Enough vector bundles on orbispaces

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14 February 2020

Abstract

We show that every orbispace satisfying certain mild hypotheses has 'enough' vector bundles. It follows that the K-theory of finite rank vector bundles on such orbispaces is well behaved. Global presentation results for smooth orbifolds and derived smooth orbifolds also follow.

1 Introduction

A *(separated) orbispace* is a topological stack X which admits a cover by open substacks of the form Y/Γ (where $\Gamma \curvearrowright Y$ is a continuous action of a finite group on a topological space) and whose diagonal $X \to X \times X$ is separated and proper. Familiar examples of orbispaces include orbifolds, graphs of groups, complexes of groups, and (the analytifications of) separated Deligne–Mumford algebraic stacks over $\mathbb{C}$.

An interesting question to ask about a given orbispace X is whether there exists a global presentation $X = Y/G$ for G a compact Lie group; let us call such an orbispace a *global quotient*. There are a number of known sufficient conditions for an orbispace to be a global quotient. If $X = Y/\Gamma$ for a discrete group Γ, then X is a global quotient by Lück–Oliver [17, Corollary 2.7]. Every paracompact smooth *effective* n-dimensional orbifold is a global quotient (of its orthonormal frame bundle by $O(n)$), and a sufficient condition for a general (not necessarily effective) orbifold to be a global quotient was given by Henriques–Metzler [13]. Henriques [14] conjectured that every compact orbispace is a global quotient, however other experts have expressed skepticism that such a general result would be true [24, §6.4]. The analogous question for algebraic stacks has been studied by Edidin–Hassett–Kresch–Vistoli [10] and Totaro [25]. It is a result of Kresch–Vistoli [16, Theorem 2] [15, Theorem 4.4] (using an unpublished result of Gabber) that smooth separated Deligne–Mumford stacks are global quotients.

In this paper, we show that all orbispaces satisfying very mild hypotheses are global quotients. In particular, we show that all compact orbispaces are global quotients.

Theorem 1.1. *Let X be an orbispace, with isotropy groups of bounded order, whose coarse space $|X|$ is coarsely finite-dimensional (every open cover has a locally finite refinement with finite-dimensional nerve). Then there exists a complex vector bundle V of rank $n > 0$ over X, whose fiber over $x \in X$ is isomorphic to a direct sum of copies of the regular representation of G_x. We may take $n = n(d, m)$ if $|X|$ is d-dimensional (every open cover has a locally finite refinement with nerve of dimension $\le d$) and has isotropy groups of order $\le m$.*

Theorem 1.1 is equivalent to the corresponding statement about real vector bundles (by tensoring from $\mathbb{R}$ to $\mathbb{C}$ and by forgetting from $\mathbb{C}$ to $\mathbb{R}$, respectively).

Corollary 1.2. *For X as in Theorem 1.1, we have $X = P/U(n)$ for a space P.*

Corollary 1.3. *Every paracompact smooth orbifold X of dimension $\leq d$ with isotropy groups of order $\leq m$ is the quotient $X = P/U(n)$ of a smooth manifold P by a smooth action of the compact Lie group $U(n)$, where $n = n(d, m)$.*

Remark 1.4. The vector bundles produced by our proof of Theorem 1.1 all have trivial Chern character. In fact, they are essentially defined by the stronger property of having trivial *inertial Chern character*. The proof of Theorem 1.1 consists of showing that the obstructions to extending vector bundles with trivial inertial Chern character are all torsion (and hence can be made to vanish by replacing V with $V^{\oplus a}$ for some sufficiently divisible $a > 0$). It follows from this proof that such vector bundles are rationally unique, in the sense that if V and W satisfy the conclusion of Theorem 1.1 and have trivial inertial Chern character, then there exist integers $a, b > 0$ such that $V^{\oplus a} \cong W^{\oplus b}$.

It seems that Theorem 1.1 does not help resolve the question of whether every separated Deligne–Mumford stack of finite type over $\mathbb{C}$ is a global quotient, since the vector bundles produced by Theorem 1.1 on its analytification have no need to be algebraic.

Corollary 1.5. *Let $X \to Y$ be a representable map of orbispaces satisfying the hypothesis of Theorem 1.1. Every vector bundle on X of bounded rank is a direct summand of a vector bundle of bounded rank pulled back from Y.*

It is the conclusion of Corollary 1.5 (or rather a special case thereof) to which the titular phrase 'having enough vector bundles' refers. It is well known that this is the key statement needed to show that the K-theory of finite rank vector bundles on orbispaces satisfies excision and exactness and is thus a cohomology theory (see [17, §3] and [14, §6.3]). We elaborate on this assertion in §6, where we observe that the suggestive reformulation of Corollary 1.5 as the statement that pullback of vector bundles is a *cofinal* map of abelian monoids (of isomorphism classes of vector bundles under direct sum) allows for clean proofs of these facts. The K-theory of finite rank vector bundles should agree (for reasonable orbispaces) with the other standard models of K-theory for orbispaces, such as using bundles of Fredholm operators [23, 18, 3, 4] or using orthogonal spectra [21, §§6.3–6.4] (compare Remark 1.7 below).

Remark 1.6. Another known (to experts) consequence of Corollary 1.5 (which we will not explain in detail) is that every paracompact quasi-smooth derived smooth orbifold with tangent and obstruction spaces of dimension $\leq d$ and isotropy groups of order $\leq m$ is the derived zero set of a smooth section of a vector bundle of rank $\leq n$ over a smooth orbifold of dimension $\leq n = n(d, m)$. ('Quasi-smooth' means locally isomorphic to the derived zero set of a smooth section of a smooth vector bundle over a smooth orbifold.)

Remark 1.7. Combined work of Schwede [22] and Gepner–Henriques [12] establishes an equivalence between certain categories of orbispaces up to homotopy and orthogonal spaces up to global equivalence (with respect to the 'global family' of all finite groups). The vector bundles produced by Theorem 1.1 allow as follows for a concrete description of the functor from orbispaces to orthogonal spaces (compare [21, Definition 1.1.27]). Let X be an orbispace, and let E be any *faithful* vector bundle over X (meaning the fibers of E are faithful representations of the isotropy groups of X). The orthogonal space corresponding to X is given by

$$V \mapsto \mathrm{Emb}_X(E, \underline{V})$$

where $\mathrm{Emb}_X(E, \underline{V})$ denotes (the total space of) the bundle of embeddings of E into V (note that $\mathrm{Emb}_X(E, \underline{V})$ is a space since E is faithful).

Schwede [21] also associates to every orthogonal spectrum a cohomology theory on orthogonal spaces, hence on orbispaces. Given an orthogonal spectrum A and an orbispace X which admits faithful vector bundles, the degree zero A-cohomology of X is (in view of the above) given by the direct limit over vector bundles E over X of the set of homotopy classes of sections of the fibration $\Omega^E A(E) \to X$. More generally, we may consider the mapping spectrum $F(X, A)$ defined by

$$n \mapsto \varinjlim_{E/X} \Gamma(X, \Omega^E A(E \oplus \mathbb{R}^n))$$

whose stable homotopy groups are the A-cohomology groups of X. If A is a global Ω-spectrum [21, Definition 4.3.8], then this direct limit is achieved at any faithful E, and the above definition of $F(X, A)$ is an Ω-spectrum. Let us also propose that the correct notion of the A-homology groups of X would be the stable homotopy groups of the spectrum

$$n \mapsto \varinjlim_{E/X} \left|\Omega^E A(E \oplus \mathbb{R}^n)\right|$$

where $|\cdot|$ indicates taking the coarse space of the total space of $\Omega^E A(E \oplus \mathbb{R}^n)$ over X.

Remark 1.8. It is natural to ask to what extent Theorem 1.1 may be generalized to the case of 'Lie orbispaces' (topological stacks locally modelled on Y/G for G a compact Lie group). The naive generalization is simply false: there are purely ineffective Lie orbispaces with isotropy group S^1 and coarse space S^3 which have no finite rank faithful vector bundles [25, §2]. It is, however, reasonable to conjecture that the proof of Theorem 1.1 could be generalized to the Lie orbispace setting to prove the following essentially optimal result: given any 'rational' cohomology class θ on the inertia stack of X whose restriction to the fiber over every point $x \in X$ is the character of a finite-dimensional representation of the isotropy group G_x, there exists a vector bundle V over X with inertial Chern character $n\theta$ for some integer $n > 0$.

Remark 1.9. We prove in Proposition 4.6 the useful fact that every orbispace satisfying the hypotheses of Theorem 1.1 admits a representable map to a simplicial complex of groups. The proof we give of this statement, while elementary, is still somewhat more nontrivial than one may expect is possible.

1.1 Acknowledgements

We would like to thank André Henriques for discussions about orbispaces and topological stacks. This research was conducted during the period the author served as a Clay Research Fellow and was partially supported by a Packard Fellowship and by the National Science Foundation under the Alan T. Waterman Award, Grant No. 1747553.

2 The main construction

The essence of Theorem 1.1 is already carried by the case of orbispaces arising from simplicial complexes of groups. Therefore the main part of this paper consists of treating this special case, which we state as Theorem 2.1.

Theorem 2.1. *Let X be (the orbispace arising from) a d-dimensional simplicial complex of groups of order $\leq m$. There exists a complex vector bundle V of rank $n = n(d, m) > 0$ over X, whose fiber over $x \in X$ is isomorphic to a direct sum of copies of the regular representation of G_x.*

Proof. A simplicial complex of groups (see also Haefliger [11], Corson [8], or Bridson–Haefliger [7]) is a pair (Z, G) consisting of a simplicial complex Z together with the following data:

- For every simplex $\sigma \subseteq Z$, a group G_σ.
- For every pair of simplices $\sigma \subseteq \tau$, an *injective* group homomorphism $G_\tau \hookrightarrow G_\sigma$.
- For every triple of simplices $\rho \subseteq \sigma \subseteq \tau$, an element of G_ρ conjugating the inclusion $G_\tau \hookrightarrow G_\rho$ to the composition of inclusions $G_\tau \hookrightarrow G_\sigma \hookrightarrow G_\rho$.
- For every quadruple of simplices $\pi \subseteq \rho \subseteq \sigma \subseteq \tau$, the resulting product of elements of G_π conjugating $G_\tau \hookrightarrow G_\pi$ to $G_\tau \hookrightarrow G_\rho \hookrightarrow G_\pi$ to $G_\tau \hookrightarrow G_\sigma \hookrightarrow G_\rho \hookrightarrow G_\pi$ to $G_\tau \hookrightarrow G_\sigma \hookrightarrow G_\pi$ and back to $G_\tau \hookrightarrow G_\pi$ must be the identity element of G_π.

A simplicial complex of finite groups naturally gives rise to an orbispace with finite stabilizers X, obtained by gluing together $\sigma \times \mathbb{B}G_\sigma$ for $\sigma \subseteq Z$ via the data given above, where $\mathbb{B}G$ denotes the stack $*/G$ for G finite. We note that injectivity of the homomorphisms $G_\tau \hookrightarrow G_\sigma$ is required for this gluing to makes sense.

We take an inductive (i.e. obstruction theoretic) approach to the construction of the desired vector bundle $V \to X$. Our orbispace X is built by attaching cells of the form $(D^k, \partial D^k) \times \mathbb{B}G$ for $k \geq 0$ and finite groups G. Note that the fact that the attaching maps are injective on isotropy groups plays well with the desired fibers of V: for an inclusion of groups $f : H \hookrightarrow G$, the pullback of the regular representation of G is isomorphic to the direct sum of $|G| / |H|$ copies of the regular representation of H. It thus would suffice, for example, to show that every complex vector bundle over $\partial D^k \times \mathbb{B}G$ extends to $D^k \times \mathbb{B}G$. A complex vector bundle V over $Y \times \mathbb{B}G$ splits canonically as a direct sum of isotypic pieces $V = \bigoplus_{\rho \in \hat{G}} V_\rho$ associated to the complex irreducible representations ρ of G, where $V_\rho = W_\rho \otimes \rho$ (canonically) for complex vector bundles W_ρ over Y. These W_ρ over ∂D^k are classified by elements of $\pi_{k-1}(BU(n_\rho))$, which form obstructions to the desired extension of W_ρ from ∂D^k to D^k. To overcome this, we will construct not only the vector bundle $V \to X$ but rather V along with some extra data to guarantee that these obstructions vanish.

Recall that $BU(n) \to BU$ induces an isomorphism on homotopy groups in degrees $\leq 2n$ and that the homotopy groups of BU are given by $\pi_{2i}(BU) = \mathbb{Z}$ and $\pi_{2i+1}(BU) = 0$ (due to Bott). Recall also that ith Chern class of a generator of $\pi_{2i}(BU)$ equals $(i-1)!$ times a generator of $H^{2i}(S^{2i})$ (also due to Bott). In particular, the Chern class detects all nontrivial elements of $\pi_i(BU)$, so the obstructions encountered just above will vanish if we can ensure that the total Chern class of V is trivial (or even just rationally trivial).

We will approach the Chern classes of V via Chern–Weil theory. Given a unitary connection θ on a complex vector bundle V over a smooth manifold X, its curvature $\Omega(V, \theta)$ is a 2-form valued in $\mathrm{End}(V)$. The Chern character form

$$\mathrm{ch}(V,\theta) := \mathrm{tr}\exp(i\Omega/2\pi) \in \Omega^{\mathrm{even}}(X;\mathbb{R})$$

is closed, and its class in cohomology $\mathrm{ch}(V) \in H^{\mathrm{even}}(X;\mathbb{R})$ is the Chern character of V, independent of θ. This theory applies equally well when X is a smooth orbifold, however it does not immediately capture all the information we need. For example, if $X = Y \times \mathbb{B}G$, so a vector bundle over X is the same thing as a vector bundle over Y with a fiberwise G action, the Chern character $\mathrm{ch}(V) \in H^{\mathrm{even}}(X;\mathbb{R}) = H^{\mathrm{even}}(Y;\mathbb{R})$ ignores the G action, whereas we need to understand the Chern characters $\mathrm{ch}(W_\rho) \in H^{\mathrm{even}}(X;\mathbb{R})$ for each of the isotypic pieces $V_\rho = W_\rho \otimes \rho$ of V.

We thus consider what we will call the *inertial Chern character*[1] of $V \to X$, which is a cohomology class on the *inertia stack*

$$IX := \mathrm{Eq}\big(X \rightrightarrows X\big) = X \times_{X\times X} X$$

of X (such classes have been studied in the past by Adem–Ruan [1]). Recall that the inertia stack of a smooth orbifold is again a smooth orbifold, being given in local coordinates by $I(M/G) = \{g \in G, x \in M \mid gx = x\}/G$. The inertial Chern character $\mathrm{ch}^I(V) \in H^{\mathrm{even}}(IX;\mathbb{C})$ is represented by the closed form

$$\mathrm{ch}^I(V,\theta) := \mathrm{tr}\big[g\exp(i\Omega/2\pi)\big] \in \Omega^{\mathrm{even}}(IX;\mathbb{C})$$

whose cohomology class is independent of θ. The inertial Chern character (form) is compatible with pullback in the sense that for a smooth map of orbifolds $f : X \to Y$, we have $f^*\mathrm{ch}^I(V) = \mathrm{ch}^I(f^*V)$ and $f^*\mathrm{ch}^I(V,\theta) = \mathrm{ch}^I(f^*V, f^*\theta)$.

To better understand the inertial Chern character, we discuss some examples. If $X = \mathbb{B}G$, a vector bundle over X is simply a representation of G, the inertia stack $IX = G/G$ is the set of conjugacy classes of G, and the inertial Chern character $\mathrm{ch}^I(V) : G/G \to \mathbb{C}$ is the character $g \mapsto \mathrm{tr}(g|V)$ of V regarded as a representation of G. More generally, if $X = Y \times \mathbb{B}G$ and $V = \bigoplus_\rho W_\rho \otimes \rho$, then $IX = Y \times G/G$ and the inertial Chern character $\mathrm{ch}^I(V) : G/G \to H^{\mathrm{even}}(Y;\mathbb{C})$ is given by $\mathrm{ch}^I(V)(g) = \sum_\rho \mathrm{tr}(g|\rho)\,\mathrm{ch}(W_\rho)$. Since the trace functions $g \mapsto \mathrm{tr}(g|\rho)$ for $\rho \in \hat{G}$ form a basis for the space of maps $G/G \to \mathbb{C}$, we see that for $X = Y \times \mathbb{B}G$, the inertial Chern character determines (and is determined by) the Chern characters of each of the associated bundles W_ρ. For arbitrary X, the degree zero component of the inertial Chern character $\mathrm{ch}^I_0(V) \in H^0(IX;\mathbb{C})$ simply records the characters of the fibers of V.

To make sense of Chern–Weil theory on X which is a finite simplicial complex of finite groups, we fix a family of (germs of) smooth retractions $\tau \to \sigma$ for every pair of simplices $\sigma \subseteq \tau$ such that the maps $\tau \to \sigma \to \rho$ and $\tau \to \rho$ agree for $\rho \subseteq \sigma \subseteq \tau$ (such a family of smooth retractions may be constructed by induction). We require everything (functions, differential forms, bundles, connections, etc.) defined on a simplex τ to be pulled back under $\tau \to \sigma$ in a neighborhood of every $\sigma \subseteq \tau$ (such neighborhoods are not fixed, rather they must simply exist). Note that for bundles, this requirement actually consists of the data

[1] It is tempting to call it the "Chern character character".

of a compatible family of isomorphisms with the pullback bundles (one could equivalently consider only vector bundles built out of transition functions which satisfy the given pullback conditions).

Let us now try again to construct inductively our desired vector bundle V, or rather V together with a trivialization of its inertial Chern character. Namely, we try to construct a triple (V, θ, γ), where θ is a unitary connection on V and $\gamma \in \Omega^{\mathrm{odd}}(IX; \mathbb{C})$ satisfies

$$\mathrm{ch}^I(V, \theta) = n\chi_1 + d\gamma$$

where $n\chi_1 \in H^0(IX; \mathbb{C}) \subseteq \Omega^0(IX; \mathbb{C})$ denotes "n times the characteristic function of the identity" as a function $G_x/G_x \to \mathbb{C}$ (i.e. the character of the direct sum of $n\,|G_x|^{-1}$ copies of the regular representation of G_x) over $x \in X$. As before, we just need to show that such a triple (V, θ, γ) defined over $\partial D^k \times \mathbb{B}G$ admits an extension to $D^k \times \mathbb{B}G$. As we saw earlier, the obstruction to extending V lies in $\prod_{\rho \in \hat{G}} \pi_{k-1}(BU)$ (we assume that n is sufficiently large in terms of k to write BU instead of $BU(n_\rho)$), and the inertial Chern character maps this group injectively into $H^{\mathrm{even}}(\partial D^k \times G/G)$. Hence this obstruction vanishes due to the trivialization γ of the inertial Chern character form of (V, θ). We may thus extend V to $D^k \times \mathbb{B}G$, the connection θ extends by a partition of unity argument, and so we are left with the problem of extending γ.

Remark 2.2. The present argument applies more generally to a sufficiently divisible multiple of any desired character in $H^0(IX; \mathbb{C})$ in place of $n\chi_1$, provided its values at points $G_x/G_x \to \mathbb{C}$ are non-negative linear combinations of characters of irreducible representations.

The obstruction to extending γ lies in $H^{\mathrm{even}}(D^k \times G/G, \partial D^k \times G/G; \mathbb{C})$, which vanishes for k odd and equals the space of functions $G/G \to \mathbb{C}$ for k even. Concretely, this obstruction is given by

$$\int_{D^k} \mathrm{ch}^I(V, \theta) - \int_{\partial D^k} \gamma.$$

To resolve this obstruction, we make use of the freedom we had in choosing how to extend V from $\partial D^k \times \mathbb{B}G$ to $D^k \times \mathbb{B}G$. The space of such extensions is a torsor for $\prod_\rho \pi_k(BU)$, and acting via an element of this group has the effect (for k even) of adding a linear combination of characters of representations of G to the obstruction class. We may also consider the operation of replacing the entire triple (V, θ, γ) with the direct sum of $a > 0$ copies of (V, θ, γ), which has the effect of multiplying the obstruction class by a. Hence it would suffice to know that the obstruction class is, as a function $G/G \to \mathbb{C}$, a rational linear combination of characters of irreducible representations of G, with denominators bounded in terms of k.

Since there can be infinitely many k-cells $(D^k, \partial D^k) \times \mathbb{B}G$ in X, we cannot actually do this direct sum replacement for each cell individually, rather we must do a single such replacement which is sufficient for extending over all k-cells simultaneously. The obstruction to this extension problem now lies in $H^{\mathrm{even}}(IX_{\leq k}, IX_{\leq k-1}; \mathbb{C})$ (which is concentrated in degree k), where $X_{\leq i} \subseteq X$ denotes the i-skeleton of X. This obstruction class is a lift of the inertial Chern character $\mathrm{ch}^I(V) \in H^{\mathrm{even}}(IX_{\leq k})$ determined by the choice of γ; by the long exact sequence of the pair $(IX_{\leq k}, IX_{\leq k-1})$, we can make the obstruction class any lift we like by choosing the appropriate γ. Hence to ensure that the obstruction class is rational with denominators bounded in terms of k, it suffices (in view of the fact that

$H^k(IX_{\leq k}, IX_{\leq k-1}) \to H^k(IX_{\leq k})$ is surjective integrally since $H^k(IX_{k-1}) = 0$) to show that the inertial Chern character is rational with denominators bounded in terms of the degree.

To make sense out of the above discussion of rationality and denominators, we should introduce a $\mathbb{Z}$-structure on the cohomology groups $H^*(IX)$. To define the $\mathbb{Z}$-structure on $H^*(IX)$, we view it as the sheaf cohomology group $H^*(X; I)$ where I is the pushforward of the constant sheaf $\underline{\mathbb{C}}$ under the projection $IX \to X$. The stalk I_x of this pushforward sheaf I over a point $x \in X$ is the space of functions $G_x/G_x \to \mathbb{C}$, which has a natural $\mathbb{Z}$-structure given by integral linear combinations of characters of complex irreducible representations of G_x. These fiberwise $\mathbb{Z}$-structures fit together to define a $\mathbb{Z}$-structure on the sheaf I on X, which thus defines the cohomology groups $H^*(X; I) = H^*(IX)$ over the integers. Note that this integral structure is different from the integral cohomology lattice of the coarse space $|IX|$ (even in the case $X = \mathbb{B}G$).

It remains to show that the inertial Chern character $\mathrm{ch}^I(V) \in H^{\mathrm{even}}(IX)$ is rational with denominators bounded in terms of the degree. Our first step is to reduce to the case of line bundles by proving the *splitting principle*. Namely, we show that for every $V \to X$, there is a map $Y \to X$ such that $H^*(IX) \to H^*(IY)$ is split-injective (integrally) and the pullback f^*V splits as a direct sum of line bundles. The usual candidate for Y is the space of decompositions of fibers of V into ordered orthogonal one-dimensional subspaces. To see that this works in our context, we consider what is the fiber of $IY \to IX$ over a given point $(x, g) \in IX$. The fiber is the space of g-invariant ordered decompositions of V_x into one-dimensional subspaces. Since the group generated by g is abelian, its irreducible representations are one-dimensional, and hence there are plenty of such decompositions which are g-invariant, namely when each one-dimensional subspace is contained in some g-isotypic piece of V_x. Thus $IY \to IX$ is an iterated projective space bundle, from which it follows that $H^*(IX) \to H^*(IY)$ is split-injective (a splitting for the pullback to a projective space bundle is given by the composition of multiplication by a power of the fiberwise hyperplane class, which is a global integral cohomology class as it is the first Chern class of the tautological bundle, and pushforward). The pullback of V to Y now splits as a direct sum of line bundles. Since the inertial Chern character is additive under direct sum of vector bundles, it suffices to show that the inertial Chern character of a line bundle is rational with denominators bounded in terms of the degree.

There is a classifying orbispace $\mathbb{B}U(n)$ for principal $U(n)$-bundles $P \to X$ over orbispaces X. We will only need the case $n = 1$ below, but for now there is no reason not to keep the discussion general. This universal bundle

$$\mathbb{E}U(n) \to \mathbb{B}U(n)$$

satisfies the following defining property: $\mathbb{E}U(n)$ is a space with a $U(n)$ action with finite stabilizers such that for every finite subgroup $G \subseteq U(n)$, the fixed set $\mathbb{E}U(n)^G$ is contractible. Note that if two finite subgroups $G, G' \subseteq U(n)$ are conjugated by $g \in U(n)$, then $g\mathbb{E}U(n)^G = \mathbb{E}U(n)^{G'}$, so this is only countably many conditions on $\mathbb{E}U(n)$ (as there are only countably many conjugacy classes of finite subgroups of $U(n)$). We may construct the space $\mathbb{E}U(n)$ in countably many steps by starting with $\mathbb{E}_{-1}U(n) = \varnothing$, defining $\mathbb{E}_iU(n)$ for $i \geq 0$ from $\mathbb{E}_{i-1}U(n)$ by adding countably many cells to kill the countably many countable groups $\pi_i E_{i-1}U(n)^G$ for representatives $G \subseteq U(n)$ of all conjugacy classes of finite subgroups, and then defining $\mathbb{E}U(n) = \bigcup_i \mathbb{E}_i U(n)$ as the ascending union. An obstruction theory argument shows that every orbispace P with a $U(n)$ action with finite stabilizers admits a $U(n)$-equivariant map to $\mathbb{E}U(n)$ (note that since $\mathbb{E}U(n)$ is a space, maps from an orbispace P to $\mathbb{E}U(n)$ are the same as maps from its coarse space $|P|$).

To show that the inertial Chern character of a line bundle is rational with denominators bounded in terms of the degree, it is enough to study the universal space $\mathbb{E}U(1) \to \mathbb{B}U(1)$. The classical universal space $EU(1) \to BU(1)$ is $\mathbb{C}^\infty \setminus 0 \to \mathbb{C}P^\infty$. The classifying space $\mathbb{E}U(1)$ is the direct limit

$$\mathbb{E}U(1) = \varinjlim \left[EU(1)_{0!} \xrightarrow{\cdot 1} EU(1)_{1!} \xrightarrow{\cdot 2} EU(1)_{2!} \xrightarrow{\cdot 3} \cdots \right]$$

where $EU(1) \xrightarrow{\cdot n} EU(1)$ indicates the map induced by the nth power map $U(1) \xrightarrow{\cdot n} U(1)$, and the subscript $EU(1)_k$ indicates equipping $EU(1)$ with k times the usual action of $U(1)$. To see that this direct limit is indeed $\mathbb{E}U(1)$, we note that the fixed set of the k-torsion subgroup of $U(1)$ is all of $EU(1)_{n!}$ (and thus contractible) once k divides $n!$. The universal base $\mathbb{B}U(1)$ is thus the corresponding direct limit of $BU(1)_{n!} := EU(1)_{n!}/U(1)$ which is $\mathbb{C}P^\infty$ equipped with a purely ineffective orbifold structure with isotropy group $\mathbb{Z}/n!\mathbb{Z}$. The space $I\mathbb{B}U(1)$ is thus given by

$$I\mathbb{B}U(1) = U(1)^\delta_{\mathrm{tors}} \times \mathbb{B}U(1)$$

where $^\delta$ indicates taking the discrete topology. Hence the universal inertial Chern character is an element of $\prod_{U(1)_{\mathrm{tors}}} H^{\mathrm{even}}(\mathbb{C}P^\infty; \mathbb{C}) = \prod_{U(1)_{\mathrm{tors}}} \mathbb{C}[H]$, and it is given by

$$\big(\zeta \cdot \exp(H)\big)_{\zeta \in U(1)_{\mathrm{tors}}}.$$

Now the $\mathbb{Z}$-structure on any finite approximation $H^i(IBU(1)_k; \mathbb{C}) = \prod_{\zeta^k=1} H^i(\mathbb{C}P^\infty; \mathbb{C})$ is given by the $\mathbb{Z}$-linear span of $(\zeta^a)_{\zeta^k=1} \cdot H^i(\mathbb{C}P^\infty; \mathbb{Z})$ for $a \in \mathbb{Z}$. Thus the universal inertial Chern character of line bundles is indeed rational with denominators bounded in terms of the degree, as desired. $\square$

Remark 2.3. One may interpret the above proof of Theorem 2.1 in homotopy theoretic terms as follows. Vector bundles are classified by maps to $\mathbb{B}U(n)$, and since $\mathbb{B}U(n)$ is not contractible, the extension problem for vector bundles has nontrivial obstructions. Vector bundles with rationally trivialized Chern character are classified by maps to the total space of a fibration over $\mathbb{B}U(n)$ whose fiber classifies odd-dimensional rational cohomology classes on the inertia stack. This total space is rationally contractible, which is reflected in our observation that the obstructions to this new extension problem are torsion. The definition of this fibration over $\mathbb{B}U(n)$ depends on the additivity of the Chern character (note that $\mathbb{E}U(n) \to \mathbb{B}U(n)$ is not suitable for this argument since $\mathbb{E}U(n)$ is a space, hence every bundle pulled back from $\mathbb{E}U(n)$ has trivial isotropy representations). It may prove interesting to intepret this argument within Schwede's framework of global homotopy theory [21].

3 Topological stacks

We review some basic facts about stacks (on the category of topological spaces), we give a precise definition of what we mean by an 'orbispace', and we establish some of their basic properties. The reader may also wish to consult Noohi [20], Gepner–Henriques [12], Metzler [19], or Behrend–Noohi [6].

Let Top denote the category of topological spaces and continuous maps, and let Grpd denote the 2-category of (essentially) small groupoids. A *stack* is a functor $F : \mathsf{Top}^{\mathsf{op}} \to \mathsf{Grpd}$ which satisfies *descent*, i.e. such that for every topological space U and every open cover $\{U_i \to U\}_i$, the natural functor

$$F(U) \to \mathrm{Eq}\Big[\prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \cap U_j) \overset{\rightarrow}{\underset{\rightarrow}{\rightarrow}} \prod_{i,j,k} F(U_i \cap U_j \cap U_k)\Big]$$

is an equivalence. Stacks form a 2-category, with morphisms given by natural transformations of functors. The 2-category of stacks is complete, meaning all (small) limits exist; furthermore these limits may be calculated pointwise in the sense that $(\lim_\alpha F_\alpha)(U) = \lim_\alpha (F_\alpha(U))$. Note that, as we are working in a 2-categorical context, all functors are 2-functors, all diagrams are 2-diagrams, all limits are 2-limits, etc. (though we will usually omit the prefix '2-').

The Yoneda lemma implies that the Yoneda functor $X \mapsto \mathrm{Hom}(-, X)$ embeds the category of topological spaces fully faithfully into the 2-category of stacks, and moreover that the natural map from $F(X)$ to the groupoid of maps of stacks $\mathrm{Hom}(-, X) \to F(-)$ is an equivalence. The category of topological spaces is complete, and the Yoneda embedding is continuous (commutes with limits). Hence we will make no distinction between a topological space X and the associated stack $\mathrm{Hom}(-, X)$ of maps to X, nor between objects of $F(X)$ and maps $\mathrm{Hom}(-, X) \to F(-)$ (which we will simply write as $X \to F$).

Every stack X has a *coarse space* $|X|$ (a topological space) which is initial in the category of maps from X to topological spaces. Concretely, the points of $|X|$ are the isomorphism classes of maps $* \to X$, and a subset $U \subseteq |X|$ is open iff for every map $Y \to X$ from a topological space Y, the inverse image of U is an open subset of Y.

A stack is called *representable* iff it is in the essential image of the Yoneda embedding (i.e. it is isomorphic to a topological space). A morphism of stacks $F \to G$ is called representable iff for every map $X \to G$ from a topological space X, the fiber product $F \times_G X$ is representable.

For any property $\mathcal{P}$ of morphisms of topological spaces which is preserved under pullback, a representable morphism of stacks $F \to G$ is said to have property $\mathcal{P}$ iff the pullback $F \times_G X \to X$ has $\mathcal{P}$ for every map $X \to G$ from a topological space X. The following are examples of properties $\mathcal{P}$ of morphisms $f : X \to Y$ which are preserved under pullback:

- f is *injective*.
- f is *surjective*.
- f is a *closed inclusion*.
- f is *open*.
- f is *étale*, meaning that for every $x \in X$ there exists an open neighborhood $x \in U \subseteq X$ such that $f|_U : U \to Y$ is an open inclusion.
- f is *separated*, meaning that for every distinct pair $x_1, x_2 \in X$ with $f(x_1) = f(x_2)$, there exist open neighborhoods $x_i \in U_i \subseteq X$ which are disjoint $U_1 \cap U_2 = \varnothing$. (This is equivalent to the relative diagonal $X \to X \times_Y X$ being a closed inclusion.)

- f is *proper*, meaning that for every $y \in Y$ and every collection of open sets $\{U_i \subseteq X\}_i$ covering $f^{-1}(y)$, there exists a finite subcollection which covers $f^{-1}(V)$ for some open neighborhood $y \in V \subseteq Y$. (This is equivalent to f being *universally closed*, meaning that $X \times_Y Z \to Z$ is closed for every $Z \to Y$. One proof of this equivalence goes via yet a third equivalent condition, namely that every net $\{x_\alpha \in X\}_\alpha$ with $f(x_\alpha) \to y$ has a subnet converging to some $x \in f^{-1}(y)$.)
- f *admits local sections*, meaning that there is an open cover $\{U_i \subseteq Y\}_i$ such that every restriction $f|_{f^{-1}(U_i)} : f^{-1}(U_i) \to U_i$ admits a section.
- f is a *finite covering space*, meaning that there is an open cover $\{U_i \subseteq Y\}_i$ such that every restriction $f|_{f^{-1}(U_i)} : f^{-1}(U_i) \to U_i$ is isomorphic to $U_i^{\sqcup n_i} \to U_i$ for some integer $n_i \geq 0$.

Each of these properties $\mathcal{P}$ is also closed under composition, and thus also under fiber products, meaning that for maps $X \to X'$ and $Y \to Y'$ over Z, if both $X \to X'$ and $Y \to Y'$ have $\mathcal{P}$ then $X \times_Z Y \to X' \times_Z Y'$ also has $\mathcal{P}$ (indeed, $X \times_Z Y \to X \times_Z Y'$ is a pullback of $Y \to Y'$).

For any stack X, open (resp. closed) inclusions $Y \hookrightarrow X$ are in natural bijection with open (resp. closed) subsets of $|X|$.

To check that a given map of spaces satisfies one of the properties $\mathcal{P}$ above, it is often helpful to make use of the fact that these $\mathcal{P}$ are all *local on the target*, meaning that for every open cover $\{U_i \subseteq Y\}_i$, if $X \times_Y U_i \to U_i$ has $\mathcal{P}$ for every i, then so does $X \to Y$. This leads to the following generalization for maps of stacks: if $F \to G$ is a representable morphism of stacks and $G' \to G$ is a representable morphism of stacks admitting local sections, then $F \to G$ has $\mathcal{P}$ iff $F \times_G G' \to G'$ has $\mathcal{P}$. In fact, in this statement we need not assume that $G' \to G$ is representable, just that it admit local sections in the generalized sense that for every map $X \to G$ from a topological space X, there exists an open cover $\{U_i \subseteq X\}_i$ such that each $G' \times_G U_i \to U_i$ admits a section. We thus say that "$\mathcal{P}$ descends along maps admitting local sections". The same descent property holds for representability itself:

Lemma 3.1 (Representability descends under maps admitting local sections)**.** *Let $F \to G$ be a map of stacks, and let $G' \to G$ be a map of stacks admitting local sections. If $F \times_G G' \to G'$ is representable, then so is $F \to G$.*

Proof. By replacing $F \to G$ and $G' \to G$ with their pullbacks under $U \to G$ for a topological space U, we may assume without loss of generality that G is representable. Since $G' \to G$ admits local sections, we may replace $G' \to G$ with the composition $\bigsqcup_i U_i \to G' \to G$ where $\{U_i \subseteq G\}_i$ is an open cover. Now each $F \times_G U_i$ is representable by assumption, and gluing these spaces together on their common overlaps $F \times_G (U_i \cap U_j)$ gives a topological space representing F. ☐

The class of so called *topological stacks* (those which admit a presentation via a topological groupoid) are somewhat better behaved than general stacks. A *topological groupoid* $M \rightrightarrows O$ consists of a pair of topological spaces O ('objects') and M ('morphisms'), two maps $M \rightrightarrows O$ ('source' and 'target'), a map $O \to M$ ('identity'), an involution $M \to M$ ('inverse'), and a map $M \times_O M \to M$ ('composition') satisfying the axioms of a groupoid. A topological groupoid $M \rightrightarrows O$ presents a stack $[M \rightrightarrows O]$ defined as follows. An object of $[M \rightrightarrows O](X)$ is an open cover $\{U_i \subseteq X\}_i$ together with maps $U_i \to O$ and $U_i \cap U_j \to M$ satisfying a compatibility condition, and an isomorphism in $[M \rightrightarrows O](X)$ consists of maps $U_i \cap U'_{i'} \to M$ satisfying a compatibility condition. There is a natural map $O \to [M \rightrightarrows O]$ (take the trivial open cover $\{X \subseteq X\}$) and the fiber product $O \times_{[M \rightrightarrows O]} O$ is naturally identified with M. The morphism $O \to [M \rightrightarrows O]$ admits local sections (by definition), so since $O \times_{[M \rightrightarrows O]} O \to O$ is representable, it follows by descent that $O \to [M \rightrightarrows O]$ is representable. Conversely, a representable map $U \to X$ admitting local sections from a topological space U to a stack X determines a topological groupoid $U \times_X U \rightrightarrows U$ presenting X. Indeed, the fiber product $U \times_X U$ is representable (since U and $U \to X$ are), it admits two maps to U (the two projections), an involution (exchanging the two factors), and a composition map $(U \times_X U) \times_U (U \times_X U) = U \times_X U \times_X U \to U \times_X U$ (forgetting the middle factor), and one can check using the stack property that the natural map $[U \times_X U \rightrightarrows U] \to X$ is an equivalence. A stack X for which there exists a representable map $U \to X$ admitting local sections from a space U is called *topological*, and such $U \to X$ is called an *atlas* for X.

For a topological stack X with atlas $U \to X$, for any property $\mathcal{P}$ which descends along maps admitting local sections, the map $U \to X$ has $\mathcal{P}$ iff both maps $U \times_X U \to U$ have $\mathcal{P}$, and the diagonal $X \to X \times X$ has $\mathcal{P}$ iff the map $U \times_X U \to U \times U$ has $\mathcal{P}$. In particular, for any topological stack X, the diagonal $X \to X \times X$ is representable, and thus every map $Z \to X$ from a topological space Z is representable. More generally, for any map of topological stacks $X \to Y$, the relative diagonal $X \to X \times_Y X$ is representable (by descent from its pullback $V \times_X V \to V \times_Y V$ for an atlas $V \to X$). If $X \to Y$ is representable and Y is topological, then so is X (if $U \to Y$ is an atlas for Y, then its pullback $V = U \times_Y X \to X$ is an atlas for X), and the relative diagonal $X \to X \times_Y X$ has $\mathcal{P}$ iff the relative diagonal $V \to V \times_U V$ of atlases has $\mathcal{P}$ (the latter is the pullback of the former under the map $V \times_U V \to X \times_Y X$, which is representable and admits local sections since it is a pullback of $U \to Y$).

For a topological space V, a topological group G, and a continuous group action $G \curvearrowright V$, we may consider the action groupoid $G \times V \rightrightarrows V$ with maps $(g, x) \mapsto x$ and $(g, x) \mapsto gx$. The stack associated to this groupoid is denoted V/G and is called the stack quotient of the action $G \curvearrowright V$. If G is discrete, the two maps $G \times V \to V$ are étale, so $V \to V/G$ is étale. If G is compact and V is Hausdorff, the map $G \times V \to V \times V$ is proper (factor as $G \times V \to G \times V \times V$ which is a closed inclusion since V is Hausdorff and $G \times V \times V \to V \times V$ which is proper since G is compact), so V/G has proper diagonal. If G is Hausdorff, then the map $G \times V \to V \times V$ is separated, so V/G has separated diagonal.

A groupoid presentation of a topological stack X also gives a description of its coarse space $|X|$ as follows. For an atlas $U \to X$, consider the equivalence relation $\sim_X$ on U given by the image of $U \times_X U \to U \times U$. There is a map $X \to U/{\sim_X}$ (which is tautological once we regard X as $[U \times_X U \rightrightarrows U]$), inducing a map $|X| \to U/{\sim_X}$, which is a bijection, essentially by definition. Since an open substack of X pulls back to an open subset of U invariant under $\sim_X$, it follows that $|X| \xrightarrow{\sim} U/{\sim_X}$ is open and is thus a homeomorphism. In particular, it follows that the coarse space $|V/G|$ of the stack quotient V/G is the usual topological quotient of V by G.

An action $G \curvearrowright V$ is called *locally trivial* iff V admits a cover by G-invariant open sets $\{G \times Z_i \subseteq V\}_i$ where $G \curvearrowright G \times Z_i$ acts by left multiplication on G and trivially on Z_i. If $G \curvearrowright V$ is locally trivial, then the natural map from the stack quotient to the topological quotient $V/G \to |V/G|$ is an equivalence. Indeed, this assertion is local on $|V/G|$, so it suffices to consider the case of $G \curvearrowright G \times Z$, where it holds by inspection.

Definition 3.2. A *(separated) orbispace* is a stack X for which there exists a representable étale surjection $U \to X$ from a topological space U (an 'étale atlas'), and the diagonal $X \to X \times X$ is separated and proper.

Proposition 3.3. *A stack X is an orbispace iff $|X|$ is Hausdorff and there exists an open cover $\{V_i/G_i \subseteq X\}_i$ where G_i are finite discrete groups acting on Hausdorff spaces V_i.*

Proof. Let X be an orbispace, and let us show that there is an open cover $\{V_i/G_i \subseteq X\}_i$. Fix an étale atlas $U \to X$, and let $u \in U$. The automorphism group $G_u := \{u\} \times_X \{u\} \subseteq U \times_X U$ is finite and discrete since $X \to X \times X$ is separated and proper. Since the two projections $U \times_X U \rightrightarrows U$ are étale, for every $g \in G$ there exists an open neighborhood $g \in U_g \subseteq U \times_X U$ such that each projection restricted to U_g is an open inclusion (this gives another proof that G has the discrete topology). Since G is finite and $U \times_X U \to U \times U$ is separated, we may take these U_g to be disjoint. Now the complement of $\bigsqcup_g U_g$ is closed and disjoint from G, so it projects to a closed (by properness) subset of $U \times U$ disjoint from (u, u). Hence there is an open neighborhood $u \in V \subseteq U$ such that $V \times_X V \subseteq \bigsqcup_g U_g$. Thus $V \times_X V$ is a disjoint union of pieces indexed by G, and each piece maps homeomorphically to V under each projection. By further shrinking V, we may assume that the map $V \times_X V \to G$ respects composition (this is possible since composition is continuous). It follows that $V \times_X V \rightrightarrows V$ is an action groupoid $G \times V \rightrightarrows V$ for an action $G \curvearrowright V$. Since $V = \{1\} \times V \hookrightarrow G \times V = V \times_X V \to V \times V$ expresses the diagonal of V as a composition of proper maps, we conclude that V is Hausdorff. Now the map of groupoids $(G \times V \rightrightarrows V) = (V \times_X V \rightrightarrows V) \to (U \times_X U \rightrightarrows U)$ induces a map of stacks $V/G \to X$. To see that this is an open inclusion, let $V^+ \subseteq U$ denote the orbit of V under the morphisms $U \times_X U \rightrightarrows U$. Since the projections $U \times_X U \rightrightarrows U$ are étale, it follows that $V^+ \subseteq U$ is open, and hence $[V^+ \times_X V^+ \rightrightarrows V^+] \to [U \times_X U \rightrightarrows U]$ is an open inclusion of stacks; denote by $Z \subseteq X$ this open substack, so $V \to Z$ is an étale atlas. Thus the topological groupoid $V \times_X V = V \times_Z V \rightrightarrows V$ presents Z, so $V/G \to X$ is an open inclusion as desired.

Let us now show that for any orbispace X, its coarse space $|X|$ is Hausdorff. We saw earlier that $|X|$ is the quotient of U by the image of $U \times_X U \to U \times U$ (which is an equivalence relation). This equivalence relation is closed since $U \times_X U \to U \times U$ is proper, so $|X|$ is Hausdorff provided the quotient map $U \to |X|$ is open. Now openness of $U \to |X|$ does not depend on which atlas $U \to X$ we take: there are étale maps $U \leftarrow U \times_X U' \to U'$ over $|X|$, which means that $U \to |X|$ is open iff $U' \to |X|$ is open. Moreover, openness of $U \to |X|$ can be checked locally on $|X|$, so we may assume without loss of generality that $X = V/G$. Hence it is enough to note that the quotient map $V \to |V/G|$ (induced by the canonical étale atlas $V \to V/G$) is open. Thus $|X|$ is Hausdorff.

Finally, let us show that if $|X|$ is Hausdorff and there is an open cover $\{V_i/G_i \subseteq X\}_i$, then X is an orbispace. The maps $V_i \to V_i/G_i$ are representable étale, so $U := \bigsqcup_i V_i \to X$ is an étale atlas. To show that the diagonal $X \to X \times X$ is separated and proper, it is equivalent to show that $U \times_X U \to U \times U$ is separated and proper. This reduces to showing that $V \times_X V' \to V \times V'$ is separated proper for any pair $V/G \hookrightarrow X \hookleftarrow V'/G'$. Since $|X|$ is Hausdorff, the map $V \times_{|X|} V' \to V \times V'$ a closed inclusion (as it is a pullback of the diagonal of $|X|$), so it follows that $V \times_X V' \to V \times V'$ is separated and proper iff $V \times_X V' \to V \times_{|X|} V'$ is separated and proper. Now for the purposes of studying the latter map $V \times_X V' \to V \times_{|X|} V'$, we may as well shrink V, V', and X so that $V/G = X = V'/G'$. Now the diagonal of $V/G = V'/G' = X$ is separated and proper, hence so is $V \times_X V' \to V \times V'$. □

Corollary 3.4. *Every orbispace X has an étale atlas $U \to X$ with U Hausdorff; equivalently, every étale atlas $U \to X$ has U locally Hausdorff.*

Proof. By Proposition 3.3 there is an open cover $\{V_i/G_i \subseteq X\}_i$ with V_i Hausdorff, so $U := \bigsqcup_i V_i \to X$ is an étale atlas with U Hausdorff. Now for any two étale atlases $U, U' \to X$, consideration of the surjective étale maps $U \leftarrow U \times_X U' \to U'$ shows that U is locally Hausdorff iff U' is locally Hausdorff. Given any étale atlas $U \to X$ with U locally Hausdorff and any open cover $\{U_i \subseteq U\}_i$ with U_i Hausdorff, the disjoint union $U' := \bigsqcup_i U_i \to X$ is an étale atlas with U' Hausdorff. □

Corollary 3.5. *For any étale atlas $U \to X$ on an orbispace, there exists an open cover $\{V_i/G_i \subseteq X\}_i$ as in Proposition 3.3 such that each map $V_i \to X$ factors through an open inclusion $V_i \to U$.*

Proof. Let $U \to X$ be given and fix any open cover $\{V_i/G_i \subseteq X\}_i$ as in Proposition 3.3. Since $U \times_X V_i \to V_i$ is étale, it admits local sections, and hence by replacing each V_i with an open cover of itself, we may assume that each projection $U \times_X V_i \to V_i$ admits a section (which is thus an open inclusion). Now the resulting maps $V_i \to U$ are étale by the factorization $V_i \to U \times_X V_i \to U$, so by again replacing each V_i with an open cover, we may assume they are open inclusions.

Alternatively, we could note that the open cover $\{V_i/G_i \subseteq X\}_i$ produced by the proof of Proposition 3.3 is in fact of the desired form. □

Corollary 3.6. *A map of orbispaces $X \to Y$ is representable iff it is injective on isotropy groups. In particular, an orbispace X is a space iff it has trivial isotropy.*

Proof. Any representable morphism of stacks is injective on isotropy groups (just test against points). Thus we are left with showing that a map of orbispaces $X \to Y$ which is injective on isotropy groups is representable.

Since representability descends under maps admitting local sections, it suffices to show that the fiber product $X' = X \times_Y Y'$ is representable for some étale atlas $Y' \to Y$. Note that X' has trivial isotropy since Y' has trivial isotropy and $X' \to Y'$ is injective on isotropy groups (being a pullback of $X \to Y$).

We claim that X' is an orbispace, provided we take Y' to be Hausdorff (which we can by Corollary 3.4). The pullback of an étale atlas $U \to X$ is an étale atlas $U' \to X'$ (since $X' \to X$ is representable, being a pullback of $Y' \to Y$). The diagonal of X' is the composition $X' \to X' \times_X X' \to X' \times X'$. The second map $X' \times_X X' \to X' \times X'$ is separated and proper as it is a pullback of $X \to X \times X$. To analyze the first map $X' \to X' \times_X X'$, note that it pulls back to $U' \to U' \times_U U'$ under the map $U' \times_U U' \to X' \times_X X'$ which admits local sections (being a pullback of $U \to X$). Since $Y \to Y \times Y$ is separated, its pullback $Y' \times_Y Y' \to Y' \times Y'$ is also separated, which implies each projection $Y' \times_Y Y' \rightrightarrows Y'$ is separated since Y' is Hausdorff, which implies $Y' \to Y$ is separated. Hence its pullback $U' \to U$ is separated, so

$U' \to U' \times_U U'$ is a closed inclusion, hence separated and proper. Thus $X' \to X' \times_X X'$ is separated and proper, and we conclude that X' is an orbispace.

We are thus reduced to showing that an orbispace Z with trivial isotropy is a space. By Proposition 3.3, we know that Z is given locally by V/G for V Hausdorff and G finite discrete acting freely (since Z has trivial isotropy). Free actions $G \curvearrowright V$ with V Hausdorff and G finite are locally trivial, so we conclude that the map $Z \to |Z|$ is an equivalence. Alternatively, we could note that the chart V/G near a given $x \in X$ constructed in the proof of Proposition 3.3 in fact satisfies $G = G_x$ by definition. □

4 Coverings and nerves

We show how Theorem 2.1 implies Theorem 1.1. The basic idea will be to map a given orbispace to the nerve of a suitable open cover (which will be a simplicial complex of groups). The implementation of this idea is somewhat more delicate than one might initially expect (see the proof of Proposition 4.6).

A *sieve* on a topological space X is a subset $S \subseteq 2^X$ consisting of open sets such that $U' \subseteq U \in S$ implies $U' \in S$. A *covering sieve* on X is a sieve S such that $\bigcup_{U \in S} U = X$. An open cover $\{U_i \subseteq X\}_i$ is said to *generate* the covering sieve on X consisting of those open sets which are contained in some U_i.

A *connection sieve* on a map of spaces $f : X \to Y$ is a covering sieve S on X such that (1) for $U \in S$, the composition $U \to X \to Y$ is an open inclusion, and (2) for $U, V \in S$ with $f(U) = f(V)$, either $U = V$ or $U \cap V = \varnothing$. Note that for sieves satisfying condition (1), condition (2) is equivalent to condition (2′) for $U, V \in S$ either $U \cap V = \varnothing$ or $f(U \cap V) = f(U) \cap f(V)$. If $S' \subseteq S$ is an inclusion of covering sieves and S is a connection sieve on $X \to Y$, then so is S'. In particular, if S and S' are connection sieves are on $X \to Y$ then so is $S \cap S'$. To check that an open cover $\{U_i \subseteq X\}_i$ generates a connection sieve, it is enough to check axioms (1) and (2′) for the open sets U_i.

Let us call a map $X \to Y$ *strongly étale* iff it admits a connection sieve. Open inclusions are strongly étale, and strongly étale maps are separated and étale (however the converse is false). Being strongly étale is preserved under pullback (take the connection sieve generated by the pullback of the original connection sieve), and the class of strongly étale maps is closed under composition (a connection sieve $S_{X/Z}$ for a composition $X \to Y \to Z$ is given by those elements of a fixed connection sieve $S_{X/Y}$ for $X \to Y$ whose image lies in a fixed connection sieve $S_{Y/Z}$ for $Y \to Z$). A disjoint union $\bigsqcup_i X_i \to \bigsqcup_i Y_i$ of strongly étale maps $X_i \to Y_i$ is strongly étale (take disjoint union of connection sieves), and the projection $A \times Y \to Y$ is strongly étale for any discrete space A. Being strongly étale is *not* local on the target (there are finite covering spaces with non-Hausdorff target which are not strongly étale).

Recall that a topological space is called *paracompact* iff every open cover admits a locally finite refinement [9]. If X is paracompact and Hausdorff, then there exists a *partition of unity* subordinate to any given locally finite open cover $\{U_i \subseteq X\}_i$, namely functions $f_i : X \to \mathbb{R}_{\geq 0}$ with $\operatorname{supp} f_i \subseteq U_i$ such that $\sum_i f_i \equiv 1$ (recall that the support $\operatorname{supp} f$ of a function $f : X \to \mathbb{R}_{\geq 0}$ is by definition the complement of the largest open set over which f vanishes identically).

Lemma 4.1. *If Y is paracompact Hausdorff, then a map $X \to Y$ is strongly étale iff there exists an open cover $\{U_i \subseteq Y\}_i$ such that each map $X \times_Y U_i \to U_i$ is strongly étale.*

Proof. Fix an open cover $\{U_i \subseteq Y\}_i$ and connection sieves S_i on $X \times_Y U_i \to U_i$. Since Y is paracompact, we may assume that our open cover $\{U_i \subseteq Y\}_i$ is locally finite. Using a partition of unity subordinate to this open cover, we may find another open cover $\{V_I \subseteq Y\}_I$ indexed by non-empty finite subsets I of the original index set, such that $V_I \subseteq \bigcap_{i\in I} U_i$ and $V_I \cap V_J = \varnothing$ unless $I \subseteq J$ or $J \subseteq I$ (explicitly, we may take V_I to be the locus where $\min_{i\in I} f_i > \max_{i\notin I} f_i$). We may now define a connection sieve on $X \to Y$ as the union over I of $2^{X\times_Y V_I} \cap \bigcap_{i\in I} S_i$. □

An orbispace will be called paracompact (resp. coarsely finite-dimensional, d-dimensional) iff its coarse space is.

Proposition 4.2. *Every paracompact orbispace X has an étale atlas $U \to X$ for which the projections $U \times_X U \rightrightarrows U$ are strongly étale. In fact, there exists such U of the form $U = \bigsqcup_i V_i$ for an open cover $\{V_i/G_i \subseteq X\}_i$ as in Proposition 3.3.*

Proof. Fix an open cover $\{V_i/G_i \hookrightarrow X\}_i$ as in Proposition 3.3. Since $|X|$ is paracompact, we may shrink the spaces V_i (G_i-equivariantly) so as to ensure that the associated open cover $\{|V_i/G_i| \subseteq |X|\}_i$ of coarse spaces is locally finite. Choose a partition of unity $\{f_i : |X| \to \mathbb{R}_{\geq 0}\}_i$ subordinate to the open cover $\{|V_i/G_i| \subseteq |X|\}_i$. Let $V_i^0 \subseteq V_i$ denote the open subset where $f_i > 0$. We will show that the étale atlas $U := \bigsqcup_i V_i^0$ has the desired property.

It suffices to show that for any pair of open inclusions $V/G \hookrightarrow X \hookleftarrow W/H$ and pair of functions $f_V, f_W : |X| \to \mathbb{R}_{\geq 0}$ supported inside $|V/G|$ and $|W/H|$, respectively, the projection $W^0 \times_X V^0 \to V^0$ is strongly étale. We begin by considering the map $W \times_X V \to V$, which is a finite covering space over its open image $W/H \times_X V \subseteq V$ (by pullback from $W \to W/H$, which is a finite covering space by descent from its pullback $H \times W \to W$). Thus every point of $W/H \times_X V$ has a neighborhood over which $W \times_X V \to V$ is strongly étale. Even better, since V is Hausdorff and G is finite, each G-orbit inside $W/H \times_X V$ has a neighborhood over which $W \times_X V \to V$ is strongly étale. In other words, each point of $|W/H| \cap |V/G| \subseteq |X|$ has a neighborhood over which $W \times_X V \to V$ is strongly étale.

Now since $|X|$ is paracompact, there exists a locally finite open cover of $|X|$ by $|X| \setminus (\operatorname{supp} f_V \cup \operatorname{supp} f_W)$ together with open subsets $\{A_i \subseteq |W/H| \cap |V/G|\}_i$ over which $W \times_X V \to V$ is strongly étale. Fix a partition of unity $g : |X| \to \mathbb{R}_{\geq 0}$ supported inside $|X| \setminus (\operatorname{supp} f_V \cup \operatorname{supp} f_W)$ and $\{g_i : |X| \to \mathbb{R}_{\geq 0}\}$ supported inside A_i, that is $g + \sum_i g_i \equiv 1$. Now the patching procedure for connection sieves from the proof of Lemma 4.1 shows that $W \times_X V \to V$ is strongly étale over the complement of $\operatorname{supp} g$. In particular, it follows by restriction that $W^0 \times_X V^0 \to V^0$ is strongly étale. □

A *simplicial complex* is a pair $X = (V, S)$ consisting of a set V ("vertices") and a set $S \subseteq 2^V \setminus \{\varnothing\}$ ("simplices") of finite subsets of V such that S contains all singletons and $\varnothing \neq A \subseteq B \in S$ implies $A \in S$. The *star* $\operatorname{st}(X, \sigma) \subseteq X$ of a simplex σ in a simplicial complex X is the subcomplex consisting of all simplices $\tau \subseteq X$ with $\sigma \cup \tau \in S(X)$.

A map of simplicial complexes $X \to Y$ is a map of vertex sets $V(X) \to V(Y)$ which maps simplices to simplices (the image of an element of $S(X)$ is an element of $S(Y)$). A map of simplicial complexes is called injective iff the map on vertex sets (hence also the map on simplices) is injective. A map of simplicial complexes $f : X \to Y$ is called *étale* (resp. *locally injective*) iff the induced maps on stars $\operatorname{st}(X, \sigma) \to \operatorname{st}(Y, f(\sigma))$ are isomorphisms (resp. injective). We will call a map of simplicial complexes $X \to Y$ *sufficiently étale* iff every simplex $\sigma \subseteq Y$ (equivalently, every vertex) is the image of a simplex $\tau \subseteq X$ at which $X \to Y$ is étale (this is a useful weakening of the condition of being surjective and étale, which in the context of simplicial complexes is too strong).

The *geometric realization* $\|X\|$ of a simplicial complex X is the set of tuples $t \in \mathbb{R}_{\geq 0}^{V(X)}$ with $\sum_v t_v = 1$ such that $\{v : t_v > 0\} \in S(X)$, topologized by declaring that the realization of the complete simplex on $k+1$ vertices has the usual topology and that a realization $\|X\|$ is given the strongest topology for which (the realization of) every map from a complete simplex to X is continuous. The geometric realization of an étale map of simplicial complexes is an étale map of spaces.

A *locally injective simplicial complex groupoid* $M \rightrightarrows O$ consists of simplicial complexes O and M together with structure maps satisfying the axioms of a groupoid, where both maps $M \rightrightarrows O$ are locally injective. Local injectivity of the two maps $M \rightrightarrows O$ implies that the natural map $\|M \times_O M\| \xrightarrow{\sim} \|M\| \times_{\|O\|} \|M\|$ is a homeomorphism, and thus the geometric realization $\|M\| \rightrightarrows \|O\|$ is a topological groupoid. If $\partial O \subseteq O$ denotes the subcomplex consisting of those simplices $\sigma \subseteq O$ for which it is not the case that the first projection $M \to O$ is étale at every simplex $\tau \subseteq M$ mapped to σ under the second projection, then the natural map $\|O\| \setminus \|\partial O\| \to [\|M\| \rightrightarrows \|O\|]$ is étale. A locally injective simplicial complex groupoid $M \rightrightarrows O$ is called *sufficiently étale* iff this map is surjective (equivalently, every vertex of O is M-isomorphic to one not in ∂O).

The *abstract simplex category* Simp has objects finite totally ordered sets and has morphisms weakly order preserving maps; every object of Simp is isomorphic to $[n] := \{0 < \cdots < n\}$ for a unique integer $n \geq 0$. A simplicial object in a category C is a functor $\mathsf{Simp}^{\mathsf{op}} \to \mathsf{C}$, and the category of simplicial objects in C is denoted sC. If C is complete (resp. cocomplete) then so is sC, and limits (resp. colimits) are calculated pointwise.

Here we will consider only simplicial sets (objects of the category sSet) and simplicial groupoids (objects of the category sGrpd). A simplicial set (or groupoid) will be denoted $X_\bullet$, where X_n is its set (or groupoid) of n-simplices. We denote by $\Delta^n_\bullet \in \mathsf{sSet}$ the standard n-simplex, given by $[m] \mapsto \operatorname{Hom}([m], [n])$. The Yoneda lemma implies that $X_n = \operatorname{Hom}(\Delta^n_\bullet, X_\bullet)$.

A map of simplicial sets $X_\bullet \to Y_\bullet$ is called injective iff it is so levelwise (i.e. every $X_n \to Y_n$ is injective). A map $X_\bullet \to Y_\bullet$ is called étale (resp. locally injective) iff for every map $[n] \to [m]$ in Simp, the induced map $X_m \xrightarrow{\sim} Y_m \times_{Y_n} X_n$ is a bijection (resp. injective) (it is equivalent to impose this condition only for $n = 0$). A map $X_\bullet \to Y_\bullet$ is called étale (resp. locally injective) at a given n-simplex σ of $X_\bullet$ iff $X_m \xrightarrow{\sim} Y_m \times_{Y_n} \{\sigma\}$ is a bijection (resp. injective) (this condition at a given σ implies the same at any preimage of σ under any structure map of $X_\bullet$). A map $X_\bullet \to Y_\bullet$ is called sufficiently étale iff the n-simplices of $X_\bullet$ at which the map is étale surject onto the n-simplices of $Y_\bullet$ (it is equivalent to impose this condition only for $n = 0$). These notions generalize to maps of simplicial groupoids by replacing 'injectivity' and 'surjectivity' for maps of sets with 'full faithfulness' and 'essential surjectivity' for functors of groupoids. These properties are all preserved under pullback and closed under composition.

The *geometric realization* $\|X_\bullet\|$ of a simplicial set $X_\bullet$ is the colimit of Δ^n over all maps

$\Delta^n_\bullet \to X_\bullet$. Geometric realization is cocontinuous, and the natural map $\|\lim_\alpha (X_\bullet)_\alpha\| \to \lim_\alpha \|(X_\bullet)_\alpha\|$ is bijective, however it need not be a homeomorphism even for finite limits. The map $\|X_\bullet \times_{Y_\bullet} Z_\bullet\| \to \|X_\bullet\| \times_{\|Y_\bullet\|} \|Z_\bullet\|$ is a homeomorphism if at least one of the maps $X_\bullet \to Y_\bullet$ and $Z_\bullet \to Y_\bullet$ is locally injective. Thus a locally injective simplicial set groupoid $M_\bullet \rightrightarrows O_\bullet$ determines a topological groupoid $\|M_\bullet\| \rightrightarrows \|O_\bullet\|$.

Let us now introduce the geometric realization $\|X_\bullet\|$ of any simplicial groupoid $X_\bullet$ which is *étale*, meaning that it admits a locally injective sufficiently étale map $U_\bullet \to X_\bullet$ from a simplicial set $U_\bullet$. For any simplicial groupoid $X_\bullet$ and any locally injective map $U_\bullet \to X_\bullet$, the pair of simplicial sets $U_\bullet \times_{X_\bullet} U_\bullet \rightrightarrows U_\bullet$ forms a locally injective simplicial set groupoid, whose geometric realization $\|U_\bullet \times_{X_\bullet} U_\bullet\| \rightrightarrows \|U_\bullet\|$ thus defines a topological stack. The geometric realization of $X_\bullet$ is defined as this topological stack $[\|U_\bullet \times_{X_\bullet} U_\bullet\| \rightrightarrows \|U_\bullet\|]$ associated to any locally injective sufficiently étale map $U_\bullet \to X_\bullet$.

Lemma 4.3. *The geometric realization of an étale simplicial groupoid $X_\bullet$ is well defined.*

Proof. Let $U_\bullet, U'_\bullet \to X_\bullet$ be locally injective and sufficiently étale. Let $U''_\bullet := U_\bullet \times_{X_\bullet} U'_\bullet \to X_\bullet$, and consider the map of simplicial set groupoids $(U''_\bullet \times_{X_\bullet} U''_\bullet \rightrightarrows U''_\bullet) \to (U_\bullet \times_{X_\bullet} U_\bullet \rightrightarrows U_\bullet)$. Since $U''_\bullet \to U_\bullet$ is locally injective and sufficiently étale, it follows that this map induces an isomorphism of topological stacks, and the same applies to $U'_\bullet$ in place of $U_\bullet$. □

Lemma 4.4. *The geometric realization $\|X_\bullet\|$ of an étale simplicial groupoid $X_\bullet$ with finite isotropy is an orbispace.*

Proof. All geometric realizations are Hausdorff, so $\|U_\bullet \times_{X_\bullet} U_\bullet\| \to \|U_\bullet\| \times \|U_\bullet\|$ is separated, hence $\|X_\bullet\|$ has separated diagonal. Since $X_\bullet$ has finite isotropy, the map $U_\bullet \times_{X_\bullet} U_\bullet \to U_\bullet \times U_\bullet$ has finite fibers, which combined with local injectivity of $U_\bullet \times_{X_\bullet} U_\bullet \rightrightarrows U_\bullet$ implies that $\|U_\bullet \times_{X_\bullet} U_\bullet\| \to \|U_\bullet\| \times \|U_\bullet\|$ is proper, hence $\|X_\bullet\|$ has proper diagonal.

To construct an étale atlas for $\|X_\bullet\|$, let $\partial U_\bullet \subseteq U_\bullet$ denote the simplicial subset consisting of those simplices of $U_\bullet$ at which $U_\bullet \to X_\bullet$ is not étale. Then $\|U_\bullet\| \setminus \|\partial U_\bullet\| \to \|X_\bullet\|$ is étale. To see that it is surjective, it suffices to show that $\|U_\bullet \times_{X_\bullet} U_\bullet\| \setminus \|U_\bullet \times_{X_\bullet} \partial U_\bullet\| \to \|U_\bullet\|$ is surjective (note that surjectivity descends under surjective maps such as $\|U_\bullet\| \to \|X_\bullet\|$), and this follows since $U_\bullet \to X_\bullet$ is sufficiently étale. □

A simplicial complex X gives rise to a simplicial set $b_\bullet X$ (its *barycentric subdivision*) whose n-simplices are chains of simplices $\sigma_0 \subseteq \cdots \subseteq \sigma_n \subseteq X$ (in other words, $b_\bullet X$ is the nerve of $S(X)$). Barycentric subdivision preserves injectivity, local injectivity, étale, and sufficiently étale. There is a natural identification of geometric realizations $\|X\| = \|b_\bullet X\|$. Moreover, for a locally injective sufficiently étale simplicial complex groupoid $M \rightrightarrows O$, there is a natural identification $[\|M\| \rightrightarrows \|O\|] = \|[b_\bullet M \rightrightarrows b_\bullet O]\|$ (the simplicial groupoid $[b_\bullet M \rightrightarrows b_\bullet O]$ is étale since $M \rightrightarrows O$ is sufficiently étale).

A simplicial complex of groups (Z, G) also admits a barycentric subdivision $b_\bullet(Z, G)$ which is a simplicial groupoid. In fact, we will define $b_\bullet(Z, G)$ for any simplicial complex Z equipped with a functor $G : S(Z)^{\mathsf{op}} \to \mathsf{Grpd}$ from the face poset to groupoids (a simplicial complex of groups (Z, G) determines such a functor which σ to $\mathbb{B}G_\sigma$). The groupoid of n-simplices in the barycentric subdivision $b_\bullet(Z, G)$ is now defined as the groupoid of functors from the category $0 \to \cdots \to n$ to the category whose objects are pairs $\sigma \in S(Z)$ and $o \in G_\sigma$ and whose morphisms are inclusions $\sigma_1 \subseteq \sigma_2$ covered by maps $o_1 \to o_2|_{\sigma_1}$. The barycentric subdivision of a simplicial complex of groups is étale, as can be seen as follows. For any $\sigma \subseteq Z$ and $o \in G_\sigma$, we consider the functor $S(\mathrm{st}(Z,\sigma))^{\mathsf{op}} \to \mathsf{Grpd}$ given by $\tau \mapsto G_{\sigma\cup\tau} \times_{G_\sigma} \{o\}$. Applying the nerve construction from just above to this functor, we obtain a simplicial set mapping to $b_\bullet(Z,G)$, which is the required locally injective map which is étale over $(\sigma, o) \in b_0(Z,G)$. An essentially equivalent discussion (albeit without barycentrically subdividing) appears in [7, 12.24–12.25]. Since $b_\bullet(Z,G)$ is étale, it has a geometric realization $\|b_\bullet(Z,G)\|$ which we also write as $\|(Z,G)\|$. When G are finite groups (or, more generally, groupoids with finite isotropy), then the geometric realization $\|(Z,G)\|$ is an orbispace by Lemma 4.4, and this is what we have been calling the orbispace presented by the simplicial complex of finite groups (Z,G).

Lemma 4.5. *Let $M \rightrightarrows O$ be a locally injective simplicial complex groupoid with the following properties:*

- *The vertices of every simplex of O are pairwise non-isomorphic via M.*
- *If simplices σ and σ' of O have vertex sets which are isomorphic via M, then σ and σ' are themselves isomorphic via M.*

Then there is a simplicial complex of groups giving rise to the same simplicial groupoid as $M \rightrightarrows O$.

Proof. The hypotheses imply that there is a simplicial complex Z whose vertices are the isomorphism classes in the vertex groupoid $V(M) \rightrightarrows O(M)$, and whose simplices are the M-isomorphism classes of simplices of O. Now $M \rightrightarrows O$ defines a functor $G : S(Z)^{\mathsf{op}} \to \mathsf{Grpd}$, and there is a natural isomorphism between $b_\bullet(Z,G)$ and the simplicial groupoid $[b_\bullet M \rightrightarrows b_\bullet O]$. By definition, all the groupoids G_σ have a single isomorphism class, and all the functors $G_\tau \to G_\sigma$ are faithful (this follows from local injectivity of $M \rightrightarrows O$). Choosing (independently) a base object of each G_σ shows (Z,G) comes from a simplicial complex of groups. □

The *nerve* $N(X, \{U_i\}_i)$ of a collection of open sets $\{U_i \subseteq X\}_i$ is the simplicial complex whose vertices V are the indices i with $U_i \neq \varnothing$, and in which a collection I of indices spans a simplex (i.e. $I \in S$) iff $\bigcap_{i\in I} U_i \neq \varnothing$. A partition of unity $\{f_i : X \to \mathbb{R}_{\geq 0}\}_i$ subordinate to a locally finite open cover $\{U_i \subseteq X\}_i$ defines a map from X to the geometric realization $\|N(X, \{U_i\}_i)\|$ of the nerve of the open cover.

Proposition 4.6. *For any paracompact orbispace X, there exists a simplicial complex of groups (Z,G) (where the groups G_z for vertices $z \in Z$ are isotropy groups of points of X) and a map $X \to \|(Z,G)\|$ which is injective on isotropy groups. If X is coarsely finite-dimensional (resp. d-dimensional), then Z may be taken to be finite-dimensional (resp. d-dimensional).*

Proof. We begin with an open cover $\{V_i/G_i \subseteq X\}_i$ with the properties guaranteed by Proposition 4.2, and we set $U := \bigsqcup_i V_i$. Since $|X|$ is paracompact, by G_i-equivariantly shrinking the spaces V_i, we may assume that the associated open cover $\{|V_i/G_i| \subseteq |X|\}_i$ of coarse spaces is locally finite. We fix a covering sieve S on $U \times_X U$ which is invariant under the 'exchange' (i.e. 'inverse') involution of $U \times_X U$ and is a connection sieve for both projections $U \times_X U \rightrightarrows U$. We also fix a partition of unity $\{f_i : |X| \to \mathbb{R}_{\geq 0}\}_i$ subordinate to the open cover $\{|V_i/G_i| \subseteq |X|\}_i$

Denote by $o(x) \subseteq U$ and $o_i(x) \subseteq V_i$ the fibers over $x \in |X|$, so $o(x) = \bigsqcup_i o_i(x)$; similarly define $m(x) \subseteq U \times_X U$ and $m_{ij}(x) \subseteq V_i \times_X V_j$ with $m(x) = \bigsqcup_{i,j} m_{ij}(x)$. These sets are finite since the G_i are finite and $\{|V_i/G_i| \subseteq |X|\}_i$ is locally finite. Furthermore, they have the discrete topology, since U is Hausdorff and $U \times_X U$ is Hausdorff (since U is Hausdorff and $X \to X \times X$ is separated).

The Hausdorff property implies that the inclusions $o_i(x) \subseteq V_i$ and $m_{ij}(x) \subseteq V_i \times_X V_j$ admit retractions defined in some open neighborhood. Now the inverse images of small open neighborhoods $x \in |Z_x| \subseteq |X|$ (i.e. open substacks $Z_x \subseteq X$) form a basis of neighborhoods of $o_i(x)$ and $m_{ij}(x)$, so for sufficiently small Z_x, these inverse images are naturally disjoint unions $U_x = \bigsqcup_{o \in o(x)} U_o$ and $U_x \times_X U_x = \bigsqcup_{m \in m(x)} U_m$. Note that this applies only inside V_i and $V_i \times_X V_j$ for which $o_i(x)$ and $m_{ij}(x)$ are non-empty: the full inverse image of Z_x inside U may intersect other V_i nontrivially. By shrinking Z_x further, we may ensure that the retraction $(U_x \times_X U_x \rightrightarrows U_x) \to (m(x) \rightrightarrows o(x))$ is a map of groupoids. For later purposes, let us also take Z_x small enough so that:

- If $x \in |V_i/G_i|$ then $|Z_x| \subseteq |V_i/G_i|$.
- If $x \notin \operatorname{supp} f_i$ then $|Z_x| \cap \operatorname{supp} f_i = \varnothing$.
- If $f_i(x) > 0$ then $f_i > 0$ over all of $|Z_x|$.

Each of these conditions can be ensured on its own, and since the open cover $\{|V_i/G_i| \subseteq |X|\}_i$ is locally finite, we can ensure all at once.

We also shrink Z_x so as to ensure that each $U_m \in S$ (our chosen connection sieve), which has the following implication: given $o, o' \in U$ with $U_o \cap U_{o'} \neq \varnothing$, the relation $U_m \cap U_{m'} \neq \varnothing$ is a partial bijection between lifts $m, m' \in U \times_X U$ of o and o'. Furthermore, the domain of this bijection is as large as possible: for $U_o \cap U_{o'} \neq \varnothing$ (so $o \in o_i(x)$ and $o' \in o_i(x')$ for some i) with $x, x' \in |V_j/G_j|$, we get a full bijection between the inverse images of o and o' inside $m_{ij}(x)$ and $m_{ij}(x')$ (this follows since the projection $V_i \times_X V_j \to V_i$ is a finite covering space of degree $|G_j|$ over $U_o \cup U_{o'}$).

We now consider the nerves $N(U, \{U_o\}_{o \in U})$ and $N(U \times_X U, \{U_m\}_{m \in U \times_X U})$. Note that for simplices in these nerves, namely subsets $O \subseteq U$ or $M \subseteq U \times_X U$ with $\bigcap_{o \in O} U_o \neq \varnothing$ or $\bigcap_{m \in M} U_m \neq \varnothing$, the maps $O \to |X|$ or $M \to |X|$ are injective. The natural maps on index sets $U \times_X U \rightrightarrows U$ determine maps of nerves

$$N(U \times_X U, \{U_m\}_{m \in U \times_X U}) \rightrightarrows N(U, \{U_o\}_{o \in U}).$$

These maps are locally injective; indeed, local injectivity means that for every $o, o' \in U$ with $U_o \cap U_{o'} \neq \varnothing$ and every $m \in U \times_X U$ projecting to o, there is at most one lift $m' \in U \times_X U$ of o' with $U_m \cap U_{m'} \neq \varnothing$, and this is a direct consequence of our assumption that every $U_m \in S$. Let us now argue that there is a natural composition map

$$N(U \times_X U, \{U_m\}_{m \in U \times_X U}) \times_{N(U, \{U_o\}_{o \in U})} N(U \times_X U, \{U_m\}_{m \in U \times_X U}) \longrightarrow N(U \times_X U, \{U_m\}_{m \in U \times_X U}).$$

More precisely, we claim that for non-empty finite subsets $M \subseteq U \times_X U$ with $\bigcap_{m \in M} U_m \neq \varnothing$ and $M' \subseteq U \times_X U$ with $\bigcap_{m \in M'} U_m \neq \varnothing$ projecting to $O \subseteq U$ (under the first and second projections, respectively), the subset $M'' \subseteq U \times_X U$ defined by applying the composition map $(U \times_X U) \times_U (U \times_X U) \to U \times_X U$ to M and M' also satisfies $\bigcap_{m \in M''} U_m \neq \varnothing$.

This claim follows from the property that every $U_m \in S$ (indeed, this property implies that $\bigcap_{m\in M} U_m \xrightarrow{\sim} \bigcap_{o\in O} U_o \xleftarrow{\sim} \bigcap_{m\in M'} U_m$). We have thus defined a locally injective simplicial complex groupoid

$$M := N(U \times_X U, \{U_m\}_{m\in U\times_X U}) \rightrightarrows N(U, \{U_o\}_{o\in U}) =: O.$$

We now show that this locally injective simplicial complex groupoid is sufficiently étale. Every isomorphism class of vertex (equivalently, every $x \in |X|$ with $|Z_x| \neq \varnothing$) has a representative $o \in V_i \subseteq U$ with $f_i(o) = f_i(x) > 0$. To show that these representatives are étale, let $m \in U \times_X U$ be a morphism with source o and target o', and let $U_{o'} \cap U_{o''} \neq \varnothing$. We must show that there is a bijection between morphisms $o \to o'$ and $o \to o''$. The morphisms in question all have source inside V_i, so we really can consider just $V_i \times U$ for the present purpose. Now the bulleted conditions on $|Z_x|$ from above imply that since $U_{o'} \cap U_{o''} \neq \varnothing$ and $f_i(o') > 0$, we have $U_{o'} \cup U_{o''} \subseteq |V_i/G_i| \subseteq |X|$. Thus over $U_{o'} \cap U_{o''}$ the map $V_i \times_X U \to U$ is a finite covering space of degree $|G_i|$. Thus all points have the same number of lifts, so it follows that the connection sieve property gives us a bijection between lifts.

We have already seen above that our sufficiently étale locally injective simplicial complex groupoid satisfies the first hypothesis of Lemma 4.5, and the second hypothesis follows from the partial bijection property derived above from the connection sieve. Thus by Lemma 4.5, there is a simplicial complex of groups (Z, G) giving rise to the same simplicial groupoid $b_\bullet(Z, G) = [b_\bullet M \rightrightarrows b_\bullet O]$ and thus (since $M \rightrightarrows O$ is sufficiently étale) to the same geometric realization $\|(Z, G)\| = [\|M\| \rightrightarrows \|O\|]$. The groups associated G_z associated to vertices $z \in Z$ are by definition isotropy groups of the vertex groupoid $V(M) \rightrightarrows V(O)$, which are by definition isotropy groups of points of X.

To conclude, it remains to define a map $X \to \|(Z, G)\|$ which is injective on isotropy groups. To define this map, we shrink the Z_x so that the open cover $\{|Z_x| \subseteq |X|\}_{x\in|X|}$ is locally finite, and choose a partition of unity $\{g_x : |X| \to \mathbb{R}_{\geq 0}\}_{x\in|X|}$ subordinate to the open cover $\{|Z_x| \subseteq |X|\}_{x\in|X|}$. These maps g_x lift to maps $g_o : U \to \mathbb{R}_{\geq 0}$ supported inside U_o and $g_m : U \times_X U \to \mathbb{R}_{\geq 0}$ supported inside U_m. The collection of these lifts defines a map of topological groupoids

$$(U^0 \times_X U^0 \rightrightarrows U^0) \to \big(\|N(U \times_X U, \{U_m\}_{m\in U\times_X U})\| \rightrightarrows \|N(U, \{U_o\}_{o\in U})\|\big)$$

where U^0 is the disjoint union of the open loci $V_i^0 \subseteq V_i$ where $f_i > 0$ (the maps g_o and g_m do not define a map over all of $U \times_X U \rightrightarrows U$ due to the fact that $U_x = \bigsqcup_{o\in o(x)} U_o$ and $U_x \times_X U_x = \bigsqcup_{m\in m(x)} U_m$ may not be the full inverse images of Z_x inside U and $U \times_X U$). It remains to check that this map is injective on isotropy groups; in other words, for $o, o' \in U^0$ we must show that the map $\{o\} \times_X \{o'\} \to \|N(U \times_X U, \{U_m\}_{m\in U\times_X U})\|$ is injective. Distinct elements of $\{o\} \times_X \{o'\}$ are, in particular, distinct lifts of o, which therefore cannot lie in any common U_m since $U_m \to U$ is injective, so we see that the map is indeed injective on isotropy groups. □

A complex vector bundle over a stack X is a representable map $V \to X$ together with maps $V \times_X V \to V$ and $\mathbb{C} \times V \to V$ (both over X) such that for every map $U \to X$ from a topological space U, there exists an open cover $\{U_i \subseteq U\}_i$ and integers $n_i \geq 0$ such that $V \times_X U_i \to U_i$ is isomorphic to $\mathbb{C}^{n_i} \times U_i \to U_i$ equipped with its fiberwise addition and scaling maps.

Proof of Theorem 1.1. Apply Proposition 4.6 to find a simplicial complex of finite groups (Z, G) and a map $X \to \|(Z, G)\|$ which is injective on isotropy groups. Now Theorem 2.1 applies to $\|(Z, G)\|$ to give a vector bundle $V \to \|(Z, G)\|$, and the pullback of this bundle to X satisfies the desired property since $X \to \|(Z, G)\|$ is injective on isotropy groups. $\square$

5 From vector bundles to principal bundles

We derive Corollaries 1.2 and 1.3 from Theorem 1.1.

A hermitian inner product on a complex vector bundle $V \to X$ is a map $h : V \times_X V \to \mathbb{C}$ satisfying $h(v, w) = \overline{h(w, v)}$, $h(v, \alpha w) = \alpha h(v, w)$ for $\alpha \in \mathbb{C}$, and $h(v, v) > 0$ for $v \neq 0$ (meaning, these conditions are imposed on the fiber $h_x : V_x \times V_x \to \mathbb{C}$ over each map $* \xrightarrow{x} X$).

Lemma 5.1. *A complex vector bundle over a paracompact orbispace X admits a hermitian inner product.*

Proof. Begin with an étale atlas $\bigsqcup_i U_i \to X$ such that the pullback of V to every U_i is trivial. By Corollary 3.5, we may refine this cover further so that each map $U_i \to X$ is the composition of $U_i \to U_i/G_i$ with an open inclusion $U_i/G_i \hookrightarrow X$. The pullback of V to each U_i is trivial, hence admits a hermitian inner product; by averaging, we may make it G_i-invariant thus giving a hermitian inner product h_i on the restriction of V to each $U_i/G_i \subseteq X$. Since $|X|$ is paracompact, there is a partition of unity φ_i subordinate to this open cover of $|X|$, and hence $\sum_i \varphi_i h_i$ is the desired hermitian inner product on V. $\square$

Corollary 5.2. *Every short exact sequence of vector bundles over a paracompact orbispace splits.* $\square$

Given a rank n complex vector bundle $V \to X$ with hermitian inner product, the associated frame bundle $P \to X$ is defined by declaring that a map $U \to P$ (U a topological space) is a map $U \to X$ together with an isomorphism $\mathbb{C}^n \times U \to V \times_X U$ under which the pullback of h is the standard hermitian inner product on $\mathbb{C}^n$. There is an action of $U(n)$ on P (by precomposition with automorphisms of $\mathbb{C}^n$ respecting its hermitian inner product), giving P the structure of a principal bundle over X, meaning that for every map $U \to X$ from a topological space, there is an open cover $\{U_i \subseteq U\}_i$ such that $P \times_X U_i \to U_i$ is isomorphic to $U(n) \times U_i \to U_i$ with $U(n)$ acting by left multiplication on the first factor.

Proof of Corollary 1.2. Let $V \to X$ be the rank n complex vector bundle produced by Theorem 1.1. By Lemma 5.1, there exists a hermitian inner product on V. The total space P of the associated principal bundle $P \to X$ has trivial isotropy since the isotropy groups of X act faithfully on the fibers of V.

We claim that P is a Hausdorff topological space. By Corollary 3.6, it is enough to show that P is an orbispace. Since $P \to X$ is representable, the pullback of an étale atlas for X is an étale atlas for P. The diagonal of P may be expressed as the composition $P \to P \times_X P \to P \times P$. The second map $P \times_X P \to P \times P$ is separated and proper, being a pullback of the diagonal of X. To check that the first map $P \to P \times_X P$ is separated and proper, it suffices by descent to show that its pullback under an étale atlas $U \to X$

is separated and proper. This pullback is separated and proper since $P \times_X U \to U$ is a principal $U(n)$ bundle (of topological spaces). We have thus shown that P is a topological space.

The map $P \to X$ is representable and admits local sections (by definition), so the topological groupoid $P \times_X P \rightrightarrows P$ presents the stack X. It can be seen by inspection that $P \times_X P \rightrightarrows P$ is the action groupoid of the $U(n)$ action on P. $\square$

Let Sm denote the category of topological spaces equipped with a maximal atlas of charts from open sets of $\bigsqcup_{n\geq 0} \mathbb{R}^n$ with smooth transition maps (a smooth manifold an object of Sm whose underlying topological space is Hausdorff). A smooth structure on a stack $F : \mathsf{Top}^{\mathsf{op}} \to \mathsf{Grpd}$ is a substack F^{sm} of the pullback of F under the forgetful functor $\mathsf{Sm} \to \mathsf{Top}$ (meaning that $F^{\mathrm{sm}}(U)$ is a full subcategory of $F(U)$ for $U \in \mathsf{Sm}$); maps $U \to F$ lying in (the essential image of) F^{sm} are then called smooth. Given a map of stacks $F \to G$ and a smooth structure on G, we may consider the pullback smooth structure on F, defined as those maps $U \to F$ whose composition with $F \to G$ is smooth. A map of stacks equipped with smooth structures $(F, F^{\mathrm{sm}}) \to (G, G^{\mathrm{sm}})$ is called smooth iff the composition of any smooth map $U \to F$ with $F \to G$ is a smooth map $U \to G$ (equivalently, F^{sm} is contained inside the pullback of G^{sm}). Stacks with smooth structures form a 2-category just like stacks. This category is complete, and limits $\lim_\alpha (F_\alpha, F_\alpha^{\mathrm{sm}})$ are calculated by taking the limit of the underlying stacks $\lim_\alpha F_\alpha$ and declaring a map $U \to \lim_\alpha F_\alpha$ to be smooth iff every induced map $U \to F_\alpha$ is smooth. The category Sm embeds fully faithfully into the category of stacks with smooth structures, by sending $X \in \mathsf{Sm}$ to the Yoneda functor of its underlying topological space, equipped with the smooth structure consisting of those maps $U \to X$ which are morphisms in Sm.

Definition 5.3. A *smooth orbifold* X is an orbispace equipped with a smooth structure such that for every (equivalently, some) étale atlas $U \to X$, we have $U \in \mathsf{Sm}$ when U is equipped with the pullback of the smooth structure on X.

By Proposition 3.3, a stack X with a smooth structure is a smooth orbifold iff $|X|$ is Hausdorff and there is an open cover of X by V_i/G_i for smooth manifolds V_i equipped with smooth actions of finite groups G_i.

Proof of Corollary 1.3. Let $V \to X$ be the complex vector bundle produced by Theorem 1.1. It suffices to define a smooth structure on V, because then we may follow the arguments of Lemma 5.1 and Corollary 1.2 in the smooth category.

We begin by arguing that a complex vector bundle V over a smooth orbifold X has a smooth structure in a neighborhood of any given point of $|X|$. We may thus assume that $X = U/G$ for a smooth manifold U acted on smoothly by a finite group G, and that the pullback of V to U is trivial. By shrinking this chart further (and adjusting G), we may assume that our given point of $|X|$ corresponds to a point $u \in U$ fixed by G. Choose any trivialization $V_u \times U \to V \times_X U$ where V_u denotes the fiber over u. By averaging, we may make this map G equivariant, and it remains an isomorphism in a neighborhood of u. We thus obtain a trivialization of $V \times_X U$ near u in which the action of G is constant (independent of the U coordinate). The standard smooth structure in this trivialization is thus, in particular, invariant under the action of G, giving the desired smooth structure on V near our given point of $|X|$.

Having shown the existence of smooth structures locally, we now show how to patch them together. To show that there exists a smooth structure on V over all of X, it suffices to show that for open subsets $A, B \subseteq |X|$ and smooth structures on $V|_A$ and $V|_B$, there exists a smooth structure on $V|_{A\cup B}$ restricting to the given smooth structure on $V|_A$ (indeed, this allows us to patch together smooth structures over arbitrary unions of open sets, by choosing a well ordering and adding one open set at a time). To prove this pairwise patching statement, it suffices to show that an isomorphism of smooth vector bundles may be approximated by a smooth isomorphism (then apply this to the identity map on $V|_{A\cap B}$ equipped with the restrictions of the two given smooth structures). Since $|X|$ is locally the quotient of Euclidean space by a finite group action, it is locally metrizable; since it is paracompact and Hausdorff, it is thus metrizable. Thus every open subset of $|X|$ is metrizable, hence paracompact. We may now conclude with a smooth partition of unity argument. □

6 K-theory of orbispaces

We derive Corollary 1.5 from Theorem 1.1. We then derive from Corollary 1.5 some basic properties of the K-theory of finite rank vector bundles on orbispaces.

Proof of Corollary 1.5. Denote our given map by $f : X \to Y$, and let E be the given vector bundle on X. Let F be any vector bundle on Y satisfying the conclusion of Theorem 1.1.

We first consider the local situation on X. Fix $x \in X$, and let $U/G \hookrightarrow X$ be an open inclusion sending $u \in U$ fixed by G to $x \in X$ and inducing an isomorphism $G \xrightarrow{\sim} G_x$ (such an open inclusion exists by the proof of Proposition 3.3). By shrinking U, we may ensure that the pullbacks of E and F to U are trivial. Choose any map on U from the pullback of E to the pullback of $F^{\oplus N}$ (some integer $N < \infty$) which is injective and G-equivariant at u, and average it to make it equivariant everywhere. This produces over U/G a map from E to $f^*F^{\oplus N}$ which is injective at x. Now since $|X|$ is paracompact Hausdorff, hence normal, there exists a continuous function $\varphi : |X| \to [0, 1]$ supported inside $|U/G|$ with $\varphi(x) = 1$. Multiplying by this function yields a map $E \to f^*F^{\oplus N}$ defined on all of X which is injective at x. Note that the integer N may be taken independent of x since the rank of E is bounded.

We now combine the above maps to give an everywhere injective map $E \to f^*F^{\oplus M}$ as follows. Fix an open cover $|X| = \bigcup_i U_i$ and maps $f_i : E \to f^*F^{\oplus N}$ defined over all of X which are injective over U_i. Since $|X|$ is coarsely finite-dimensional, we may refine this open cover so that it is locally finite and has nerve of dimension $\leq d$ for some integer $d < \infty$. As in the proof of Lemma 4.1, there is yet another open cover of $|X|$ by open sets V_I indexed by the non-empty subsets I of the index set of the U_i, such that $V_I \subseteq \bigcap_{i\in I} U_i$ and $V_I \cap V_J = \varnothing$ unless $I \subseteq J$ or $J \subseteq I$. Note that these conditions imply that $V_I = \varnothing$ unless $|I| \leq d + 1$ and that $V_I \cap V_J = \varnothing$ if $|I| = |J|$ and $I \neq J$. Now let $\sum_I \varphi_I \equiv 1$ be a partition of unity subordinate to the open cover $|X| = \bigcup_I V_I$. We may now define our desired everywhere injective map $E \to f^*F^{\oplus(d+1)N}$ (so $M = (d+1)N$) by the formula $\sum_I \alpha_{|I|}\varphi_I f_{i(I)}$, where $i(I)$ is any choice of index in the set I, and $\alpha_r : F^{\oplus N} \hookrightarrow (F^{\oplus N})^{\oplus(d+1)} = F^{\oplus(d+1)N}$ is the inclusion of the rth direct summand.

We have thus constructed an injection $E \to f^*F^{\oplus M}$ over X, which splits by Corollary 5.2 since X is paracompact. □

For any stack X, denote by $\mathrm{Vect}(X)$ the set of isomorphism classes of vector bundles on X *of bounded rank*. Direct sum of vector bundles equips $\mathrm{Vect}(X)$ with the structure of an abelian monoid. A map of stacks $X \to Y$ induces a pullback map $\mathrm{Vect}(Y) \to \mathrm{Vect}(X)$. The reason we restrict attention to vector bundles of bounded rank is so that we may appeal to Corollary 1.5, which we now restate in a way which suggests how it will eventually be used. Let us call a map of abelian monoids $f : M \to N$ *cofinal* iff for every $n \in N$ there exist $m \in M$ and $n' \in N$ with $f(m) = n + n'$. Now Corollary 1.5 may be stated as follows.

Corollary 6.1. *For any representable map $Y \to X$ of orbispaces satisfying the hypothesis of Theorem 1.1, the induced map* $\mathrm{Vect}(X) \to \mathrm{Vect}(Y)$ *is cofinal.* □

To explain the significance of cofinality, we must recall some definitions. Recall that a category is called *filtered* iff (1) it is non-empty, (2) for every pair of objects x, y, there exist morphisms $x \to z \leftarrow y$, and (3) for every pair of morphisms $x \rightrightarrows y$ there exists a morphism $y \to z$ such that both compositions $x \to z$ coincide. Colimits over filtered categories (called direct limits) are *exact*, in the sense that they preserve finite limits and colimits. Also recall that a functor F between filtered categories is called *cofinal* iff (1) for every x in the target there exists a morphism $x \to F(y)$, and (2) for every two morphisms $x \rightrightarrows F(y)$, there exists a morphism $y \to z$ such that the compositions $x \to F(z)$ coincide. A direct limit may be calculated by pulling back under any cofinal functor.

To any abelian monoid M, we may associate a filtered category $\hat{M}$ as follows. The set of objects of $\hat{M}$ is the underlying set of M, and a morphism $a \to b$ ($a, b \in M$) is an element $c \in M$ satisfying $a + c = b$. The category $\hat{M}$ is filtered: (1) holds since $0 \in M$, (2) holds by taking $z = x + y$, and (3) holds by taking $y \to z$ to be addition by x. A map of monoids $f : M \to N$ induces a functor $\hat{f} : \hat{M} \to \hat{N}$, and if f is cofinal then so is $\hat{f}$. Indeed: (1) is immediate, and for (2) observe that if $x + b = f(y) = x + b'$ are the two morphisms, then the morphism $y + y = z$ has the desired property since $b + f(y) = b + x + b' = f(y) + b'$.

For an abelian monoid M, its group completion $M[-M]$ is the quotient of the free abelian group on the underlying set of M by the subgroup generated by $[a]+[b]-[a+b]$ for $a, b \in M$. The map $M \to M[-M]$ is universal (initial) among maps from M to an abelian group. The group completion $M[-M]$ is the direct limit of the directed system over $\hat{M}$ which associates to every object $x \in \hat{M}$ the set M and to a morphism $x \xrightarrow{+z} y$ the map $M \xrightarrow{+z} M$:

$$M[-M] = \varinjlim_{y \in \hat{M}} M.$$

The element x in the copy of M over an object $y \in \hat{M}$ represents the formal difference $x - y$.

The group $K^0(X)$ is defined as the group completion of $\mathrm{Vect}(X)$ for any stack X. A map of stacks $X \to Y$ induces a pullback map $K^0(Y) \to K^0(X)$. The functor K^0 is finitely additive, in the sense that $K^0(\varnothing) = 0$ and the natural map $K^0(X \sqcup Y) \to K^0(X) \oplus K^0(Y)$ is an isomorphism. In the present context of vector bundles of bounded rank, additivity does not hold for general infinite disjoint unions.

Eventually, we will restrict attention to the K-theory of orbispaces satisfying the hypothesis of Theorem 1.1. However, we will impose such hypotheses gradually, as they become relevant.

To show that K-theory is homotopy invariant, the following is the key assertion.

Lemma 6.2. *For any paracompact orbispace X, every vector bundle on $X \times [0,1]$ is pulled back from X.*

Proof. Let a vector bundle V over $X \times [0,1]$ be given.

We first discuss the local structure around a given point $x \in X$. Fix an open embedding $Y/G \hookrightarrow X$ and a lift $y \in Y$ of x fixed by G. We consider the pullback bundle $V|_{Y\times[0,1]}$ on $Y \times [0,1]$, which is a G-equivariant vector bundle. For each $t \in [0,1]$, there exist open neighborhoods $t \in T \subseteq [0,1]$ and $y \in U \subseteq Y$ such that $V|_{U\times T}$ is trivial. By compactness of $[0,1]$, we may assume U is independent of t. Replacing Y with U, we may assume that for each $t \in [0,1]$ there exists an open neighborhood $t \in T \subseteq [0,1]$ such that $V|_{Y\times T}$ is trivial. A trivialization induces a map $V|_{Y\times T} \to p_T^* V|_{\{y\}\times T}$ which is the identity over $\{y\} \times T$. Averaging makes this map G-equivariant, so it descends to a map $V|_{Y/G\times T} \to p_T^* V|_{\{y\}/G\times T}$ which is still the identity over $\{y\} \times T$. It is thus an isomorphism over some $U \times T'$; another compactness argument and shrinking of Y ensures that we have a collection of isomorphisms $V|_{Y/G\times T} \to p_T^* V|_{\{y\}/G\times T}$ which are the identity over $\{y\} \times T$. Patching these together via a partition of unity on $[0,1]$ and further shrinking Y produces an isomorphism

$$V|_{Y/G\times[0,1]} \xrightarrow{\sim} p_{[0,1]}^* V|_{\{y\}/G\times[0,1]}$$

which is the identity over $\{y\} \times [0,1]$. Any vector bundle over $\mathbb{B}G \times [0,1]$ is pulled back from $\mathbb{B}G$, and hence we conclude that $V|_{Y/G\times[0,1]}$ is pulled back from Y/G, for some neighborhood Y/G of $x \in X$.

We now globalize. Begin with an open cover $|X| = \bigcup_i U_i$ and over each U_i an isomorphism $\xi_i : V|_{U_i\times[0,1]} \to (p_X^* V|_{X\times\{0\}})|_{U_i\times[0,1]}$ which is the identity over $U_i \times \{0\}$. Note that for any $x \in U_i$ and any $t, t' \in [0,1]$, the map ξ_i determines an isomorphism

$$V_{(x,t)} \xrightarrow{\xi_i^{-1}\circ\xi_i} V_{(x,t')}$$

which is the specialization of an isomorphism between the pullbacks of V under the two projections $X \times [0,1]^2 \to X \times [0,1]$. Using this observation, we may now patch together the ξ_i into an isomorphism $V \to p_X^* V|_{X\times\{0\}}$ as follows. Fix a partition of unity $\varphi_i : |X| \to \mathbb{R}_{\geq 0}$ subordinate to the open cover $|X| = \bigcup_i U_i$, and fix an arbitrary total ordering of the index set of the open cover. For $(x,t) \in X \times [0,1]$, let $i = 1, \ldots, k$ denote the indices of the open sets U_i containing x, ordered as in the fixed total order, and consider the composition of isomorphisms

$$V_{(x,t)} \xrightarrow{\xi_1^{-1}\circ\xi_1} V_{(x,(1-\varphi_1(x))\cdot t)} \xrightarrow{\xi_2^{-1}\circ\xi_2} V_{(x,(1-\varphi_1(x)-\varphi_2(x))\cdot t)} \xrightarrow{\xi_3^{-1}\circ\xi_3} \cdots \xrightarrow{\xi_k^{-1}\circ\xi_k} V_{(x,0)}.$$

This fiberwise description now translates into the desired isomorphism of vector bundles $V \to p_X^*|_{X\times\{0\}}$. ∎

Two maps $X \to Y$ will be called homotopic iff there exists a map $X \times [0,1] \to Y$ whose restrictions to $X \times \{0\}$ and $X \times \{1\}$ are the two given maps. A map $X \to Y$ is called a homotopy equivalence iff there exists a map $Y \to X$ such that the compositions $X \to Y \to X$ and $Y \to X \to Y$ are homotopic to the respective identity maps.

Corollary 6.3 (Homotopy Invariance)**.** *If X is a paracompact orbispace, then homotopic maps $X \to Y$ induce the same map $K^0(Y) \to K^0(X)$.* □

For any inclusion of abelian monoids $M' \subseteq M$, the quotient M/M' is, as a set, the quotient of M by the equivalence relation $x \sim_{M'} y$ iff there exist $a, b \in M'$ with $x+a = y+b$, equipped with the descent of the monoid operation from M. The map $M \to M/M'$ is initial among maps from M to an abelian monoid sending M' to zero. The group completion $M[-M]$ coincides with the quotient of $M \times M$ by the diagonal submonoid M. In the special case that there is a retraction $r : M \to M'$, the quotient M/M' may be described as the direct limit

$$M/M' = \varinjlim_{e \in \hat{M}'} r^{-1}(e).$$

This observation will be used later.

We now introduce the relative K-theory $K^0(X, Y)$ for any map of stacks $f : Y \to X$. We consider the set of isomorphism classes of triples (E_0, E_1, i) where E_0, E_1 are vector bundles on X and $i : f^*E_0 \to f^*E_1$ is an isomorphism between their pullbacks to Y. This set is an abelian monoid under direct sum, and $K^0(X, Y)$ is the quotient of this monoid by the submonoid of triples of the form $(E, E, f^* \operatorname{id}_E)$ for vector bundles E on X. Note that *a priori* $K^0(X, Y)$ is merely an abelian monoid, not an abelian group. A map $(Y \to X) \to (Y' \to X')$ (i.e. a commutative square) induces a map $K^0(X', Y') \to K^0(X, Y)$. There is a natural map $K^0(X, Y) \to K^0(X)$ given by sending (E_0, E_1, i) to the formal difference $[E_0] - [E_1]$, and the composition $K^0(X, Y) \to K^0(X) \to K^0(Y)$ is zero.

A *pair* (X, A) shall mean that A is a closed substack of X. A map of pairs $(X, A) \to (Y, B)$ means a map $X \to Y$ sending A inside B. Note that paracompactness, coarse finite-dimensionality, being an orbispace, and the hypothesis of Theorem 1.1 all pass to closed substacks.

Remark 6.4. Even for very nice orbispace pairs (X, A), we *cannot* in general form the quotient X/A in a reasonable way. For example, for an orbispace X, if we could reasonably define $(X \times [0, 1])/(X \times \{0\})$ as an orbispace, it would follow that X is a global quotient of a topological space by a finite group action (a very special property).

For any pair (X, A), let $\operatorname{cyl}(X, A)$ denote the pair $((X \times \{0\}) \cup (A \times [0, 1]), A \times \{1\})$. Here $(X \times \{0\}) \cup (A \times [0, 1])$ denotes the union of closed substacks of $X \times [0, 1]$ (recall that closed substacks of a stack Z are in bijective correspondence with closed subsets of $|Z|$, so by union of closed substacks we mean union of subsets of $|Z|$). There is a natural map of pairs $\operatorname{cyl}(X, A) \to (X, A)$. A map of pairs $f : (X, A) \to (X', A')$ induces a map $\operatorname{cyl}(f) : \operatorname{cyl}(X, A) \to \operatorname{cyl}(X', A')$.

Lemma 6.5. *For any pair (X, A) where A is a paracompact orbispace, $K^0(\operatorname{cyl}(X, A))$ is an abelian group.*

Proof. Let $(E_0, E_1, i : E_0|_{A \times \{1\}} \to E_1|_{A \times \{1\}})$ be a triple representing an arbitrary element of $K^0(\operatorname{cyl}(X, A))$. We claim that the triple $(E_1, E_0, -i)$ is an inverse to it. It suffices to show that

$$(E_0 \oplus E_1, E_0 \oplus E_1, \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)) \quad \text{and} \quad (E_0 \oplus E_1, E_0 \oplus E_1, \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right))$$

are isomorphic. In other words, it suffices to show that $\left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$ is the restriction to $A\times\{1\}$ of an automorphism of $E_0\oplus E_1$. By Lemma 6.2, there exist isomorphisms $E_i|_{A\times[0,1]} = p_A^*(E_i|_{A\times\{1\}})$. In such coordinates, the desired automorphism of $E_0 \oplus E_1$ may be given by

$$\begin{pmatrix} \cos\frac{\pi}{2}t & -i\sin\frac{\pi}{2}t \\ i\sin\frac{\pi}{2}t & \cos\frac{\pi}{2}t \end{pmatrix}$$

on $A\times[0,1]$ and the identity on $X\times\{0\}$ (note that to specify a map of vector bundles over a stack, it suffices to specify its restriction to each member of a finite cover by closed substacks, subject to the requirement that these restrictions agree on their pairwise overlaps). ☐

Lemma 6.6. *For any pair (X, A) where A is a paracompact orbispace, the map $K^0(X, A) \to K^0(\operatorname{cyl}(X, A))$ is surjective.*

Proof. It suffices to show that every vector bundle on $(X \times \{0\}) \cup (A \times [0, 1])$ is pulled back from X. Let E be a vector bundle on $\operatorname{cyl}(X, A)$, let F denote its restriction to $X = X \times \{0\}$, and let us show that $E = p_X^*F$. The identity map is an identification of E and p_X^*F over $X \times \{0\}$, and by Lemma 6.2 there is an identification of E and p_X^*F over $A \times [0, 1]$ which agrees with the identity over $A \times \{0\}$. These isomorphisms $E \to p_X^*F$ thus patch together to define the desired isomorphism. ☐

Lemma 6.7. *For every vector bundle E over an orbispace X, there exists an open cover $X = \bigcup_i Z_i/G_i$ such that the restriction of E to each Z_i/G_i is pulled back from $*/G_i$ (equivalently, the pullback of E to Z_i is G_i-equivariantly trivial).*

Proof. Every point $x \in X$ has an open neighborhood of the form Z/G by Proposition 3.3. Choose a lift $z \in Z$ of x, and by shrinking Z and replacing G with the stabilizer of z, assume that G fixes z. Choose a map from $E|_Z$ to the trivial bundle $E_z\times Z$ which is the identity at z, average this map to make it G-equivariant, and shrink Z so that this map is an isomorphism over all of Z. ☐

Lemma 6.8. *Let $G \curvearrowright Z$ be a finite group action on a Hausdorff topological space. If Z/G is paracompact, then so is Z.*

Proof. Let an open covering $Z = \bigcup_\alpha V_\alpha$ be given. For every $z \in Z$, there exists an open neighborhood U of z which is G_z-invariant and whose G/G_z-translates are disjoint and each contained in some (possibly different) V_α. The images of such U in Z/G form an open covering of Z/G. Refining this to a locally finite covering and taking inverse images in Z gives the desired locally finite refinement of the original covering. ☐

Lemma 6.9. *Let X be a paracompact orbispace, let E be a vector bundle over X, and let $Y \subseteq X$ be a closed substack. Every section of $E|_Y$ extends to a section of E.*

Proof. Fix an open cover $X = \bigcup_i Z_i/G_i$ such that the pullback of E to each Z_i is trivialized G_i-equivariantly (Lemma 6.7). Let $\varphi_i : |X| \to [0, 1]$ be a partition of unity subordinate to this covering. Now $\operatorname{supp}\varphi_i$ is a closed subset of a paracompact Hausdorff space, hence paracompact Hausdorff; its inverse image inside Z_i is thus paracompact Hausdorff by Lemma 6.8 (recall Z_i is Hausdorff). Thus by the Tietze extension theorem, our given section on

$Z_i \times_X (Y \cap \operatorname{supp} \varphi_i)$ extends to $Z_i \times_X \operatorname{supp} \varphi_i$. We can make it G_i-equivariant by averaging, so our given section on $Y \cap \operatorname{supp} \varphi_i$ extends to $\operatorname{supp} \varphi_i$. Now $(\operatorname{supp} \varphi_i)^\circ$ is an open cover of X, so pick another partition of unity ψ_i subordinate to this cover, and use it to patch together the extended sections on each $\operatorname{supp} \varphi_i$. □

Lemma 6.10. *For any paracompact orbispace pair (X, A), the map $K^0(X, A) \to K^0(\operatorname{cyl}(X, A))$ is an isomorphism.*

Proof. We know from the proof of Lemma 6.6 that every vector bundle on $(X \times \{0\}) \cup (A \times [0, 1])$ is pulled back from X. Given this fact, it suffices to show that pullback from (X, A) to $\operatorname{cyl}(X, A)$ induces an injective map on sets of triples (E_0, E_1, i). Thus suppose that we are given two triples $(E_0, E_1, i : E_0|_A \to E_1|_A)$ and $(E_0', E_1', i' : E_0'|_A \to E_1'|_A)$ whose pullbacks to $\operatorname{cyl}(X, A)$ coincide, meaning that there are isomorphisms $\alpha_i : p_X^* E_i \to p_X^* E_i'$ on $(X \times \{0\}) \cup (A \times [0, 1])$ intertwining i and i' over $A \times \{1\}$. Now these maps α_i extend to all of $X \times [0, 1]$ by Lemma 6.9, and these extended maps are isomorphisms over open neighborhood of $(X \times \{0\}) \cup (A \times [0, 1])$. Since $X \times [0, 1] \to X$ is proper (being a pullback of $[0, 1] \to *$), this neighborhood contains $U \times [0, 1]$ for some open neighborhood U of $A \subseteq X$. Now a paracompact Hausdorff space is normal, so by Urysohn's lemma there exists a continuous function $\varphi : |X| \to [0, 1]$ supported inside U which is identically 1 on A. Pulling back α_i under the graph of φ defines isomorphisms $E_i \to E_i'$ whose restrictions to A intertwine i and i', showing that the original triples on (X, A) are isomorphic. □

Proposition 6.11 (Relative Homotopy Invariance)**.** *For any map of paracompact orbispace pairs $(X, A) \to (X', A')$ whose constituent maps $X \to X'$ and $A \to A'$ are individually homotopy equivalences, the map $K^0(X', A') \to K^0(X, A)$ is an isomorphism.*

Proof. Denote the components of our map $(X, A) \to (X', A')$ by $f_X : X \to X'$ and $f_A : A \to A'$. Let $g_X : X' \to X$ and $g_A : A' \to A$ be homotopy inverses of f_X and f_A. We claim that $g_X|_{A'}$ and g_A are homotopic as maps $A' \to X$. To see this, it suffices to further compose with the homotopy equivalence f_X, after which they are both homotopic to the given inclusion $A' \to X$. Now g_X and g_A together with a choice of homotopy between $g_X|_{A'}$ and g_A define a map $\operatorname{cyl}(X', A') \to \operatorname{cyl}(X, A)$.

We now consider the composition $\operatorname{cyl}(X, A) \to \operatorname{cyl}(X', A') \to \operatorname{cyl}(X, A)$. The restrictions of this map to $X \times \{0\}$ and to $A \times \{1\}$ are homotopic to id_X and id_A. By embedding these homotopies into $A \times [0, 1]$, we see that our map $\operatorname{cyl}(X, A) \to \operatorname{cyl}(X, A)$ is in fact homotopic to a map which is the identity on X and is the identity on A. Such a map $\operatorname{cyl}(X, A) \to \operatorname{cyl}(X, A)$ need not be homotopic to the identity (it contains a choice of homotopy from the inclusion $A \to X$ to itself), but it is certainly a homotopy equivalence of pairs. It follows that the composition

$$K^0(\operatorname{cyl}(X, A)) \to K^0(\operatorname{cyl}(X', A')) \to K^0(\operatorname{cyl}(X, A))$$

is the identity. In particular, $K^0(\operatorname{cyl}(X', A')) \to K^0(\operatorname{cyl}(X, A))$ is surjective, so by Lemma 6.10 the map $K^0(X', A') \to K^0(X, A)$ is surjective. Now the map $\operatorname{cyl}(X', A') \to \operatorname{cyl}(X, A)$ satisfies the hypotheses of the result we are proving, so $K^0(\operatorname{cyl}(X, A)) \to K^0(\operatorname{cyl}(X', A'))$ is surjective. We thus have two surjective maps whose composition is an isomorphism, so both are isomorphisms. Applying Lemma 6.10 again, we see that $K^0(X', A') \to K^0(X, A)$ is an isomorphism. □

Lemma 6.12. *Let X be a paracompact orbispace. Every vector bundle E over a closed substack $Y \subseteq X$ is the restriction of a vector bundle over some open substack $U \subseteq X$ containing Y.*

Proof. Fix a locally finite open cover $X = \bigcup_i Z_i/G_i$ such that the pullback of E to each $(Z_i)_Y := Z_i \times_X Y \subseteq Z_i$ is trivialized G_i-equivariantly (Lemma 6.7). Such trivializations evidently extend $E|_{Y\cap(Z_i/G_i)}$ to Z_i/G_i. We patch together these extensions E_i on Z_i/G_i as follows.

Choose closed substacks $K_i \subseteq X$ with $K_i \subseteq Z_i/G_i$ and whose interiors cover X. We thus have a collection of transition functions $\alpha_{ij} : K_i \cap K_j \cap Y \to \mathrm{Hom}(E_i, E_j)$ satisfying $\alpha_{ii} = \mathrm{id}$ and $\alpha_{ij}\alpha_{jk}\alpha_{ki} = \mathrm{id}$ over $K_i \cap K_j \cap K_k \cap Y$. We now execute the following operation for every pair (i, j) inductively according to an arbitrary well-ordering of such pairs. For $i = j$, do nothing. For $i \neq j$, choose an extension of α_{ij} from $K_i \cap K_j \cap Y$ to $K_i \cap K_j$ using Lemma 6.9. This extension remains an isomorphism in a neighborhood of $K_i \cap K_j \cap Y$. Since (the coarse space of) $K_i \cap K_j$ is paracompact Hausdorff, hence normal, we may choose a closed substack $A \subseteq K_i \cap K_j$ whose interior contains $K_i \cap K_j \cap Y$ and over which the extension of α_{ij} is an isomorphism. Now this new α_{ij} over A gives unique extensions of the remaining $\alpha_{k\ell}$ to $K_k \cap K_\ell \cap (Y \cup A)$ such that the cocycle condition is satisfied. We now replace Y with $Y \cup A$ and go on to the next pair of indices. Note that since our cover $X = \bigcup_i Z_i/G_i$ is locally finite, the set Y remains closed even after possibly infinitely many steps.

After processing every pair (i, j), our extended transition functions define a vector bundle on a closed substack $\bar{Y} \subseteq X$ whose restriction to $Y \subseteq \bar{Y}$ is E, and by construction Y is contained in the interior of $\bar{Y}$. □

We now come to the excision and exactness properties of K-theory, where we will finally make use of Corollary 1.5 (in its reformulation as Corollary 6.1).

We will make crucial use of the following direct limit description of relative K-theory. For any map of stackcs $f : Y \to X$ and any vector bundle F on Y, let $\mathrm{Vect}(X, F/Y)$ denote the set of isomorphism classes of pairs (E, i) consisting of a vector bundle E on X and an isomorphism $i : f^*E \xrightarrow{\sim} F$. Now $K^0(X, Y)$ may be expressed as the direct limit

$$K^0(X,Y) = \varinjlim_{E\in\widehat{\mathrm{Vect}}(X)} \mathrm{Vect}(X, f^*E/Y).$$

A triple (E_0, E_1, i) corresponds in this description to the element $(E_0, i) \in \mathrm{Vect}(X, f^*E_1/Y)$.

Proposition 6.13 (Excision)**.** *Let $X' \to X$ be a map of orbispaces satisfying the hypothesis of Theorem 1.1. Let $Y \to X$ be arbitrary, and let $Y' \to X'$ denote its pullback along $X' \to X$. If there is an open cover $X = U \cup V$ such that $Y \to X$ is an isomorphism over U and $X' \to X$ is an isomorphism over V, then the natural map $K^0(X, Y) \to K^0(X', Y')$ is an isomorphism.*

Proof. We may write the map in question in terms of direct limits as

$$\varinjlim_{E\in\widehat{\mathrm{Vect}}(X)} \mathrm{Vect}(X, f^*E/Y) \to \varinjlim_{E'\in\widehat{\mathrm{Vect}}(X')} \mathrm{Vect}(X', (f')^*E'/Y').$$

By Corollary 6.1, we may replace the second direct limit with the corresponding direct limit over $\widehat{\mathrm{Vect}}(X)$. It thus suffices to show that the pullback map $\mathrm{Vect}(X, f^*E/Y) \to$

$\mathrm{Vect}(X', (f')^*E'/Y')$ is an isomorphism for vector bundles E on X where $E' := E \times_X X'$. Since $Y \to X$ is an isomorphism over U and $X = U \cup V$, this coincides with the map $\mathrm{Vect}(V, f^*E|_V/Y \times_X V) \to \mathrm{Vect}(V \times_X X', (f')^*E|_V/Y' \times_X V)$, which is a bijection since $X' \to X$ is an isomorphism over V. $\square$

Proposition 6.14 (Exactness)**.** *Let X be an orbispace satisfying the hypothesis of Theorem 1.1, let $A \subseteq X \supseteq Y$ be closed substacks, let $B := A \cap Y$ be their intersection, and let $A \cup_B Y \subseteq X$ be their union. The following sequence is exact:*

$$K^0(X, A \cup_B Y) \to K^0(X, A) \to K^0(Y, B).$$

Proof. We may write the sequence in question in terms of direct limits as

$$\varinjlim_{E \in \widehat{\mathrm{Vect}}(X)} \mathrm{Vect}(X, E|_{A\cup_B Y}) \to \varinjlim_{E \in \widehat{\mathrm{Vect}}(X)} \mathrm{Vect}(X, E|_A) \to \varinjlim_{E \in \widehat{\mathrm{Vect}}(Y)} \mathrm{Vect}(Y, E|_B)$$

By Corollary 6.1, we may replace $\widehat{\mathrm{Vect}}(Y)$ with $\widehat{\mathrm{Vect}}(X)$ in the final direct limit. It thus suffices to show that for any vector bundle E over X, the fiber of the map $\mathrm{Vect}(X, E|_A) \to \mathrm{Vect}(Y, E|_B)$ over E coincides with the image of $\mathrm{Vect}(X, E|_{A\cup_B Y}) \to \mathrm{Vect}(X, E|_A)$. This is simply the fact that given a vector bundle F over X and isomorphisms $F|_A \xrightarrow{\sim} E|_A$ and $F|_Y \xrightarrow{\sim} E|_Y$ which agree over $B = A \cap Y$, they glue together into an isomorphism $F|_{A\cup_B Y} \xrightarrow{\sim} E|_{A\cup_B Y}$. $\square$

Given the significance of the hypothesis of Theorem 1.1, it is essential to show that this property is preserved under various natural operations.

Lemma 6.15. *For any topological stack X and any locally compact Hausdorff space R, the natural map $|X \times R| \to |X| \times R$ is an isomorphism.*

Proof. For any atlas $U \to X$, the induced map $U \to |X|$ is a topological quotient map, meaning it is surjective and a subset of the target is open iff its inverse image in the source is open. Topological quotient maps are preserved by taking product with a locally compact Hausdorff space [26, Lemma 4], so $U \times R \to |X| \times R$ is also a topological quotient map. Since $U \times R \to X \times R$ is also an atlas, the map $U \times R \to |X \times R|$ is also a topological quotient map. Now $|X \times R| \to |X| \times R$ is a bijection, so we are done. $\square$

Lemma 6.16. *If a topological space X is coarsely finite-dimensional, then so is $X \times R$ for any closed subset $R \subseteq \mathbb{R}^n$.*

Proof. We begin with the case $R = [0, 1]$. Let a cover $X \times [0, 1] = \bigcup_\alpha U_\alpha$ be given. For every $x \in X$ and $t \in [0, 1]$, there exists a pair of open neighborhoods $x \in V \subseteq X$ and $t \in W \subseteq [0, 1]$ such that $V \times W$ is contained in some U_α. Fixing $x \in X$ and using compactness of $[0, 1]$, we see that there are finitely many such $V_i \times W_i$ covering $\{x\} \times [0, 1]$. Hence there exists an open neighborhood $x \in V \subseteq X$ and an $\varepsilon > 0$ such that $V \times ((t - \varepsilon, t + \varepsilon) \cap [0, 1])$ is contained in some U_α for every $t \in [0, 1]$. We consider the collection of all such pairs (V, ε). Since X is coarsely finite-dimensional, there is a collection of such pairs $(V_\alpha, \varepsilon_\alpha)$ such that the nerve of the covering $X = \bigcup_\alpha V_\alpha$ is finite-dimensional. Now cover $X \times [0, 1]$ by $V_\alpha \times ([0, 1] \cap \varepsilon_\alpha \cdot (k, k + 2))$ for integers k.

Next, we consider the case $R = \mathbb{R}$. Let a cover of $X \times \mathbb{R}$ be given. The intersection of this cover with $X \times [n, n+2]$ has a refinement with finite-dimensional nerve by the case $R = [0, 1]$. Consider this refinement intersected with $X \times (n, n+2)$, and take union over all integers n to obtain a refinement of the original cover of $X \times \mathbb{R}$ with finite-dimensional nerve.

Finally, by induction we obtain the case $R = \mathbb{R}^n$, and the general case follows since coarse finite-dimensionality passes to closed subsets. $\square$

Let us now recall the Puppe sequence, which produces from Proposition 6.14 a long exact sequence. Let (X, A) be any pair of orbispaces satisfying the hypothesis of Theorem 1.1. Let $I := [0, 1]$. We now have maps of pairs

$$\begin{array}{l} (A, \varnothing) \to (X, \varnothing) \to (X, A) \\ (A \times I, Y \times \partial I) \to (X \times I, X \times \partial I) \to (X \times I, (X \times \partial I) \cup (A \times I)) \\ (A \times I^2, A \times \partial I^2) \to (X \times I^2, X \times \partial I^2) \to (X \times I^2, (X \times \partial I^2) \cup (A \times I^2)) \\ \quad \vdots \end{array}$$

We also have maps $(X \times I^k, (X \times \partial I^k) \cup (A \times I^k)) \to (A \times I^{k+1}, A \times \partial I^{k+1})$ up to inverting maps inducing isomorphisms on K^0. Namely, these 'connecting maps' are given by

$$\begin{array}{c} (X \times I^k \times I, (X \times \partial I^k \times I) \cup (A \times I^k \times \{1\})) \\ \downarrow \\ (X \times I^k \times I, (X \times \partial I^k \times I) \cup (A \times I^k \times \{1\}) \cup (X \times I^k \times \{0\})) \end{array}$$

where we note that the domain admits a natural map to $(X \times I^k, (X \times \partial I^k) \cup (A \times I^k))$ (which induces an isomorphism on K^0 by Proposition 6.11) and the target a natural map from $(A \times I^{k+1}, A \times \partial I^{k+1})$. To see that this second map also induces an isomorphism on K^0, factor it as

$$\begin{array}{c} (A \times I^k \times I, (A \times \partial I^k \times I) \cup (A \times I^k \times \{1\}) \cup (A \times I^k \times \{0\})) \\ \downarrow \\ ((X \times I^k \times \{0\}) \cup (A \times I^k \times I), (A \times \partial I^k \times I) \cup (A \times I^k \times \{1\}) \cup (X \times I^k \times \{0\})) \\ \downarrow \\ ((X \times I^k \times \{0\}), (X \times \partial I^k \times I) \cup (A \times I^k \times \{1\}) \cup (X \times I^k \times \{0\})) \end{array}$$

and observe that the first map is an isomorphism by excision Proposition 6.13 (and some manipulation involving adding $A \times I^k \times [0, \frac{1}{2}]$ to the second space of both pairs, which does not change their homotopy type but makes it possibly to apply Proposition 6.13) and the second map is an isomorphism by Proposition 6.11 (a homotopy inverse to $(X \times I^k \times \{0\}) \cup (A \times I^k \times I) \to X \times I^k \times I$ is given by projection to $X \times I^k \times \{0\}$, and the second terms are both homotopy equivalent to $(X \times D^k) \cup (A \times S^k)$).

We may now define[2]

$$K^{-n}(X) := K^0(X \times I^n, X \times \partial I^n),$$
$$K^{-n}(X, A) := K^0(X \times I^n, (X \times \partial I^n) \cup (A \times I^n)),$$

so the Puppe sequence above gives a sequence of maps

$$\cdots \to K^{-2}(A) \to K^{-1}(X, A) \to K^{-1}(X) \to K^{-1}(A) \to K^0(X, A) \to K^0(X) \to K^0(A).$$

functorial in the pair (X, A).

Proposition 6.17 (Long Exact Sequence)**.** *The sequence above is exact.*

Proof. Each of the following three triples satisfies the hypotheses of Proposition 6.14:

$$(A \times I^k, A \times \partial I^k) \to (X \times I^k, X \times \partial I^k) \to (X \times I^k, (X \times \partial I^k) \cup (A \times I^k))$$
$$(X \times I^k, X \times \partial I^k) \xrightarrow{\times\{0\}} (X \times I^k \times I, (X \times \partial I^k \times I) \cup (A \times I^k \times \{1\}))$$
$$\to (X \times I^k \times I, (X \times \partial I^k \times I) \cup (A \times I^k \times \{1\}) \cup (X \times I^k \times \{0\}))$$
$$(X \times I^k \times [\tfrac{1}{2}, 1], (X \times \partial I^k \times [\tfrac{1}{2}, 1]) \cup (A \times I^k \times \{1\}))$$
$$\to (X \times I^k \times I, (X \times \partial I^k \times I) \cup (X \times I^k \times \{0\}) \cup (A \times I^k \times \{1\}))$$
$$\to (X \times I^k \times I, (X \times \partial I^k \times I) \cup (X \times I^k \times \{0\}) \cup (X \times I^k \times [\tfrac{1}{2}, 1]))$$

and the last pair is homotopy equivalent to $(X \times I^k \times [0, \frac{1}{2}], (X \times \partial I^k \times [0, \frac{1}{2}]) \cup (X \times I^k \times \{0\}) \cup (X \times I^k \times \{\frac{1}{2}\}))$ ☐

We now define a multiplicative structure on K-theory. There is a natural map $K^0(X) \otimes K^0(Y) \to K^0(X \times Y)$ sending $[E] \otimes [F]$ to $[E \boxtimes F]$. To define the tensor product map on relative K^0, we must work a bit harder.

Let us consider the following alternative model of relative K-theory which we denote by $K^0_{\mathrm{kom}}(X, Y)$. For a map $f : Y \to X$, we consider the set of isomorphism classes of pairs of vector bundles E_0 and E_1 on X together with an acyclic differential $d : f^*E_0 \rightleftarrows f^*E_1$ (i.e. d consists of two maps, $d_0 : f^*E_0 \to f^*E_1$ and $d_1 : f^*E_1 \to f^*E_0$, such that $\operatorname{im} d_0 = \ker d_1$ and $\operatorname{im} d_1 = \ker d_0$). This set is an abelian monoid under direct sum, and we consider the submonoid consisting of triples of the form $(E, E, d_{F,G})$ where $f^*E = F \oplus G$ is a direct sum decomposition and $d_{F,G} := (d_0 = \mathrm{id}_F \oplus 0, d_1 = 0 \oplus \mathrm{id}_G)$. The quotient by this submonoid is by definition $K^0_{\mathrm{kom}}(X, Y)$. There is a natural map $K^0(X, Y) \to K^0_{\mathrm{kom}}(X, Y)$ which regards an isomorphism $i : f^*E_0 \to f^*E_1$ as an acyclic differential d_i by declaring the map in the other direction to be zero.

To study K^0_{kom}, let us first contemplate the meaning of the acyclic differential $d : f^*E_0 \rightleftarrows f^*E_1$. Fix a triple $(E_0, E_1, d : f^*E_0 \rightleftarrows f^*E_1)$. The function $\dim \ker d_0 : Y \to \mathbb{Z}_{\geq 0}$ is upper semicontinuous, as is $\dim \ker d_1$. Since d is acyclic, we have $\dim \ker d_0 = \dim \operatorname{im} d_1 = \dim E_1 - \dim \ker d_1$, so both sides are continuous. Thus $\dim \ker d_0$ and $\dim \ker d_1$ are both locally constant on Y, which implies that $\ker d_0 \subseteq f^*E_0$ and $\ker d_1 \subseteq f^*E_1$ are both vector

[2] To keep track of signs, one should really write this as $K^{-n}(X) \otimes H^n(I^n, \partial I^n) = K^0(X \times I^n, X \times \partial I^n)$ etc., however for the purposes of our presentation here, we will not be so precise.

bundles. Thus the data of an acyclic differential $d : f^*E_0 \rightleftarrows f^*E_1$ is equivalent to the data of two vector bundles F and G on Y and two short exact sequences

$$0 \to G \to f^*E_0 \to F \to 0,$$
$$0 \to F \to f^*E_1 \to G \to 0.$$

A choice of splitting of both sequences (which may or may not exist) is equivalent to a choice of isomorphisms $f^*E_0 = F \oplus G = f^*E_1$ with respect to which $d = d_{F,G} = (\mathrm{id}_F \oplus 0, 0 \oplus \mathrm{id}_G)$. A choice of null homotopy $h : f^*E_0 \rightleftarrows f^*E_1$ (meaning $dh+hd = \mathrm{id}$) determines a splitting of the sequences above (dh and hd are orthogonal idempotents which induce the relevant direct sum decompositions of f^*E_0 and f^*E_1). A choice of splitting comes from a canonical (but not unique!) null homotopy $h = (0 \oplus \mathrm{id}_G, \mathrm{id}_F \oplus 0)$. A null homotopy h is the canonical one associated to a splitting iff $h^2 = 0$, and in this case the associated isomorphism $f^*E_0 = f^*E_1$ is given by $d + h$.

Lemma 6.18. *For a paracompact orbispace pair (X, A), the map $K^0(X, A) \to K^0_{\mathrm{kom}}(X, A)$ is an isomorphism.*

Proof. We define a map $K^0_{\mathrm{kom}}(X, A) \to K^0(X, A)$ as follows. Fix a triple (E_0, E_1, d) representing an element of $K^0_{\mathrm{kom}}(X, A)$. Since A is a paracompact orbispace, the short exact sequences above are split by Corollary 5.2. A choice of splitting determines, in particular, an isomorphism $E_0|_A = E_1|_A$, and thus an element of $K^0(X, A)$. To see that this element of $K^0(X, A)$ is independent of the choice of splitting, suppose given two choices of splittings, consider their associated null homotopies, use the fact that the space null homotopies is convex to interpolate between these null homotopies on $(X \times [0, 1], A \times [0, 1])$, pass to the associated splitting of the pullback of our triple (E_0, E_1, d), and conclude since $K^0(X, A) \to K^0(X \times [0, 1], A \times [0, 1])$ is an isomorphism by Proposition 6.11. We thus have associated to any triple (E_0, E_1, d) a well defined element of $K^0(X, A)$. Triples of the form $(E, E, d_{F,G})$ are mapped to zero by definition, so this map defined on triples descends to $K^0_{\mathrm{kom}}(X, A)$.

The composition $K^0(X, A) \to K^0_{\mathrm{kom}}(X, A) \to K^0(X, A)$ is evidently the identity. It thus suffices to show that $K^0(X, A) \to K^0_{\mathrm{kom}}(X, A)$ is surjective.

Let (E_0, E_1, d) be a triple representing an arbitrary element of $K^0_{\mathrm{kom}}(X, A)$. Choose arbitrarily a splitting of d as above, which determines isomorphisms $E_0|_A = E_1|_A = F \oplus G$ with respect to which $d = d_{F,G}$. By Lemma 6.9, the isomorphism $E_0|_A = E_1|_A$ extends to an isomorphism i defined over an open neighborhood U of A. Let $\varphi : X \to \mathbb{R}_{\geq 0}$ be a cutoff function supported inside U whose restriction to A is identically 1. We now observe that there is an isomorphism of triples

$$(E_0, E_0, d_{F,G}) \oplus (E_0, E_1, d_i) \to (E_0, E_1, d_{F,G}) \oplus (E_0, E_0, d_{E_0,0})$$

given by

$$\left(\begin{pmatrix} \mathrm{id}_{E_0} & 0 \\ 0 & \mathrm{id}_{E_0} \end{pmatrix}, \begin{pmatrix} \varphi\, i & -(1-\varphi)\,\mathrm{id}_{E_0} \\ (1-\varphi)\,\mathrm{id}_{E_0} & \varphi\, i \end{pmatrix}\right).$$

This shows that the element of $K^0_{\mathrm{kom}}(X, A)$ represented by our original arbitrary triple $(E_0, E_1, d_{F,G})$ is in fact also represented by the triple (E_0, E_1, d_i), which is by definition in the image of $K^0(X, A)$. □

We may now define an associative tensor product map

$$K^0(X,A)\otimes K^0(Y,B)\to K^0(X\times Y,(A\times Y)\cup(X\times B))$$

for paracompact orbispace pairs (X,A) and (Y,B) for which $X\times Y$ is paracompact. Note that for any triple (E_0,E_1,d) representing an element of $K^0_{\mathrm{kom}}(X,A)$, the differential d always extends as a differential to all of X (indeed, choose a splitting of d on A, extend the resulting pair of vector bundles to a neighborhood of A using Lemma 6.12, extend their inclusions into E_0 and E_1 using Lemma 6.9, and finally multiply the resulting extension of d to an open neighborhood of A by a cutoff function). Now given triples (E_0,E_1,d) and (E_0',E_1',d') representing elements of $K^0_{\mathrm{kom}}(X,A)$ and $K^0_{\mathrm{kom}}(Y,B)$, respectively, we extend the differentials d and d' arbitrarily to X and Y, so the tensor product of these $\mathbb{Z}/2$-graded complexes (which will be acyclic over $(A\times Y)\cup(X\times B)$) now defines an element of $K^0_{\mathrm{kom}}(X\times Y,(A\times Y)\cup(X\times B))$. This map is associative by definition, provided we show that it is well defined. First, let us show that the result in K^0_{kom} is independent of how we extend d and d'. Let (E_0,E_1,d) be a triple representing an element of $K^0_{\mathrm{kom}}(X,A)$, and fix an extension of d. Since $\operatorname{im} d\subseteq\ker d$ with equality on A and their dimensions are lower and upper semicontinuous, respectively, it follows that $\operatorname{im} d=\ker d$ over an open neighborhood of A. A choice of splitting of d over this neighborhood expresses $d=d_{F,G}$ for isomorphisms $E_0=E_1=F\oplus G$. Now suppose we have two different extensions of d to all of X, and we choose splittings for both. The vector bundles F and G are identified over A, hence over a neighborhood of A by Lemma 6.9, from which we may argue that the two extensions are homotopic (through differentials) on a neighborhood of A. Multiplying by a cutoff function, we see that the two extensions of d are homotopic on all of X. It thus follows from Proposition 6.11 that the tensor product in the target K^0_{kom} is independent of the choice of extension of d. Finally, we should show that if one of the tensor factors is of the form $(E,E,d_{F,G})$, then the tensor product vanishes in the target K^0_{kom}. Note that to do this, we are free to choose nice extensions of $d_{F,G}$ and of the differential of the second factor, since we have already shown independence of the choice of extension. Thus let us consider the tensor product $(E,E,\varphi\, d_{F,G})\otimes(J,K,d':J\to K,d'':K\to J)$. The first factor has a null homotopy in a neighborhood of A squaring to zero given by $(0\oplus\varphi^{-1}\mathrm{id}_G,\varphi^{-1}\mathrm{id}_F\oplus 0)$, and we may assume that the second factor also has a square zero null homotopy $h':J\to K$ and $h'':K\to J$ in a neighborhood of B. The tensor product is given by

$$(E\otimes J)\oplus(E\otimes K)\;\underset{\begin{pmatrix}\varphi\,\mathrm{id}_G & d''\\ -d' & \varphi\,\mathrm{id}_F\end{pmatrix}}{\overset{\begin{pmatrix}\varphi\,\mathrm{id}_F & -d''\\ d' & \varphi\,\mathrm{id}_G\end{pmatrix}}{\rightleftarrows}}\;(E\otimes J)\oplus(E\otimes K).$$

Over the locus where $\varphi>0$, there is a square zero null homotopy

$$(E\otimes J)\oplus(E\otimes K)\;\underset{\begin{pmatrix}\varphi^{-1}\,\mathrm{id}_F & 0\\ 0 & \varphi^{-1}\,\mathrm{id}_G\end{pmatrix}}{\overset{\begin{pmatrix}\varphi^{-1}\,\mathrm{id}_G & 0\\ 0 & \varphi^{-1}\,\mathrm{id}_F\end{pmatrix}}{\rightleftarrows}}\;(E\otimes J)\oplus(E\otimes K),$$

and over X times a neighborhood of B, there is a square zero null homotopy

$$(E\otimes J)\oplus(E\otimes K)\underset{\begin{pmatrix}0 & h'' \\ -h' & 0\end{pmatrix}}{\overset{\begin{pmatrix}0 & -h'' \\ h' & 0\end{pmatrix}}{\rightleftarrows}}(E\otimes J)\oplus(E\otimes K).$$

Convex combinations of these null homotopies also square to zero by inspection, so we may define a square zero null homotopy in a neighborhood of $(A\times Y)\cup(X\times B)$ by patching together the two null homotopies above using a partition of unity. This patched square zero null homotopy determines an isomorphism between $(E\otimes J)\oplus(E\otimes K)$ and itself over $(A\times Y)\cup(X\times B)$ (given by the sum of the differential and the null homotopy), and it suffices (in view of Proposition 6.11) to show that this isomorphism is homotopic (through isomorphisms) to the identity map. It thus suffices to show that convex combinations of the identity map and the isomorphisms

$$\begin{pmatrix}\varphi\operatorname{id}_F+\varphi^{-1}\operatorname{id}_G & -d'' \\ d' & \varphi^{-1}\operatorname{id}_F+\varphi\operatorname{id}_G\end{pmatrix}\qquad\begin{pmatrix}\varphi\operatorname{id}_F & -d''-h'' \\ d'+h' & \varphi\operatorname{id}_G\end{pmatrix}$$

are isomorphisms, which holds by the calculation

$$\begin{aligned}&\begin{pmatrix}(a+(b+c)\varphi)\operatorname{id}_F+(a+b\varphi^{-1})\operatorname{id}_G & -(b+c)d''-ch'' \\ (b+c)d'+ch' & (a+b\varphi^{-1})\operatorname{id}_F+(a+(b+c)\varphi)\operatorname{id}_G\end{pmatrix}\\ &\quad\times\begin{pmatrix}(a+b\varphi^{-1})\operatorname{id}_F+(a+(b+c)\varphi)\operatorname{id}_G & (b+c)d''+ch'' \\ -(b+c)d'-ch' & (a+(b+c)\varphi)\operatorname{id}_F+(a+b\varphi^{-1})\operatorname{id}_G\end{pmatrix}\\ &\qquad=\big((a+(b+c)\varphi)(a+b\varphi^{-1})+(b+c)c\big)\begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix}\end{aligned}$$

(the first factor on the left side is the relevant convex combination).

The tensor product on relative K^0 now gives associate product maps

$$K^{-n}(X,A)\otimes K^{-m}(Y,B)\to K^{-n-m}(X\times Y,(A\times Y)\cup(X\times B))$$

by inspection.

We now come to Bott periodicity. So far, our discussion has applied equally to complex vector bundles as to real vector bundles, however we now restrict to complex vector bundles. The *Bott element* $\beta\in K^{-2}(*)=K^0(D^2,\partial D^2)$ is represented by the complex of vector bundles $\underline{\mathbb{C}}\xrightarrow{\cdot z}\underline{\mathbb{C}}$ on $\mathbb{C}$ (which contains D^2 as the unit disk). It is well known that multiplication by β induces an isomorphism $K^{-i}(*)\to K^{-i-2}(*)$ for all $i\geq 0$ [2, 5]. It follows from the long exact sequence, excision, and the five lemma that multiplication by β induces an isomorphism $K^{-i}(X,A)\to K^{-i-2}(X,A)$ for any finite simplicial complex pair (X,A) (X a finite simplicial complex, A a subcomplex). Now $K^{-i}(\mathbb{B}G)=\bigoplus_{\rho\in\hat{G}}K^{-i}(*)$, since a vector bundle over $Y\times\mathbb{B}G$ (Y a topological space) is the same thing as a bundle of G representations over Y. It then follows by the same argument that multiplication by β induces an isomorphism $K^{-i}(X,A)\to K^{-i-2}(X,A)$ for any finite simplicial complex of groups X and subcomplex

A. One can show similarly that for any complex line bundle L over such X, pullback and pairing with the relative Bott class in $K^0(L, \infty)$ (represented by the complex $\underline{\mathbb{C}} \xrightarrow{\cdot p} L$ on L) defines an isomorphism $K^{-i}(X) \to K^{-i}(L, \infty)$.

We may define periodic K-theory $K^i_{\text{per}}(X) := \varinjlim_k K^{i-2k}(X)$ for all integers i, where the maps are by multiplication by β. Thus for finite simplicial complexes of groups, K^*_{per} extends K^* to all integer gradings. Since direct limits are exact, K^*_{per} satisfies exactness for all orbispace pairs (X, A) satisfying the hypothesis of Theorem 1.1.

References

[1] Alejandro Adem and Yongbin Ruan, *Twisted orbifold K-theory*, Comm. Math. Phys. **237** (2003), no. 3, 533–556. MR 1993337

[2] M. F. Atiyah, *Bott periodicity and the index of elliptic operators*, Quart. J. Math. Oxford Ser. (2) **19** (1968), 113–140. MR 228000

[3] ______, *Algebraic topology and operators in Hilbert space*, Lectures in Modern Analysis and Applications. I, Springer, Berlin, 1969, pp. 101–121. MR 0248803

[4] M. F. Atiyah and I. M. Singer, *Index theory for skew-adjoint Fredholm operators*, Inst. Hautes Études Sci. Publ. Math. (1969), no. 37, 5–26. MR 285033

[5] Michael Atiyah and Raoul Bott, *On the periodicity theorem for complex vector bundles*, Acta Math. **112** (1964), 229–247. MR 178470

[6] Kai Behrend and Behrang Noohi, *Uniformization of Deligne-Mumford curves*, J. Reine Angew. Math. **599** (2006), 111–153. MR 2279100

[7] Martin R. Bridson and André Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 319, Springer-Verlag, Berlin, 1999. MR 1744486

[8] Jon Michael Corson, *Complexes of groups*, Proc. London Math. Soc. (3) **65** (1992), no. 1, 199–224. MR 1162493

[9] Jean Dieudonné, *Une généralisation des espaces compacts*, J. Math. Pures Appl. (9) **23** (1944), 65–76. MR 0013297

[10] Dan Edidin, Brendan Hassett, Andrew Kresch, and Angelo Vistoli, *Brauer groups and quotient stacks*, Amer. J. Math. **123** (2001), no. 4, 761–777. MR 1844577

[11] André Haefliger, *Complexes of groups and orbihedra*, Group theory from a geometrical viewpoint (Trieste, 1990), World Sci. Publ., River Edge, NJ, 1991, pp. 504–540. MR 1170375

[12] Andre Henriques and David Gepner, *Homotopy Theory of Orbispaces*, Preprint **arxiv:math/0701916** (2007), 1–52.

[13] Andre Henriques and David S. Metzler, *Presentations of noneffective orbifolds*, Trans. Amer. Math. Soc. **356** (2004), no. 6, 2481–2499. MR 2048526

[14] Andre Gil Henriques, *Orbispaces*, ProQuest LLC, Ann Arbor, MI, 2005, Thesis (Ph.D.)–Massachusetts Institute of Technology. MR 2717316

[15] Andrew Kresch, *On the geometry of Deligne-Mumford stacks*, Algebraic geometry—Seattle 2005. Part 1, Proc. Sympos. Pure Math., vol. 80, Amer. Math. Soc., Providence, RI, 2009, pp. 259–271. MR 2483938

[16] Andrew Kresch and Angelo Vistoli, *On coverings of Deligne-Mumford stacks and surjectivity of the Brauer map*, Bull. London Math. Soc. **36** (2004), no. 2, 188–192. MR 2026412

[17] Wolfgang Lück and Bob Oliver, *The completion theorem in K-theory for proper actions of a discrete group*, Topology **40** (2001), no. 3, 585–616. MR 1838997

[18] Takao Matumoto, *Equivariant K-theory and Fredholm operators*, J. Fac. Sci. Univ. Tokyo Sect. I A Math. **18** (1971), 109–125. MR 0290354

[19] David Metzler, *Topological and Smooth Stacks*, Preprint **arxiv:math/0306176** (2003), 1–47.

[20] Behrang Noohi, *Foundations of Topological Stacks I*, Preprint **arxiv:math/0503247** (2005), 1–81.

[21] Stefan Schwede, *Global homotopy theory*, New Mathematical Monographs, vol. 34, Cambridge University Press, Cambridge, 2018. MR 3838307

[22] ______, *Orbispaces, orthogonal spaces, and the universal compact lie group*, Mathematische Zeitschrift (to appear) (accepted 2019), 1–37.

[23] Graeme Segal, *Fredholm complexes*, Quart. J. Math. Oxford Ser. (2) **21** (1970), 385–402. MR 271930

[24] Bernd Siebert, *Gromov–Witten invariants for general symplectic manifolds*, Preprint **arxiv:dg-ga/9608005v2** (1998), 1–74.

[25] Burt Totaro, *The resolution property for schemes and stacks*, J. Reine Angew. Math. **577** (2004), 1–22. MR 2108211

[26] J. H. C. Whitehead, *Note on a theorem due to Borsuk*, Bull. Amer. Math. Soc. **54** (1948), 1125–1132. MR 29503