generated, by work of Neeman adapted to the dg/\( A_\infty \) case (for a definition of this notion see [40, §8], and for a proof of Theorem A.3 in that setting, see [40, Prop. 8.4.2 and Thm. 8.4.4]).

The following is a useful criterion for when a functor preserves compact objects.

Lemma A.4. If a functor \( f : \mathsf{C} \to \mathsf{D} \) has a co-continuous right adjoint \( g \), then \( f \) preserves compact objects.

Proof. For \( c \in \mathsf{C} \) a compact object, we have

\[
\text{Hom}_\mathsf{D}(f(c), \bigoplus \alpha d_\alpha) = \text{Hom}_\mathsf{C}(c, g(\bigoplus \alpha d_\alpha)) = \text{Hom}_\mathsf{C}(c, \bigoplus \alpha g(d_\alpha)) = \bigoplus \alpha \text{Hom}_\mathsf{C}(c, g(d_\alpha)) = \bigoplus \alpha \text{Hom}_\mathsf{D}(f(c), d_\alpha) \quad (A.12)
\]

as desired. □

A.3 Quotients and localization

Given a (small) \( A_\infty \) (or dg) category \( \mathsf{C} \) and a full subcategory \( \mathsf{D} \subseteq \mathsf{C} \), there is a well-defined notion of the quotient (dg or \( A_\infty \)) category \( \mathsf{C}/\mathsf{D} \)

which comes equipped with a functor \( q : \mathsf{C} \to \mathsf{C}/\mathsf{D} \) (see [13, 32] for an explicit model in the dg and \( A_\infty \) cases respectively, also discussed in [19, §3.1.3]). The pair \( \mathsf{C}/\mathsf{D} \) and \( q \) satisfy the following universal property: any functor \( \mathsf{C} \to \mathsf{E} \) which sends \( \mathsf{D} \) to 0 factors essentially uniquely through \( \mathsf{C}/\mathsf{D} \) via \( q \); more precisely, the pre-composition \( q^* : \text{Fun}(\mathsf{C}/\mathsf{D}, \mathsf{E}) \hookrightarrow \text{Fun}(\mathsf{C}, \mathsf{E}) \) fully faithfully embeds the former category as the full subcategory \( \text{Fun}_{\text{Ann}(\mathsf{D})}(\mathsf{C}, \mathsf{E}) \) of the latter consisting of functors from \( \mathsf{C} \) to \( \mathsf{E} \) which annihilate \( \mathsf{D} \). Taking \( \mathsf{E} \) to be \( \text{Mod} \mathsf{Z} \) op, we note in particular that the pullback map

\[
q^* : \text{Mod}(\mathsf{C}/\mathsf{D}) \to \text{Mod} \mathsf{C}
\]

is a fully faithful embedding whose essential image is the \( \mathsf{C} \) modules which annihilate \( \mathsf{D} \) (see [19, Lemmas 3.12 and 3.13]).

In light of (A.13), for a large category \( \mathsf{C} \) and a large subcategory \( \mathsf{D} \subset \mathsf{C} \) which is compactly generated by a subset of \( \mathsf{C} \)'s compact objects \( \mathsf{D} \subseteq \mathsf{C} = (\mathsf{C})^c \), we can define the...
quotient of \( C \) by \( D \), also denoted \( C / D \), as the quotient constructed as before or, equivalently, as the full subcategory of \( C \) that is right-orthogonal to \( D \) (these are equivalent because the map \( C \to C / D \) now has a fully faithful right adjoint onto precisely the right-orthogonal to \( D \)). Implicitly, we may sometimes use \( C / D \) to refer to the corresponding large quotient \( C / D \). The category \( C / D \) is also compactly generated, and this large quotient operation is compatible (by passing to compact objects) with small quotients, at least up to idempotent completion:

**Lemma A.5** (Compatibility of large quotients with compact objects, compare [38, Thm. 2.1]). If \( C \) is compactly generated by \( \mathcal{C} = C^c \), and \( D \subseteq \mathcal{C} \) is a full subcategory, then there is a natural fully faithful functor \( \mathcal{C} / D \to (C / D)^c \) which exhibits the latter as the idempotent completion of the former.

In particular, we note that quotients of (small) categories need not preserve idempotent completeness: if \( \mathcal{C} \) is idempotent complete and pre-triangulated, then \( \mathcal{C} / D \) is pre-triangulated but need not be idempotent complete. We denote by \( (\mathcal{C} / D)^\pi \) the idempotent-completed quotient of \( \mathcal{C} \) by \( D \).

If \( \mathcal{C} \) is a pre-triangulated dg/\( A_\infty \) category and \( Z \) is a set of morphisms in \( H^0(\mathcal{C}) \), one can form the localization of \( \mathcal{C} \) with respect to \( Z \) by taking the quotient

\[
\mathcal{C}[Z^{-1}] := \mathcal{C} / \text{cones } Z
\]

where \( \text{cones } Z \) denotes any set of cones of morphisms in \( \mathcal{C} \) representing the elements in \( Z \) (regardless of how one chooses such a subset, one notices that \( \text{cones } Z \) is a well-defined full subcategory of \( \mathcal{C} \), and in particular, \( \mathcal{C}[Z^{-1}] \) is unaffected by the choice). If \( \mathcal{C} \) is not pre-triangulated, one can still define this localization by taking the essential image of \( \mathcal{C} \) under

\[
\mathcal{C} \to \text{Perf } \mathcal{C} \to \text{Perf } \mathcal{C} / (\text{cones } Z)
\]

(see §A.5 below for the definition of Perf). The tautological localization map \( \mathcal{C} \to \mathcal{C}[Z^{-1}] \) possesses a host of nice properties, simply as a special case of the properties of quotients discussed above; we leave it to the reader to spell out the details.

**A.4 (Split-)generation**

We say a full subcategory \( A \subseteq \mathcal{C} \) (split-)generates a module \( M \in \text{Mod } \mathcal{C} \) if \( M \) is isomorphic to (a retract of) an iterated extension of Yoneda modules of objects of \( A \). We say \( A \subseteq \mathcal{C} \) (split-)generates an object \( X \in \mathcal{C} \) if it (split-)generates the Yoneda module over \( X \).

**A.5 Perfect and proper modules**

We define the category of perfect modules

\[
\text{Perf } \mathcal{C} \subseteq \text{Mod } \mathcal{C}
\]

to be the idempotent completed pre-triangulated hull of (the image under the Yoneda embedding of) \( \mathcal{C} \) in \( \text{Mod } \mathcal{C} \), i.e. it is the full subcategory consisting of all modules which are
split-generated by representable modules.\textsuperscript{12} A fundamental feature of Perf $\mathcal{C}$ is that it is precisely the collection of compact objects in $\text{Mod} \mathcal{C}$:

$$\text{Perf} \mathcal{C} = (\text{Mod} \mathcal{C})^c.$$ \hfill (A.14)

In particular, the inclusion $\mathcal{C} \subseteq \text{Perf} \mathcal{C}$ induces an equivalence $\text{Mod} \text{Perf} \mathcal{C} = \text{Mod} \mathcal{C}$. We similarly say that a bimodule is perfect if it is split-generated by representable bimodules.

As a special case of the above construction, denote by

$$\text{Perf} \mathbb{Z} \subseteq \text{Mod} \mathbb{Z}$$

the subcategory of perfect $\mathbb{Z}$-linear chain complexes, namely those chain complexes which are quasi-isomorphic to a bounded complex of finite projective $\mathbb{Z}$-modules. We say a module or bimodule is proper (sometimes called pseudo-perfect in the literature) if as a functor to $\text{Mod} \mathbb{Z}$, it takes values in the full subcategory $\text{Perf} \mathbb{Z}$ (i.e. for a module $M$ if $M(X)$ is a perfect chain complex for every $X \in \mathcal{C}$). Denote by

$$\text{Prop} \mathcal{C} := \text{Fun}(\mathcal{C}^{\text{op}}, \text{Perf} \mathbb{Z}) \subseteq \text{Mod} \mathcal{C}$$

the full subcategory of proper modules.

### A.6 Morita equivalence

We say small categories $\mathcal{C}$ and $\mathcal{D}$ are Morita equivalent if there exists a $(\mathcal{C}, \mathcal{D})$ bimodule $\mathcal{B}$ and a $(\mathcal{D}, \mathcal{C})$ bimodule $\mathcal{P}$ inducing, via convolution, an inverse pair of equivalences

$$\text{Mod} \mathcal{C} \leftrightarrow \text{Mod} \mathcal{D}.$$ \hfill (A.15)

**Lemma A.6.** $\mathcal{C}$ and $\mathcal{D}$ are Morita equivalent iff there is an equivalence $\text{Perf} \mathcal{C} = \text{Perf} \mathcal{D}$.

**Proof.** By Theorem A.1, the convolution functors are co-continuous. As they form an inverse pair of equivalences, they are each others adjoints (in both directions). Thus by Theorem A.4, the convolution functors preserve compact objects, thus restricting to form an equivalence $\text{Perf} \mathcal{C} = \text{Perf} \mathcal{D}$.

Conversely, any equivalence $\text{Perf} \mathcal{C} = \text{Perf} \mathcal{D}$ induces an equivalence $\text{Mod} \mathcal{C} = \text{Mod} \text{Perf} \mathcal{C} = \text{Mod} \text{Perf} \mathcal{D} = \text{Mod} \mathcal{D}$ (which, being an equivalence, must be co-continuous, and thus given by convolution by Theorem A.1). \qed

In particular, the canonical inclusion $\mathcal{C} \hookrightarrow \text{Perf} \mathcal{C}$ is a Morita equivalence. In light of the above Lemma, we will also refer to an equivalence $\text{Perf} \mathcal{C} = \text{Perf} \mathcal{D}$ as a Morita equivalence between $\mathcal{C}$ and $\mathcal{D}$. We say a property of $\mathcal{C}$ is “a Morita-invariant notion” if its validity only depends on $\text{Perf} \mathcal{C}$ up to equivalence.

\textsuperscript{12}Note that, as $\text{Perf} \mathcal{C}$ satisfies a universal property, it is quasi-equivalent to any other model of the split-closed pre-triangulated hull, for instance idempotent-completed twisted complexes over $\mathcal{C}$ (see e.g. \cite[§(4c)]{46} where it is denoted $\Pi \text{Tw} \mathcal{C}$).
A.7 Smooth and proper categories

We say a category $C$ is smooth (sometimes called homologically smooth) if its diagonal bimodule $C \Delta$ is perfect. We say $C$ is proper (sometimes called compact) if its diagonal bimodule $C \Delta$ is proper, or if equivalently $\text{hom}_C(X,Y)$ is a perfect $\mathbb{Z}$-module for any two objects $X, Y \in C$. Smoothness and properness are Morita-invariant notions; in particular $C$ is smooth (resp. proper) if and only if $\text{Perf}_C$ is.

In general, the subcategories of modules $\text{Perf}_C$ and $\text{Prop}_C$ do not coincide, however they are related under the above finiteness assumptions on $C$:

Lemma A.7. If $C$ is proper, then $\text{Perf}_C \subseteq \text{Prop}_C$ and if $C$ is smooth, then $\text{Prop}_C \subseteq \text{Perf}_C$. In particular, if $C$ is smooth and proper, then $\text{Prop}_C = \text{Perf}_C$.

Lemma A.8. Properness is inherited by full subcategories, and smoothness passes to quotients/localizations.

A.8 Exceptional collections

We say a (full) subcategory of finitely many objects $A \subseteq C$ is an exceptional collection if there exists a partial ordering of the objects of $A$ such that

\begin{align*}
\text{hom}(X, X) &= \mathbb{Z}(\text{id}_X) \quad \text{(A.16)} \\
\text{hom}(X, Y) &= 0 \quad \text{unless } X \leq Y. \quad \text{(A.17)}
\end{align*}

Lemma A.9. If $X \in \text{Mod}_C$ is split-generated by an exceptional collection $A$, then $X$ is generated by $A$ (i.e. it is not necessary to add idempotent summands).

Proof. Let $X \in A$ be any maximal (with respect to the given partial order) object. We consider the functor

\[ F_X : \text{Mod}A \to \text{Mod} \mathbb{Z} \]

\[ M \mapsto M(X). \]  

(A.18) (A.19)

Certainly if $M$ is generated by $A$ (i.e. by the Yoneda modules $\text{hom}_C(\cdot, A)$ for $A \in A$), then $F_X(M) \in \text{Perf} \mathbb{Z}$ by maximality of $X$, as all of the Yoneda modules except $\text{hom}_C(\cdot, X)$, contribute trivially $F_X$, and each $\text{hom}_C(\cdot, X)$ contributes a perfect $\mathbb{Z}$ module.

There is a tautological map of $C$ modules $\text{hom}_C(\cdot, X) \otimes F_X(M) \to M(\cdot)$; denote its cone by $M|_{A - \{X\}}$. Now given any maximal object $Y$ of $A - \{X\}$, we may define a functor

\[ F_Y : \text{Mod}A \to \text{Mod} \mathbb{Z} \]

\[ M \mapsto M|_{A - \{X\}}(Y). \]  

(A.20) (A.21)

Again, if $M$ is generated by $A$ then $F_Y(M) \in \text{Perf} \mathbb{Z}$. To see this, simply note that given a twisted complex $M$ of objects of $A$, the object $M|_{A - \{X\}}$ is just the same twisted complex but with all instances of $X$ deleted. We may now similarly define $M|_{A - \{X,Y\}}$ to be the cone of $\text{hom}_C(\cdot, Y) \otimes F_Y(M) \to M|_{A - \{X\}}(\cdot)$.

---

13Rather, they are in some sense ‘Morita dual’ in that $\text{Prop}_C = \text{Fun}(\text{Perf}_C^{\text{op}}, \text{Perf} \mathbb{Z})$.  

43
Iterating this procedure defines a sequence of functors $F_X : \text{Mod}\mathcal{A} \to \text{Mod}\mathcal{Z}$ for all $X \in \mathcal{A}$ (in fact, these are independent of the order in which we pick off maximal elements, however we won’t use this). The above arguments show that for any $\mathcal{M}$ generated by $\mathcal{A}$, all $F_X(\mathcal{M})$ are in $\text{Perf}\mathcal{Z}$.

In fact, the above arguments verify the converse assertion as well: if all $F_X(\mathcal{M})$ are in $\text{Perf}\mathcal{Z}$ then $\mathcal{M}$ is generated by $\mathcal{A}$. To see this, note that whenever $Y$ was a maximal element of $\mathcal{B} \subseteq \mathcal{A}$ with $F_Y(\mathcal{M}) \in \text{Perf}\mathcal{Z}$, the same reasoning above allowed us to decompose $\mathcal{M}|_{\mathcal{B}}$ in terms of the module $\text{hom}_\mathcal{C}(-, Y) \otimes F_Y(\mathcal{M})$ (which is generated by $\text{hom}_\mathcal{C}(-, Y)$ as $\text{Perf}\mathcal{Z}$ is generated, not just split-generated, by $\mathcal{Z}$) and a module $\mathcal{M}|_{\mathcal{B}\setminus\{Y\}}$, with eventually $\mathcal{M}|_{\emptyset} = 0$ (as it tests trivially against every object of $\mathcal{A}$).

Now, the property of all $F_X(\mathcal{M}) \in \text{Mod}\mathcal{Z}$ lying in $\text{Perf}\mathcal{Z}$ is clearly preserved under passing to direct summands, so the proof is complete.

**Lemma A.10.** If $\mathcal{A}$ is an exceptional collection which is proper, then it is smooth.

**Proof.** In the case $\mathcal{A}$ has one object, this is true because $\mathcal{Z}$ is trivially smooth. Now inductively apply the following assertion: If $\mathcal{C}$ and $\mathcal{D}$ are both smooth, and $\mathcal{E}$ denotes the semi-orthogonal gluing of $\mathcal{C}$ with $\mathcal{D}$ along a $(\mathcal{C}, \mathcal{D})$ bimodule $\mathcal{B}$ which is perfect, then $\mathcal{E}$ is smooth as well (see [28, Prop. 3.11] and [29, Thm. 3.24] for the dg case, which immediately extends to this setting). In the assertion observe it suffices that $\mathcal{B}$ be proper, since proper bimodules over smooth categories are automatically perfect (by the bimodule version of Lemma A.7). Hence, one can apply the assertion to $\mathcal{A}$.

**References**


