The Hilbert scheme of points in the plane and triply-graded homology

Joshua Wang

June 2026

Outline

Last time, we saw that triply-graded homology is highly structured. Today: a model for $\overline{\mathrm{H}}(T_{n,n+1})$ coming from the geometry of the Hilbert scheme of n points in $\mathbf{C}^2$.

Outline

Last time, we saw that triply-graded homology is highly structured. Today: a model for $\overline{\mathrm{H}}(T_{n,n+1})$ coming from the geometry of the Hilbert scheme of n points in $\mathbf{C}^2$.

- $\mathrm{Hilb}^n(\mathbf{C}^2)$: the Hilbert scheme of n points in $\mathbf{C}^2$

Outline

Last time, we saw that triply-graded homology is highly structured. Today: a model for $\overline{\mathrm{H}}(T_{n,n+1})$ coming from the geometry of the Hilbert scheme of n points in $\mathbf{C}^2$.

- $\mathrm{Hilb}^n(\mathbf{C}^2)$: the Hilbert scheme of n points in $\mathbf{C}^2$
- $Z_n \subseteq \mathrm{Hilb}^n(\mathbf{C}^2)$: the zero fiber

Outline

Last time, we saw that triply-graded homology is highly structured. Today: a model for $\overline{\mathrm{H}}(T_{n,n+1})$ coming from the geometry of the Hilbert scheme of n points in $\mathbf{C}^2$.

- $\mathrm{Hilb}^n(\mathbf{C}^2)$: the Hilbert scheme of n points in $\mathbf{C}^2$
- $Z_n \subseteq \mathrm{Hilb}^n(\mathbf{C}^2)$: the zero fiber
- The (reduced) tautological vector bundle $T_0 \to Z_n$ of rank $n-1$.

Outline

Last time, we saw that triply-graded homology is highly structured. Today: a model for $\overline{\mathrm{H}}(T_{n,n+1})$ coming from the geometry of the Hilbert scheme of n points in $\mathbf{C}^2$.

- $\mathrm{Hilb}^n(\mathbf{C}^2)$: the Hilbert scheme of n points in $\mathbf{C}^2$
- $Z_n \subseteq \mathrm{Hilb}^n(\mathbf{C}^2)$: the zero fiber
- The (reduced) tautological vector bundle $T_0 \to Z_n$ of rank $n-1$.
- The bottom a-layer of $\overline{\mathrm{H}}(T_{n,n+1})$ is modeled by the space of global sections of the determinant line bundle $\det T_0$ over Z_n.

Outline

Last time, we saw that triply-graded homology is highly structured. Today: a model for $\overline{\mathrm{H}}(T_{n,n+1})$ coming from the geometry of the Hilbert scheme of n points in $\mathbf{C}^2$.

- $\mathrm{Hilb}^n(\mathbf{C}^2)$: the Hilbert scheme of n points in $\mathbf{C}^2$
- $Z_n \subseteq \mathrm{Hilb}^n(\mathbf{C}^2)$: the zero fiber
- The (reduced) tautological vector bundle $T_0 \to Z_n$ of rank $n-1$.
- The bottom a-layer of $\overline{\mathrm{H}}(T_{n,n+1})$ is modeled by the space of global sections of the determinant line bundle $\det T_0$ over Z_n.
- The a^d-layer of $\overline{\mathrm{H}}(T_{n,n+1})$ is modeled by the space of global sections of the exterior power bundle $\Lambda^{n-1-d} T_0$ over Z_n.

Outline

Last time, we saw that triply-graded homology is highly structured. Today: a model for $\overline{\mathrm{H}}(T_{n,n+1})$ coming from the geometry of the Hilbert scheme of n points in $\mathbf{C}^2$.

- $\mathrm{Hilb}^n(\mathbf{C}^2)$: the Hilbert scheme of n points in $\mathbf{C}^2$
- $Z_n \subseteq \mathrm{Hilb}^n(\mathbf{C}^2)$: the zero fiber
- The (reduced) tautological vector bundle $T_0 \to Z_n$ of rank $n-1$.
- The bottom a-layer of $\overline{\mathrm{H}}(T_{n,n+1})$ is modeled by the space of global sections of the determinant line bundle $\det T_0$ over Z_n.
- The a^d-layer of $\overline{\mathrm{H}}(T_{n,n+1})$ is modeled by the space of global sections of the exterior power bundle $\Lambda^{n-1-d} T_0$ over Z_n.
- We see the trigrading, the $d_{N|M}$ differentials, and the $\mathfrak{sl}_2$ action in this model.

Outline

Last time, we saw that triply-graded homology is highly structured. Today: a model for $\overline{\mathrm{H}}(T_{n,n+1})$ coming from the geometry of the Hilbert scheme of n points in $\mathbf{C}^2$.

- $\mathrm{Hilb}^n(\mathbf{C}^2)$: the Hilbert scheme of n points in $\mathbf{C}^2$
- $Z_n \subseteq \mathrm{Hilb}^n(\mathbf{C}^2)$: the zero fiber
- The (reduced) tautological vector bundle $T_0 \to Z_n$ of rank $n-1$.
- The bottom a-layer of $\overline{\mathrm{H}}(T_{n,n+1})$ is modeled by the space of global sections of the determinant line bundle $\det T_0$ over Z_n.
- The a^d-layer of $\overline{\mathrm{H}}(T_{n,n+1})$ is modeled by the space of global sections of the exterior power bundle $\Lambda^{n-1-d} T_0$ over Z_n.
- We see the trigrading, the $d_{N|M}$ differentials, and the $\mathfrak{sl}_2$ action in this model.

This story was conjectured by Gorsky–Neguţ–Rasmussen '16 and extended/refined/mostly proven by Oblomkov–Rozansky in a series of papers.

Hilb$^n(\mathbf{C}^2)$

Two ways of describing Hilb$^n(\mathbf{C}^2)$:

$\mathsf{Hilb}^n(\mathbf{C}^2)$

Two ways of describing $\mathsf{Hilb}^n(\mathbf{C}^2)$:

- $\mathsf{Hilb}^n(\mathbf{C}^2)$ is the set of ideals $I \subseteq \mathbf{C}[x, y]$ for which $\dim_{\mathbf{C}} \mathbf{C}[x, y]/I = n$.

$\mathrm{Hilb}^n(\mathbf{C}^2)$

Two ways of describing $\mathrm{Hilb}^n(\mathbf{C}^2)$:

- $\mathrm{Hilb}^n(\mathbf{C}^2)$ is the set of ideals $I \subseteq \mathbf{C}[x, y]$ for which $\dim_{\mathbf{C}} \mathbf{C}[x, y]/I = n$.
- Let $\widetilde{\mathrm{Hilb}}^n(\mathbf{C}^2)$ be the set of triples (X, Y, v) where X, Y are $n \times n$ complex matrices and $v \in \mathbf{C}^n$ such that

$\mathrm{Hilb}^n(\mathbf{C}^2)$

Two ways of describing $\mathrm{Hilb}^n(\mathbf{C}^2)$:

- $\mathrm{Hilb}^n(\mathbf{C}^2)$ is the set of ideals $I \subseteq \mathbf{C}[x, y]$ for which $\dim_{\mathbf{C}} \mathbf{C}[x, y]/I = n$.
- Let $\widetilde{\mathrm{Hilb}}^n(\mathbf{C}^2)$ be the set of triples (X, Y, v) where X, Y are $n \times n$ complex matrices and $v \in \mathbf{C}^n$ such that
  - X and Y commute

$\mathrm{Hilb}^n(\mathbf{C}^2)$

Two ways of describing $\mathrm{Hilb}^n(\mathbf{C}^2)$:

- $\mathrm{Hilb}^n(\mathbf{C}^2)$ is the set of ideals $I \subseteq \mathbf{C}[x, y]$ for which $\dim_{\mathbf{C}} \mathbf{C}[x, y]/I = n$.
- Let $\widetilde{\mathrm{Hilb}}^n(\mathbf{C}^2)$ be the set of triples (X, Y, v) where X, Y are $n \times n$ complex matrices and $v \in \mathbf{C}^n$ such that
  - X and Y commute
  - the vectors $v, Xv, Yv, X^2v, XYv, Y^2v, X^3v, \ldots$ span $\mathbf{C}^n$.

$\mathrm{Hilb}^n(\mathbf{C}^2)$

Two ways of describing $\mathrm{Hilb}^n(\mathbf{C}^2)$:

- $\mathrm{Hilb}^n(\mathbf{C}^2)$ is the set of ideals $I \subseteq \mathbf{C}[x, y]$ for which $\dim_{\mathbf{C}} \mathbf{C}[x, y]/I = n$.
- Let $\widetilde{\mathrm{Hilb}}^n(\mathbf{C}^2)$ be the set of triples (X, Y, v) where X, Y are $n \times n$ complex matrices and $v \in \mathbf{C}^n$ such that
  - X and Y commute
  - the vectors $v, Xv, Yv, X^2v, XYv, Y^2v, X^3v, \ldots$ span $\mathbf{C}^n$.

  Let $g \in \mathrm{GL}_n(\mathbf{C})$ act by $g(X, Y, v) = (gXg^{-1}, gYg^{-1}, gv)$.

$\mathrm{Hilb}^n(\mathbf{C}^2)$

Two ways of describing $\mathrm{Hilb}^n(\mathbf{C}^2)$:

- $\mathrm{Hilb}^n(\mathbf{C}^2)$ is the set of ideals $I \subseteq \mathbf{C}[x, y]$ for which $\dim_{\mathbf{C}} \mathbf{C}[x, y]/I = n$.
- Let $\widetilde{\mathrm{Hilb}}^n(\mathbf{C}^2)$ be the set of triples (X, Y, v) where X, Y are $n \times n$ complex matrices and $v \in \mathbf{C}^n$ such that
  - X and Y commute
  - the vectors $v, Xv, Yv, X^2v, XYv, Y^2v, X^3v, \ldots$ span $\mathbf{C}^n$.

  Let $g \in \mathrm{GL}_n(\mathbf{C})$ act by $g(X, Y, v) = (gXg^{-1}, gYg^{-1}, gv)$. $\mathrm{Hilb}^n(\mathbf{C}^2)$ is the quotient of $\widetilde{\mathrm{Hilb}}^n(\mathbf{C}^2)$ by $\mathrm{GL}_n(\mathbf{C})$. Let $[X, Y, v]$ be the equivalence class of (X, Y, v).

$\mathsf{Hilb}^n(\mathbf{C}^2)$

Two ways of describing $\mathsf{Hilb}^n(\mathbf{C}^2)$:

- $\mathsf{Hilb}^n(\mathbf{C}^2)$ is the set of ideals $I \subseteq \mathbf{C}[x, y]$ for which $\dim_{\mathbf{C}} \mathbf{C}[x, y]/I = n$.
- Let $\widetilde{\mathsf{Hilb}}^n(\mathbf{C}^2)$ be the set of triples (X, Y, v) where X, Y are $n \times n$ complex matrices and $v \in \mathbf{C}^n$ such that
  - X and Y commute
  - the vectors $v, Xv, Yv, X^2v, XYv, Y^2v, X^3v, \ldots$ span $\mathbf{C}^n$.

  Let $g \in \mathsf{GL}_n(\mathbf{C})$ act by $g(X, Y, v) = (gXg^{-1}, gYg^{-1}, gv)$. $\mathsf{Hilb}^n(\mathbf{C}^2)$ is the quotient of $\widetilde{\mathsf{Hilb}}^n(\mathbf{C}^2)$ by $\mathsf{GL}_n(\mathbf{C})$. Let $[X, Y, v]$ be the equivalence class of (X, Y, v).

Bijection: assign to (X, Y, v) the ideal $I = \{p(x, y) \in \mathbf{C}[x, y] \mid p(X, Y)v = 0\}$.

$\mathrm{Hilb}^n(\mathbf{C}^2)$

Two ways of describing $\mathrm{Hilb}^n(\mathbf{C}^2)$:

- $\mathrm{Hilb}^n(\mathbf{C}^2)$ is the set of ideals $I \subseteq \mathbf{C}[x, y]$ for which $\dim_{\mathbf{C}} \mathbf{C}[x, y]/I = n$.
- Let $\widetilde{\mathrm{Hilb}}^n(\mathbf{C}^2)$ be the set of triples (X, Y, v) where X, Y are $n \times n$ complex matrices and $v \in \mathbf{C}^n$ such that
  - X and Y commute
  - the vectors $v, Xv, Yv, X^2v, XYv, Y^2v, X^3v, \ldots$ span $\mathbf{C}^n$.

  Let $g \in \mathrm{GL}_n(\mathbf{C})$ act by $g(X, Y, v) = (gXg^{-1}, gYg^{-1}, gv)$. $\mathrm{Hilb}^n(\mathbf{C}^2)$ is the quotient of $\widetilde{\mathrm{Hilb}}^n(\mathbf{C}^2)$ by $\mathrm{GL}_n(\mathbf{C})$. Let $[X, Y, v]$ be the equivalence class of (X, Y, v).

Bijection: assign to (X, Y, v) the ideal $I = \{p(x, y) \in \mathbf{C}[x, y] \mid p(X, Y)v = 0\}$.

Theorem (Fogarty '68): $\mathrm{Hilb}^n(\mathbf{C}^2)$ is a $2n$-dim smooth irreducible variety.

$\mathrm{Hilb}^n(\mathbf{C}^2)$

How is $\mathrm{Hilb}^n(\mathbf{C}^2)$ related to "n points in $\mathbf{C}^2$"? Fix $[X, Y, v] \in \mathrm{Hilb}^n(\mathbf{C}^2)$.

$\mathrm{Hilb}^n(\mathbf{C}^2)$

How is $\mathrm{Hilb}^n(\mathbf{C}^2)$ related to "n points in $\mathbf{C}^2$"? Fix $[X, Y, v] \in \mathrm{Hilb}^n(\mathbf{C}^2)$.

- Commuting complex matrices may be simultaneously triangularized, so we may assume X and Y are upper triangular.

$$X = \begin{pmatrix} x_1 & * & * \\ & x_2 & * \\ & & x_3 \end{pmatrix} \quad Y = \begin{pmatrix} y_1 & * & * \\ & y_2 & * \\ & & y_3 \end{pmatrix}$$

The unordered collection of n points $(x_1, y_1), \ldots, (x_n, y_n)$ in $\mathbf{C}^2$ is uniquely determined by the class $[X, Y, v]$.

$\mathrm{Hilb}^n(\mathbf{C}^2)$

How is $\mathrm{Hilb}^n(\mathbf{C}^2)$ related to "n points in $\mathbf{C}^2$"? Fix $[X, Y, v] \in \mathrm{Hilb}^n(\mathbf{C}^2)$.

- Commuting complex matrices may be simultaneously triangularized, so we may assume X and Y are upper triangular.

$$X = \begin{pmatrix} x_1 & * & * \\ & x_2 & * \\ & & x_3 \end{pmatrix} \quad Y = \begin{pmatrix} y_1 & * & * \\ & y_2 & * \\ & & y_3 \end{pmatrix}$$

  The unordered collection of n points $(x_1, y_1), \ldots, (x_n, y_n)$ in $\mathbf{C}^2$ is uniquely determined by the class $[X, Y, v]$.

- Alternatively, X and Y preserve each other's generalized eigenspaces, so they have a joint spectrum $\{(x_1, y_1), \ldots, (x_n, y_n)\}$.

$\mathrm{Hilb}^n(\mathbf{C}^2)$

How is $\mathrm{Hilb}^n(\mathbf{C}^2)$ related to "n points in $\mathbf{C}^2$"? Fix $[X, Y, v] \in \mathrm{Hilb}^n(\mathbf{C}^2)$.

- Commuting complex matrices may be simultaneously triangularized, so we may assume X and Y are upper triangular.

$$X = \begin{pmatrix} x_1 & * & * \\ & x_2 & * \\ & & x_3 \end{pmatrix} \quad Y = \begin{pmatrix} y_1 & * & * \\ & y_2 & * \\ & & y_3 \end{pmatrix}$$

  The unordered collection of n points $(x_1, y_1), \ldots, (x_n, y_n)$ in $\mathbf{C}^2$ is uniquely determined by the class $[X, Y, v]$.

- Alternatively, X and Y preserve each other's generalized eigenspaces, so they have a joint spectrum $\{(x_1, y_1), \ldots, (x_n, y_n)\}$.

$\mathrm{Hilb}^n(\mathbf{C}^2) \to (\mathbf{C}^2)^n/\mathfrak{S}_n$ is a resolution of singularities. It is bijective on the set where the n points are distinct.

The zero fiber Z_n

Special subvariety $Z_n \subseteq \mathrm{Hilb}^n(\mathbf{C}^2)$ called the *zero fiber* or the *punctual* Hilbert scheme of n points in $\mathbf{C}^2$: the preimage of $n \cdot 0 \in \mathrm{Sym}^n(\mathbf{C}^2)$.

The zero fiber Z_n

Special subvariety $Z_n \subseteq \mathrm{Hilb}^n(\mathbf{C}^2)$ called the *zero fiber* or the *punctual* Hilbert scheme of n points in $\mathbf{C}^2$: the preimage of $n \cdot 0 \in \mathrm{Sym}^n(\mathbf{C}^2)$. It is the $\mathrm{GL}_n(\mathbf{C})$-quotient of

$$\begin{aligned} \widetilde{Z}_n &= \left\{ (X, Y, v) \in \widetilde{Hilb}^n(\mathbf{C}^2) \,\middle|\, X, Y \text{ are nilpotent} \right\} \\ &= \left\{ (X, Y, v) \in \mathbf{C}^{n\times n} \times \mathbf{C}^{n\times n} \times \mathbf{C}^n \,\middle|\, \begin{matrix} [X, Y] = 0, \mathbf{C}[X, Y]v = \mathbf{C}^n \\ X, Y \text{ are nilpotent} \end{matrix} \right\} \end{aligned}$$

The zero fiber Z_n

Special subvariety $Z_n \subseteq \mathrm{Hilb}^n(\mathbf{C}^2)$ called the *zero fiber* or the *punctual* Hilbert scheme of n points in $\mathbf{C}^2$: the preimage of $n \cdot 0 \in \mathrm{Sym}^n(\mathbf{C}^2)$. It is the $\mathrm{GL}_n(\mathbf{C})$-quotient of

$$\begin{aligned} \widetilde{Z}_n &= \left\{ (X, Y, v) \in \widetilde{Hilb}^n(\mathbf{C}^2) \,\middle|\, X, Y \text{ are nilpotent} \right\} \\ &= \left\{ (X, Y, v) \in \mathbf{C}^{n \times n} \times \mathbf{C}^{n \times n} \times \mathbf{C}^n \,\middle|\, \begin{matrix} [X, Y] = 0, \mathbf{C}[X, Y]v = \mathbf{C}^n \\ X, Y \text{ are nilpotent} \end{matrix} \right\} \end{aligned}$$

If X, Y are nilpotent and commute, then $X^a Y^b = 0$ whenever $a + b \geq n$.

The zero fiber Z_n

Special subvariety $Z_n \subseteq \mathrm{Hilb}^n(\mathbf{C}^2)$ called the *zero fiber* or the *punctual* Hilbert scheme of n points in $\mathbf{C}^2$: the preimage of $n \cdot 0 \in \mathrm{Sym}^n(\mathbf{C}^2)$. It is the $\mathrm{GL}_n(\mathbf{C})$-quotient of

$$\begin{aligned} \widetilde{Z}_n &= \left\{ (X, Y, v) \in \widetilde{Hilb}^n(\mathbf{C}^2) \;\middle|\; X, Y \text{ are nilpotent} \right\} \\ &= \left\{ (X, Y, v) \in \mathbf{C}^{n\times n} \times \mathbf{C}^{n\times n} \times \mathbf{C}^n \;\middle|\; \begin{array}{c} [X, Y] = 0, \mathbf{C}[X, Y]v = \mathbf{C}^n \\ X, Y \text{ are nilpotent} \end{array} \right\} \end{aligned}$$

If X, Y are nilpotent and commute, then $X^a Y^b = 0$ whenever $a + b \geq n$. So

$$\begin{array}{ccccc} Y^{n-1}v & & & & \\ \vdots & XY^{n-2}v & & & \\ Y^2v & \vdots & \ddots & & \\ Yv & XYv & \cdots & X^{n-2}Yv & \\ v & Xv & X^2v & \cdots & X^{n-1}v \end{array}$$

must span $\mathbf{C}^n$.

The zero fiber Z_n

Example: $n = 2$. Then $Z_2 = \{[X, Y, v]\}$ where $v \in \mathbf{C}^2$, $X, Y\colon \mathbf{C}^2 \to \mathbf{C}^2$ commuting and nilpotent, and

$$\begin{matrix} Yv & \\ v & Xv \end{matrix}$$

span $\mathbf{C}^2$. Also $[X, Y, v] = [gXg^{-1}, gYg^{-1}, gv]$ for all $g \in \mathsf{GL}_2(\mathbf{C})$.

The zero fiber Z_n

Example: $n = 2$. Then $Z_2 = \{[X, Y, v]\}$ where $v \in \mathbf{C}^2$, $X, Y\colon \mathbf{C}^2 \to \mathbf{C}^2$ commuting and nilpotent, and

$$\begin{matrix} Yv & \\ v & Xv \end{matrix}$$

span $\mathbf{C}^2$. Also $[X, Y, v] = [gXg^{-1}, gYg^{-1}, gv]$ for all $g \in \mathrm{GL}_2(\mathbf{C})$.

- Clearly $v \neq 0$ so we have a 1-dim subspace $\mathbf{C}\langle v\rangle \subseteq \mathbf{C}^2$.

The zero fiber Z_n

Example: $n = 2$. Then $Z_2 = \{[X, Y, v]\}$ where $v \in \mathbf{C}^2$, $X, Y : \mathbf{C}^2 \to \mathbf{C}^2$ commuting and nilpotent, and

$$\begin{matrix} Yv & \\ v & Xv \end{matrix}$$

span $\mathbf{C}^2$. Also $[X, Y, v] = [gXg^{-1}, gYg^{-1}, gv]$ for all $g \in \mathsf{GL}_2(\mathbf{C})$.

- Clearly $v \neq 0$ so we have a 1-dim subspace $\mathbf{C}\langle v \rangle \subseteq \mathbf{C}^2$.
- Next, consider $\mathbf{C}\langle Xv, Yv \rangle \subseteq \mathbf{C}^2$.

The zero fiber Z_n

Example: $n = 2$. Then $Z_2 = \{[X, Y, v]\}$ where $v \in \mathbf{C}^2$, $X, Y\colon \mathbf{C}^2 \to \mathbf{C}^2$ commuting and nilpotent, and

$$\begin{matrix} Yv & \\ v & Xv \end{matrix}$$

span $\mathbf{C}^2$. Also $[X, Y, v] = [gXg^{-1}, gYg^{-1}, gv]$ for all $g \in \mathrm{GL}_2(\mathbf{C})$.

- Clearly $v \neq 0$ so we have a 1-dim subspace $\mathbf{C}\langle v \rangle \subseteq \mathbf{C}^2$.
- Next, consider $\mathbf{C}\langle Xv, Yv \rangle \subseteq \mathbf{C}^2$.
  - It is not 0-dim because v, Xv, Yv span $\mathbf{C}^2$.

The zero fiber Z_n

Example: $n = 2$. Then $Z_2 = \{[X, Y, v]\}$ where $v \in \mathbf{C}^2$, $X, Y : \mathbf{C}^2 \to \mathbf{C}^2$ commuting and nilpotent, and

$$\begin{array}{ll} Yv & \\ v & Xv \end{array}$$

span $\mathbf{C}^2$. Also $[X, Y, v] = [gXg^{-1}, gYg^{-1}, gv]$ for all $g \in \mathrm{GL}_2(\mathbf{C})$.

- Clearly $v \neq 0$ so we have a 1-dim subspace $\mathbf{C}\langle v \rangle \subseteq \mathbf{C}^2$.
- Next, consider $\mathbf{C}\langle Xv, Yv \rangle \subseteq \mathbf{C}^2$.
  - It is not 0-dim because v, Xv, Yv span $\mathbf{C}^2$.
  - If $v \in \mathbf{C}\langle Xv, Yv \rangle$, then $Xv \in \mathbf{C}\langle X^2 v, XYv \rangle$ and $Yv \in \mathbf{C}\langle YXv, Y^2 v \rangle$.

The zero fiber Z_n

Example: $n = 2$. Then $Z_2 = \{[X, Y, v]\}$ where $v \in \mathbf{C}^2$, $X, Y \colon \mathbf{C}^2 \to \mathbf{C}^2$ commuting and nilpotent, and

$$\begin{array}{ll} Yv & \\ v & Xv \end{array}$$

span $\mathbf{C}^2$. Also $[X, Y, v] = [gXg^{-1}, gYg^{-1}, gv]$ for all $g \in \mathsf{GL}_2(\mathbf{C})$.

- Clearly $v \neq 0$ so we have a 1-dim subspace $\mathbf{C}\langle v \rangle \subseteq \mathbf{C}^2$.
- Next, consider $\mathbf{C}\langle Xv, Yv \rangle \subseteq \mathbf{C}^2$.
  - It is not 0-dim because v, Xv, Yv span $\mathbf{C}^2$.
  - If $v \in \mathbf{C}\langle Xv, Yv \rangle$, then $Xv \in \mathbf{C}\langle X^2v, XYv \rangle$ and $Yv \in \mathbf{C}\langle YXv, Y^2v \rangle$.
  - So $v \notin \mathbf{C}\langle Xv, Yv \rangle$ so $\mathbf{C}\langle Xv, Yv \rangle \subseteq \mathbf{C}^2$ is 1-dimensional.

The zero fiber Z_n

Example: $n = 2$. Then $Z_2 = \{[X, Y, v]\}$ where $v \in \mathbf{C}^2$, $X, Y\colon \mathbf{C}^2 \to \mathbf{C}^2$ commuting and nilpotent, and

$$\begin{matrix} Yv & \\ v & Xv \end{matrix}$$

span $\mathbf{C}^2$. Also $[X, Y, v] = [gXg^{-1}, gYg^{-1}, gv]$ for all $g \in \mathrm{GL}_2(\mathbf{C})$.

- Clearly $v \neq 0$ so we have a 1-dim subspace $\mathbf{C}\langle v\rangle \subseteq \mathbf{C}^2$.
- Next, consider $\mathbf{C}\langle Xv, Yv\rangle \subseteq \mathbf{C}^2$.
  - It is not 0-dim because v, Xv, Yv span $\mathbf{C}^2$.
  - If $v \in \mathbf{C}\langle Xv, Yv\rangle$, then $Xv \in \mathbf{C}\langle X^2v, XYv\rangle$ and $Yv \in \mathbf{C}\langle YXv, Y^2v\rangle$.
  - So $v \notin \mathbf{C}\langle Xv, Yv\rangle$ so $\mathbf{C}\langle Xv, Yv\rangle \subseteq \mathbf{C}^2$ is 1-dimensional.

Let $w \in \mathbf{C}\langle Xv, Yv\rangle$ be nonzero, and express $Xv = \alpha w$ and $Yv = \beta w$ for $(\alpha, \beta) \in \mathbf{C}^2 \setminus 0$. The ratio $[\alpha : \beta] \in \mathbf{CP}^1$ does not depend on w.

The zero fiber Z_n

Example: $n = 2$. Then $Z_2 = \{[X, Y, v]\}$ where $v \in \mathbf{C}^2$, $X, Y \colon \mathbf{C}^2 \to \mathbf{C}^2$ commuting and nilpotent, and

$$\begin{matrix} Yv & \\ v & Xv \end{matrix}$$

span $\mathbf{C}^2$. Also $[X, Y, v] = [gXg^{-1}, gYg^{-1}, gv]$ for all $g \in \mathrm{GL}_2(\mathbf{C})$.

- Clearly $v \neq 0$ so we have a 1-dim subspace $\mathbf{C}\langle v \rangle \subseteq \mathbf{C}^2$.
- Next, consider $\mathbf{C}\langle Xv, Yv \rangle \subseteq \mathbf{C}^2$.
  - It is not 0-dim because v, Xv, Yv span $\mathbf{C}^2$.
  - If $v \in \mathbf{C}\langle Xv, Yv \rangle$, then $Xv \in \mathbf{C}\langle X^2 v, XYv \rangle$ and $Yv \in \mathbf{C}\langle YXv, Y^2 v \rangle$.
  - So $v \notin \mathbf{C}\langle Xv, Yv \rangle$ so $\mathbf{C}\langle Xv, Yv \rangle \subseteq \mathbf{C}^2$ is 1-dimensional.

Let $w \in \mathbf{C}\langle Xv, Yv \rangle$ be nonzero, and express $Xv = \alpha w$ and $Yv = \beta w$ for $(\alpha, \beta) \in \mathbf{C}^2 \setminus 0$. The ratio $[\alpha : \beta] \in \mathbf{CP}^1$ does not depend on w.

The map $[X, Y, v] \mapsto [\alpha : \beta]$ is an isomorphism $\boxed{Z_2 \cong \mathbf{CP}^1}$ with inverse

$$[\alpha : \beta] \mapsto \left[\begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]$$

The tautological bundle $T \longrightarrow Z_n$

Consider the product $\widetilde{Z}_n \times \mathbf{C}^n = \left\{ (X, Y, v, w) \;\middle|\; \begin{matrix} [X, Y] = 0, \mathbf{C}[X, Y]v = \mathbf{C}^n \\ X, Y \text{ are nilpotent} \end{matrix} \right\}$ and $\mathsf{GL}_n(\mathbf{C})$ action $g(X, Y, v, w) = (gXg^{-1}, gYg^{-1}, gv, gw)$.

The tautological bundle $T \longrightarrow Z_n$

Consider the product $\widetilde{Z}_n \times \mathbf{C}^n = \left\{ (X, Y, v, w) \;\middle|\; \begin{matrix} [X, Y] = 0, \mathbf{C}[X, Y]v = \mathbf{C}^n \\ X, Y \text{ are nilpotent} \end{matrix} \right\}$ and $\mathsf{GL}_n(\mathbf{C})$ action $g(X, Y, v, w) = (gXg^{-1}, gYg^{-1}, gv, gw)$. Let T be the quotient, a rank n vector bundle over Z_n called the *tautological vector bundle.*

$$\begin{array}{ccc} T & & [X, Y, v, w] \\ \downarrow & & \downarrow \\ Z_n & & [X, Y, v] \end{array}$$

The tautological bundle $T \longrightarrow Z_n$

Consider the product $\widetilde{Z}_n \times \mathbf{C}^n = \left\{ (X, Y, v, w) \,\middle|\, \begin{matrix} [X, Y] = 0, \mathbf{C}[X, Y]v = \mathbf{C}^n \\ X, Y \text{ are nilpotent} \end{matrix} \right\}$ and $\mathrm{GL}_n(\mathbf{C})$ action $g(X, Y, v, w) = (gXg^{-1}, gYg^{-1}, gv, gw)$. Let T be the quotient, a rank n vector bundle over Z_n called the *tautological vector bundle.*

$$\begin{array}{ccc} T & & [X, Y, v, w] \\ \downarrow & & \downarrow \\ Z_n & & [X, Y, v] \end{array}$$

What is a global section of T?

The tautological bundle $T \longrightarrow Z_n$

Consider the product $\widetilde{Z}_n \times \mathbf{C}^n = \left\{ (X, Y, v, w) \;\middle|\; \begin{matrix} [X, Y] = 0, \mathbf{C}[X, Y]v = \mathbf{C}^n \\ X, Y \text{ are nilpotent} \end{matrix} \right\}$ and $\mathsf{GL}_n(\mathbf{C})$ action $g(X, Y, v, w) = (gXg^{-1}, gYg^{-1}, gv, gw)$. Let T be the quotient, a rank n vector bundle over Z_n called the *tautological vector bundle.*

$$\begin{array}{ccc} T & & [X, Y, v, w] \\ \big\downarrow & & \big\downarrow \\ Z_n & & [X, Y, v] \end{array}$$

What is a global section of T? An assignment $(X, Y, v) \mapsto w \in \mathbf{C}^n$ such that $(gXg^{-1}, gYg^{-1}, gv) \mapsto gw$.

The tautological bundle $T \longrightarrow Z_n$

Consider the product $\widetilde{Z}_n \times \mathbf{C}^n = \left\{ (X, Y, v, w) \;\middle|\; \begin{matrix} [X, Y] = 0, \mathbf{C}[X, Y]v = \mathbf{C}^n \\ X, Y \text{ are nilpotent} \end{matrix} \right\}$ and $\mathrm{GL}_n(\mathbf{C})$ action $g(X, Y, v, w) = (gXg^{-1}, gYg^{-1}, gv, gw)$. Let T be the quotient, a rank n vector bundle over Z_n called the *tautological vector bundle.*

$$\begin{array}{ccc} T & & [X, Y, v, w] \\ \big\downarrow & & \big\downarrow \\ Z_n & & [X, Y, v] \end{array}$$

What is a global section of T? An assignment $(X, Y, v) \mapsto w \in \mathbf{C}^n$ such that $(gXg^{-1}, gYg^{-1}, gv) \mapsto gw$. E.g. $w = v$ or $[X, Y, v] \mapsto [X, Y, v, v]$.

The tautological bundle $T \longrightarrow Z_n$

Consider the product $\widetilde{Z}_n \times \mathbf{C}^n = \left\{ (X, Y, v, w) \;\middle|\; \begin{matrix} [X, Y] = 0, \mathbf{C}[X, Y]v = \mathbf{C}^n \\ X, Y \text{ are nilpotent} \end{matrix} \right\}$ and $\mathsf{GL}_n(\mathbf{C})$ action $g(X, Y, v, w) = (gXg^{-1}, gYg^{-1}, gv, gw)$. Let T be the quotient, a rank n vector bundle over Z_n called the *tautological vector bundle.*

$$\begin{array}{ccc} T & \qquad & [X, Y, v, w] \\ \downarrow & & \downarrow \\ Z_n & & [X, Y, v] \end{array}$$

What is a global section of T? An assignment $(X, Y, v) \mapsto w \in \mathbf{C}^n$ such that $(gXg^{-1}, gYg^{-1}, gv) \mapsto gw$. E.g. $w = v$ or $[X, Y, v] \mapsto [X, Y, v, v]$.

Here are $\binom{n+1}{2}$ global sections of T:

$$\begin{array}{ccccc} \underline{Y^{n-1}v} & & & & \\ \vdots & \underline{XY^{n-2}v} & & & \\ \underline{Y^2 v} & \vdots & \ddots & & \\ \underline{Yv} & \underline{XYv} & \cdots & \underline{X^{n-2}Yv} & \\ \underline{v} & \underline{Xv} & \underline{X^2 v} & \cdots & \underline{X^{n-1}v} \end{array}$$

$\underline{X^a Y^b v}$ is a global section

$X^a Y^b v$ is a vector in the fiber over $[X, Y, v] \in Z_n$

The tautological bundle $T \longrightarrow Z_n$

Example: $n = 2$. Then $T \longrightarrow Z_2$ is a rank 2 vector bundle with global sections

$$\begin{array}{cc} \underline{Yv} & \\ \underline{v} & \underline{Xv} \end{array}$$

The tautological bundle $T \longrightarrow Z_n$

Example: $n = 2$. Then $T \longrightarrow Z_2$ is a rank 2 vector bundle with global sections

$$\begin{array}{ll} \underline{Yv} & \\ \underline{v} & \underline{Xv} \end{array}$$

Although Xv and Yv are linearly dependent vectors for any fixed $[X, Y, v] \in Z_2$, the sections $\underline{Xv}$ and $\underline{Yv}$ are not linearly dependent (the coefficients of $\beta Xv - \alpha Yv = 0$ are functions of $[X, Y, v]$, not constants).

The tautological bundle $T \longrightarrow Z_n$

Example: $n = 2$. Then $T \longrightarrow Z_2$ is a rank 2 vector bundle with global sections

$$\begin{array}{ll} \underline{Yv} & \\ \underline{v} & \underline{Xv} \end{array}$$

Although Xv and Yv are linearly dependent vectors for any fixed $[X, Y, v] \in Z_2$, the sections $\underline{Xv}$ and $\underline{Yv}$ are not linearly dependent (the coefficients of $\beta Xv - \alpha Yv = 0$ are functions of $[X, Y, v]$, not constants).

The bundle T splits as a direct sum of line bundles $T = \mathbf{C}\langle v \rangle \oplus \mathbf{C}\langle Xv, Yv \rangle$.

The tautological bundle $T \longrightarrow Z_n$

Example: $n = 2$. Then $T \longrightarrow Z_2$ is a rank 2 vector bundle with global sections

$$\begin{array}{ll} \underline{Yv} & \\ \underline{v} & \underline{Xv} \end{array}$$

Although Xv and Yv are linearly dependent vectors for any fixed $[X, Y, v] \in Z_2$, the sections $\underline{Xv}$ and $\underline{Yv}$ are not linearly dependent (the coefficients of $\beta Xv - \alpha Yv = 0$ are functions of $[X, Y, v]$, not constants).

The bundle T splits as a direct sum of line bundles $T = \mathbf{C}\langle v \rangle \oplus \mathbf{C}\langle Xv, Yv \rangle$.

- The first line bundle is trivial (global nowhere-vanishing section $\underline{v}$)

The tautological bundle $T \to Z_n$

Example: $n = 2$. Then $T \to Z_2$ is a rank 2 vector bundle with global sections

$$\begin{array}{cc} \underline{Yv} & \\ \underline{v} & \underline{Xv} \end{array}$$

Although Xv and Yv are linearly dependent vectors for any fixed $[X, Y, v] \in Z_2$, the sections $\underline{Xv}$ and $\underline{Yv}$ are not linearly dependent (the coefficients of $\beta Xv - \alpha Yv = 0$ are functions of $[X, Y, v]$, not constants).

The bundle T splits as a direct sum of line bundles $T = \mathbf{C}\langle v\rangle \oplus \mathbf{C}\langle Xv, Yv\rangle$.

- The first line bundle is trivial (global nowhere-vanishing section $\underline{v}$)
- The second is nontrivial, with a 2-dim space of sections $\mathbf{C}\langle \underline{Xv}, \underline{Yv}\rangle$.

The tautological bundle $T \to Z_n$

Example: $n = 2$. Then $T \to Z_2$ is a rank 2 vector bundle with global sections

$$\begin{array}{cc} \underline{Yv} & \\ \underline{v} & \underline{Xv} \end{array}$$

Although Xv and Yv are linearly dependent vectors for any fixed $[X, Y, v] \in Z_2$, the sections $\underline{Xv}$ and $\underline{Yv}$ are not linearly dependent (the coefficients of $\beta Xv - \alpha Yv = 0$ are functions of $[X, Y, v]$, not constants).

The bundle T splits as a direct sum of line bundles $T = \mathbf{C}\langle v \rangle \oplus \mathbf{C}\langle Xv, Yv \rangle$.

- The first line bundle is trivial (global nowhere-vanishing section $\underline{v}$)
- The second is nontrivial, with a 2-dim space of sections $\mathbf{C}\langle \underline{Xv}, \underline{Yv} \rangle$.

Viewing $Z_2 = \mathbf{CP}^1$, we see that $T = \mathcal{O} \oplus \mathcal{O}(1)$.

The reduced tautological bundle $T_0 \to Z_n$

For any n, there is a direct sum splitting of the rank n vector bundle T as

$$T = \mathbf{C}\langle v\rangle \oplus T_0 \qquad T_0 = \mathbf{C}\langle X^a Y^b v \mid 0 < a + b < n\rangle$$

where $\mathbf{C}\langle v\rangle$ is a trivial rank 1 bundle and T_0 is a rank $n-1$ vector bundle called the *reduced* tautological bundle.

The reduced tautological bundle $T_0 \to Z_n$

For any n, there is a direct sum splitting of the rank n vector bundle T as

$$T = \mathbf{C}\langle v\rangle \oplus T_0 \qquad T_0 = \mathbf{C}\langle X^a Y^b v \mid 0 < a + b < n\rangle$$

where $\mathbf{C}\langle v\rangle$ is a trivial rank 1 bundle and T_0 is a rank $n-1$ vector bundle called the *reduced* tautological bundle.

Geometric model of $\overline{\mathrm{H}}(T_{n,n+1})$: for $d = 0, \ldots, n-1$, the a^d-layer of $\overline{\mathrm{H}}(T_{n,n+1})$ is the space of global sections of $\Lambda^{n-1-d} T_0 \to Z_n$.

The reduced tautological bundle $T_0 \to Z_n$

For any n, there is a direct sum splitting of the rank n vector bundle T as

$$T = \mathbf{C}\langle v\rangle \oplus T_0 \qquad T_0 = \mathbf{C}\langle X^a Y^b v \mid 0 < a + b < n\rangle$$

where $\mathbf{C}\langle v\rangle$ is a trivial rank 1 bundle and T_0 is a rank $n-1$ vector bundle called the *reduced* tautological bundle.

Geometric model of $\overline{\mathrm{H}}(T_{n,n+1})$: for $d = 0, \ldots, n-1$, the a^d-layer of $\overline{\mathrm{H}}(T_{n,n+1})$ is the space of global sections of $\Lambda^{n-1-d} T_0 \to Z_n$.

For $n = 2$, $(\underline{at})^{-2}\overline{\mathscr{P}}(T_{2,3})$ is

The reduced tautological bundle $T_0 \to Z_n$

For any n, there is a direct sum splitting of the rank n vector bundle T as

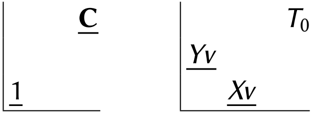
$$T = \mathbf{C}\langle v\rangle \oplus T_0 \qquad T_0 = \mathbf{C}\langle X^a Y^b v \mid 0 < a+b < n\rangle$$

where $\mathbf{C}\langle v\rangle$ is a trivial rank 1 bundle and T_0 is a rank $n-1$ vector bundle called the *reduced* tautological bundle.

Geometric model of $\overline{\mathrm{H}}(T_{n,n+1})$: for $d = 0, \ldots, n-1$, the a^d-layer of $\overline{\mathrm{H}}(T_{n,n+1})$ is the space of global sections of $\Lambda^{n-1-d} T_0 \to Z_n$.

For $n = 2$, $(\underline{at})^{-2}\overline{\mathscr{P}}(T_{2,3})$ is

Bases for the global sections of $\Lambda^0 T_0 = \underline{\mathbf{C}}$ and $\Lambda^1 T_0 = T_0$ over Z_2 are



Geometric model of $\overline{\mathrm{H}}(T_{3,4})$

$(\underline{at})^{-6}\overline{\mathscr{P}}(T_{3,4})$ is

			a^2
1			

			a^1
1			
1	1		
	1	1	

1			a^0
	1		
	1	1	
			1

Geometric model of $\overline{\mathrm{H}}(T_{3,4})$

$(\underline{at})^{-6}\overline{\mathscr{P}}(T_{3,4})$ is

$$
\begin{array}{|cccc}
 & & & a^2 \\
 & & & \\
 & & & \\
1 & & & \\ \hline
\end{array}
\qquad
\begin{array}{|cccc}
 & & & a^1 \\
1 & & & \\
1 & 1 & & \\
 & 1 & 1 & \\ \hline
\end{array}
\qquad
\begin{array}{|cccc}
1 & & & a^0 \\
 & 1 & & \\
 & 1 & 1 & \\
 & & & 1 \\ \hline
\end{array}
$$

Bases for the global sections of $\Lambda^0 T_0 = \underline{\mathbf{C}}$ and $\Lambda^1 T_0 = T_0$ are

$$
\begin{array}{|cc}
 & \underline{\mathbf{C}} \\
 & \\
 & \\
\underline{1} & \\ \hline
\end{array}
\qquad
\begin{array}{|cccc}
 & & & T_0 \\
\underline{Y^2 v} & & & \\
\underline{Yv} & \underline{XYv} & & \\
 & \underline{Xv} & \underline{X^2 v} & \\ \hline
\end{array}
$$

Geometric model of $\overline{\mathrm{H}}(T_{3,4})$

$(\underline{at})^{-6}\overline{\mathscr{P}}(T_{3,4})$ is

$$
\begin{array}{|cccc}
 & & & a^2 \\
 & & & \\
 & & & \\
1 & & & \\ \hline
\end{array}
\qquad
\begin{array}{|cccc}
 & & & a^1 \\
1 & & & \\
1 & 1 & & \\
 & 1 & 1 & \\ \hline
\end{array}
\qquad
\begin{array}{|cccc}
1 & & & a^0 \\
 & 1 & & \\
 & 1 & 1 & \\
 & & & 1 \\ \hline
\end{array}
$$

Bases for the global sections of $\Lambda^0 T_0 = \underline{\mathbf{C}}$ and $\Lambda^1 T_0 = T_0$ are

$$
\begin{array}{|ccc}
 & & \underline{\mathbf{C}} \\
 & & \\
 & & \\
\underline{1} & & \\ \hline
\end{array}
\qquad
\begin{array}{|cccc}
 & & & T_0 \\
\underline{Y^2 v} & & & \\
\underline{Yv} & \underline{XYv} & & \\
 & \underline{Xv} & \underline{X^2 v} & \\ \hline
\end{array}
$$

The space of global sections of $\Lambda^2 T_0 = \det T_0$ is spanned by the $\binom{5}{2}$ wedge products

$$
\begin{array}{cccc}
 & \underline{Y^2 v \wedge XYv} \quad \underline{Y^2 \wedge X^2 v} \quad \underline{XYv \wedge X^2 v} & & \\
\underline{Yv \wedge Y^2 v} & \begin{array}{c}\underline{Xv \wedge Y^2 v} \\ \underline{Yv \wedge XYv}\end{array} \quad \begin{array}{c}\underline{Xv \wedge XYv} \\ \underline{Yv \wedge X^2 v}\end{array} & \underline{Xv \wedge X^2 v} & \\
 & \underline{Xv \wedge Yv} & &
\end{array}
$$

Geometric model of $\overline{\mathrm{H}}(T_{3,4})$

$$\underline{Y^2v \wedge XYv} \qquad \underline{Y^2 \wedge X^2v} \qquad \underline{XYv \wedge X^2v}$$

$$\underline{Yv \wedge Y^2v} \qquad \frac{Xv \wedge Y^2v}{\underline{Yv \wedge XYv}} \qquad \frac{Xv \wedge XYv}{\underline{Yv \wedge X^2v}} \qquad \underline{Xv \wedge X^2v}$$

$$\underline{Xv \wedge Yv}$$

Let's fix $[X, Y, v] \in Z_3$ and analyze the 5 vectors Y^2v, XYv, X^2v, Xv, Yv in the 2-dim fiber of T_0

Geometric model of $\overline{\mathrm{H}}(T_{3,4})$

$$\underline{Y^2v \wedge XYv} \qquad \underline{Y^2 \wedge X^2v} \qquad \underline{XYv \wedge X^2v}$$

$$\underline{Yv \wedge Y^2v} \qquad \frac{Xv \wedge Y^2v}{\underline{Yv \wedge XYv}} \qquad \frac{Xv \wedge XYv}{\underline{Yv \wedge X^2v}} \qquad \underline{Xv \wedge X^2v}$$

$$\underline{Xv \wedge Yv}$$

Let's fix $[X, Y, v] \in Z_3$ and analyze the 5 vectors Y^2v, XYv, X^2v, Xv, Yv in the 2-dim fiber of T_0

- Claim: $\mathbf{C}\langle Y^2v, XYv, X^2v\rangle$ is at most 1-dim.

Geometric model of $\overline{\mathrm{H}}(T_{3,4})$

$$\underline{Y^2v \wedge XYv} \qquad \underline{Y^2 \wedge X^2v} \qquad \underline{XYv \wedge X^2v}$$

$$\underline{Yv \wedge Y^2v} \qquad \frac{Xv \wedge Y^2v}{\underline{Yv \wedge XYv}} \qquad \frac{Xv \wedge XYv}{\underline{Yv \wedge X^2v}} \qquad \underline{Xv \wedge X^2v}$$

$$\underline{Xv \wedge Yv}$$

Let's fix $[X, Y, v] \in Z_3$ and analyze the 5 vectors Y^2v, XYv, X^2v, Xv, Yv in the 2-dim fiber of T_0

- Claim: $\mathbf{C}\langle Y^2v, XYv, X^2v\rangle$ is at most 1-dim. Otherwise $Xv, Yv \in \mathbf{C}\langle Y^2v, XYv, X^2v\rangle$ which implies Y^2v, XYv, X^2v are zero.

Geometric model of $\overline{\mathrm{H}}(T_{3,4})$

$$\underline{Yv \wedge Y^2 v} \qquad \frac{Xv \wedge Y^2 v}{\underline{Yv \wedge XYv}} \qquad \frac{Xv \wedge XYv}{\underline{Yv \wedge X^2 v}} \qquad \underline{Xv \wedge X^2 v}$$

$$\underline{Xv \wedge Yv}$$

Let's fix $[X, Y, v] \in Z_3$ and analyze the 5 vectors $Y^2 v, XYv, X^2 v, Xv, Yv$ in the 2-dim fiber of T_0

- Claim: $\mathbf{C}\langle Y^2 v, XYv, X^2 v\rangle$ is at most 1-dim. Otherwise $Xv, Yv \in \mathbf{C}\langle Y^2 v, XYv, X^2 v\rangle$ which implies $Y^2 v, XYv, X^2 v$ are zero.

Geometric model of $\overline{\mathrm{H}}(T_{3,4})$

$$\underline{Yv \wedge Y^2 v} \qquad \underline{\frac{Xv \wedge Y^2 v}{Yv \wedge XYv}} \qquad \underline{\frac{Xv \wedge XYv}{Yv \wedge X^2 v}} \qquad \underline{Xv \wedge X^2 v}$$

$$\underline{Xv \wedge Yv}$$

Let's fix $[X, Y, v] \in Z_3$ and analyze the 5 vectors $Y^2 v, XYv, X^2 v, Xv, Yv$ in the 2-dim fiber of T_0

- Claim: $\mathbf{C}\langle Y^2 v, XYv, X^2 v\rangle$ is at most 1-dim. Otherwise $Xv, Yv \in \mathbf{C}\langle Y^2 v, XYv, X^2 v\rangle$ which implies $Y^2 v, XYv, X^2 v$ are zero.
- Claim: $Xv \wedge Y^2 v = Yv \wedge XYv$ (and similarly $Xv \wedge XYv = Yv \wedge X^2 v$)

Geometric model of $\overline{\mathrm{H}}(T_{3,4})$

$$\underline{Yv \wedge Y^2 v} \qquad \frac{Xv \wedge Y^2 v}{\underline{Yv \wedge XYv}} \qquad \frac{Xv \wedge XYv}{\underline{Yv \wedge X^2 v}} \qquad \underline{Xv \wedge X^2 v}$$

$$\underline{Xv \wedge Yv}$$

Let's fix $[X, Y, v] \in Z_3$ and analyze the 5 vectors $Y^2 v, XYv, X^2 v, Xv, Yv$ in the 2-dim fiber of T_0

- Claim: $\mathbf{C}\langle Y^2 v, XYv, X^2 v\rangle$ is at most 1-dim. Otherwise $Xv, Yv \in \mathbf{C}\langle Y^2 v, XYv, X^2 v\rangle$ which implies $Y^2 v, XYv, X^2 v$ are zero.
- Claim: $Xv \wedge Y^2 v = Yv \wedge XYv$ (and similarly $Xv \wedge XYv = Yv \wedge X^2 v$)
  - If $Y^2 v \neq 0$, then $Yv, Y^2 v$ form a basis for this 2-dim fiber. Expressing $Xv = aYv + bY^2 v$, we get $Xv \wedge Y^2 v = a(Yv \wedge Y^2 v) = Yv \wedge XYv$.

Geometric model of $\overline{\mathrm{H}}(T_{3,4})$

$$\underline{Yv \wedge Y^2 v} \qquad \frac{Xv \wedge Y^2 v}{\underline{Yv \wedge XYv}} \qquad \frac{Xv \wedge XYv}{\underline{Yv \wedge X^2 v}} \qquad \underline{Xv \wedge X^2 v}$$

$$\underline{Xv \wedge Yv}$$

Let's fix $[X, Y, v] \in Z_3$ and analyze the 5 vectors Y^2v, XYv, X^2v, Xv, Yv in the 2-dim fiber of T_0

- Claim: $\mathbf{C}\langle Y^2v, XYv, X^2v\rangle$ is at most 1-dim. Otherwise $Xv, Yv \in \mathbf{C}\langle Y^2v, XYv, X^2v\rangle$ which implies Y^2v, XYv, X^2v are zero.
- Claim: $Xv \wedge Y^2v = Yv \wedge XYv$ (and similarly $Xv \wedge XYv = Yv \wedge X^2v$)
  - If $Y^2v \neq 0$, then Yv, Y^2v form a basis for this 2-dim fiber. Expressing $Xv = aYv + bY^2v$, we get $Xv \wedge Y^2v = a(Yv \wedge Y^2v) = Yv \wedge XYv$.
  - If $Y^2v = 0$, then $Yv \wedge XYv = 0$. Otherwise $Xv \in \mathbf{C}\langle Yv, XYv\rangle$, which implies $YXv \in \mathbf{C}\langle Y^2v, XY^2v\rangle = 0$.

Geometric model of $\overline{\mathrm{H}}(T_{3,4})$

$$\underline{Yv \wedge Y^2 v} \qquad \underline{Yv \wedge XYv} \qquad \underline{Xv \wedge XYv} \qquad \underline{Xv \wedge X^2 v}$$

$$\underline{Xv \wedge Yv}$$

Let's fix $[X, Y, v] \in Z_3$ and analyze the 5 vectors $Y^2 v, XYv, X^2 v, Xv, Yv$ in the 2-dim fiber of T_0

- Claim: $\mathbf{C}\langle Y^2 v, XYv, X^2 v\rangle$ is at most 1-dim. Otherwise $Xv, Yv \in \mathbf{C}\langle Y^2 v, XYv, X^2 v\rangle$ which implies $Y^2 v, XYv, X^2 v$ are zero.
- Claim: $Xv \wedge Y^2 v = Yv \wedge XYv$ (and similarly $Xv \wedge XYv = Yv \wedge X^2 v$)
  - If $Y^2 v \neq 0$, then $Yv, Y^2 v$ form a basis for this 2-dim fiber. Expressing $Xv = aYv + bY^2 v$, we get $Xv \wedge Y^2 v = a(Yv \wedge Y^2 v) = Yv \wedge XYv$.
  - If $Y^2 v = 0$, then $Yv \wedge XYv = 0$. Otherwise $Xv \in \mathbf{C}\langle Yv, XYv\rangle$, which implies $YXv \in \mathbf{C}\langle Y^2 v, XY^2 v\rangle = 0$.

Geometric model of $\overline{\mathrm{H}}(T_{3,4})$

$$\underline{Yv \wedge Y^2 v} \qquad \underline{Yv \wedge XYv} \qquad \underline{Xv \wedge XYv} \qquad \underline{Xv \wedge X^2 v}$$

$$\underline{Xv \wedge Yv}$$

Let's fix $[X, Y, v] \in Z_3$ and analyze the 5 vectors Y^2v, XYv, X^2v, Xv, Yv in the 2-dim fiber of T_0

- Claim: $\mathbf{C}\langle Y^2v, XYv, X^2v\rangle$ is at most 1-dim. Otherwise $Xv, Yv \in \mathbf{C}\langle Y^2v, XYv, X^2v\rangle$ which implies Y^2v, XYv, X^2v are zero.
- Claim: $Xv \wedge Y^2v = Yv \wedge XYv$ (and similarly $Xv \wedge XYv = Yv \wedge X^2v$)
  - If $Y^2v \neq 0$, then Yv, Y^2v form a basis for this 2-dim fiber. Expressing $Xv = aYv + bY^2v$, we get $Xv \wedge Y^2v = a(Yv \wedge Y^2v) = Yv \wedge XYv$.
  - If $Y^2v = 0$, then $Yv \wedge XYv = 0$. Otherwise $Xv \in \mathbf{C}\langle Yv, XYv\rangle$, which implies $YXv \in \mathbf{C}\langle Y^2v, XY^2v\rangle = 0$.

The remaining 5 wedge product sections are linearly independent.

Geometric model of $\overline{\mathrm{H}}(T_{n,n+1})$

A basis for the global sections of $\Lambda^2 T_0 = \det T_0$ is

$$\begin{array}{|llll}
Yv \wedge Y^2 v & & & \\
& Yv \wedge XYv & & \\
& Xv \wedge Yv & Xv \wedge XYv & \\
& & & Xv \wedge X^2 v \\ \hline
\end{array}$$

$$\begin{array}{|cccc}
1 & & & a^0 \\
& 1 & & \\
& 1 & 1 & \\
& & & 1 \\ \hline
\end{array}$$

Geometric model of $\overline{\mathrm{H}}(T_{n,n+1})$

A basis for the global sections of $\Lambda^2 T_0 = \det T_0$ is

$Yv \wedge Y^2 v$			
	$Yv \wedge XYv$		
	$Xv \wedge Yv$	$Xv \wedge XYv$	
			$Xv \wedge X^2 v$

1			a^0
	1		
	1	1	
			1

The q, t-gradings are the X- and Y-degrees:

Geometric model of $\overline{\mathrm{H}}(T_{n,n+1})$

A basis for the global sections of $\Lambda^2 T_0 = \det T_0$ is

$Yv \wedge Y^2 v$			
	$Yv \wedge XYv$		
	$Xv \wedge Yv$	$Xv \wedge XYv$	
			$Xv \wedge X^2 v$

1			a^0
	1		
	1	1	
			1

The q, t-gradings are the X- and Y-degrees: there is a $(\mathbf{C}^\times)^2$-action on Z_n given by $(q, t)[X, Y, v] = [qX, tY, v]$ which induces a $(\mathbf{C}^\times)^2$-action/bigrading on $H^0(\Lambda^{n-1-d} T_0)$

Geometric model of $\overline{\mathrm{H}}(T_{n,n+1})$

A basis for the global sections of $\Lambda^2 T_0 = \det T_0$ is

$Yv \wedge Y^2 v$			
	$Yv \wedge XYv$		
	$Xv \wedge Yv$	$Xv \wedge XYv$	
			$Xv \wedge X^2 v$

1			a^0
	1		
	1	1	
			1

The q, t-gradings are the X- and Y-degrees: there is a $(\mathbf{C}^\times)^2$-action on Z_n given by $(q,t)[X,Y,v] = [qX, tY, v]$ which induces a $(\mathbf{C}^\times)^2$-action/bigrading on $H^0(\Lambda^{n-1-d} T_0)$

Theorem (Haiman '02 and Haglund '03): The q, t-Hilbert polynomial of $H^0(\Lambda^{n-1-d} T_0)$ is the q, t-Schröder number. In particular, the q, t-Hilbert polynomial of $H^0(\det T_0)$ is the q, t-Catalan number.

Further structure on $H^0(\Lambda^* T_0)$

$\overline{\mathrm{H}}(T_{n,n+1})$ and $H^0(\Lambda^* T_0)$ match as trigraded vector spaces. More structure:

Further structure on $H^0(\Lambda^* T_0)$

$\overline{\mathrm{H}}(T_{n,n+1})$ and $H^0(\Lambda^* T_0)$ match as trigraded vector spaces. More structure:

- $\mathfrak{sl}_2(\mathbf{C})$-representation: $\mathsf{SL}_2(\mathbf{C})$ acts on Z_n by

  $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} [X, Y, v] = [aX + bY, cX + dY, v]$$

  which induces an action of $\mathfrak{sl}_2(\mathbf{C})$ on $H^0(\Lambda^* T_0)$. Basically $E = X\frac{\partial}{\partial Y}$ and $F = Y\frac{\partial}{\partial X}$.

Further structure on $H^0(\Lambda^* T_0)$

$\overline{\mathrm{H}}(T_{n,n+1})$ and $H^0(\Lambda^* T_0)$ match as trigraded vector spaces. More structure:

- $\mathfrak{sl}_2(\mathbf{C})$-representation: $\mathrm{SL}_2(\mathbf{C})$ acts on Z_n by

  $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} [X, Y, v] = [aX + bY, cX + dY, v]$$

  which induces an action of $\mathfrak{sl}_2(\mathbf{C})$ on $H^0(\Lambda^* T_0)$. Basically $E = X\frac{\partial}{\partial Y}$ and $F = Y\frac{\partial}{\partial X}$.
- $d_{N|M}$ differentials: the map $H^0(\Lambda^* T_0) \to H^0(\Lambda^{*+1} T_0)$ given by wedge product with $X^N Y^M v$.

Further structure on $H^0(\Lambda^* T_0)$