The structure of triply-graded homology

Joshua Wang

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Background on $\mathfrak{sl}(N)$ homology

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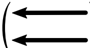
$\underline{a}\overline{P}\Big($  $\Big) - \underline{a}^{-1}\overline{P}\Big($  $\Big) = (\underline{q} - \underline{q}^{-1})\overline{P}\Big($  $\Big).$

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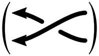
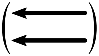
$$\underline{a}\overline{P}\left(\right.$$  $$\left.\right) - \underline{a}^{-1}\overline{P}\left(\right.$$  $$\left.\right) = (\underline{q} - \underline{q}^{-1})\overline{P}\left(\right.$$  $$\left.\right).$$

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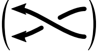
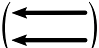
$$\underline{a}\overline{P}\left(\right.$$  $$\left.\right) - \underline{a}^{-1}\overline{P}\left(\right.$$  $$\left.\right) = (\underline{q} - \underline{q}^{-1})\overline{P}\left(\right.$$  $$\left.\right).$$

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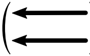
$\underline{a}\overline{P}$ () $- \underline{a}^{-1}\overline{P}$ () $= (\underline{q} - \underline{q}^{-1})\overline{P}$ ().

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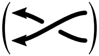
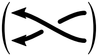
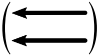
$\underline{a}\overline{P}$ () $- \underline{a}^{-1}\overline{P}$ () $= (\underline{q} - \underline{q}^{-1})\overline{P}$ ().

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$\overline{\mathrm{H}}_N(T_{3,4})$ for $N = 1, 2, 3$:

	0	2	4	6	8	10	12
8							
7							
6							
5							
4							
3							
2							
1							
0	$\mathbf{C}^1$						

	6	8	10	12	14	16	18	20
8								
7								
6								
5						$\mathbf{C}^1$		
4				$\mathbf{C}^1$				
3				$\mathbf{C}^1$				
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1								
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8										$\mathbf{C}^1$
7								$\mathbf{C}^1$	$\mathbf{C}^1$	
6							$\mathbf{C}^1$			
5						$\mathbf{C}^1$	$\mathbf{C}^1$			
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1										
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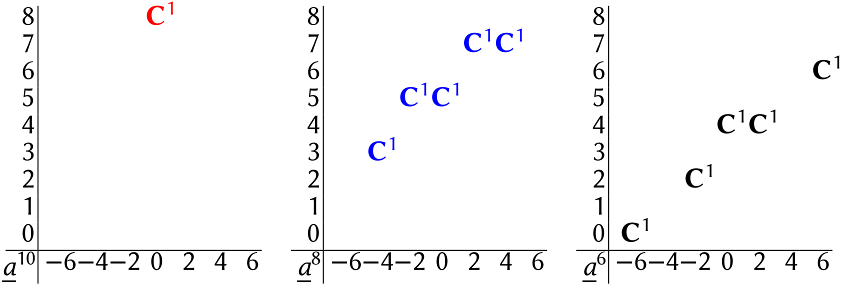
Comparing $\overline{\mathrm{H}}(L)$ and $\overline{\mathrm{H}}_N(L)$

Compare $\overline{\mathrm{H}}_N(L)$ with the trigrading-collapse of $\overline{\mathrm{H}}(T_{3,4})$ by $\underline{a} = \underline{q}^N$



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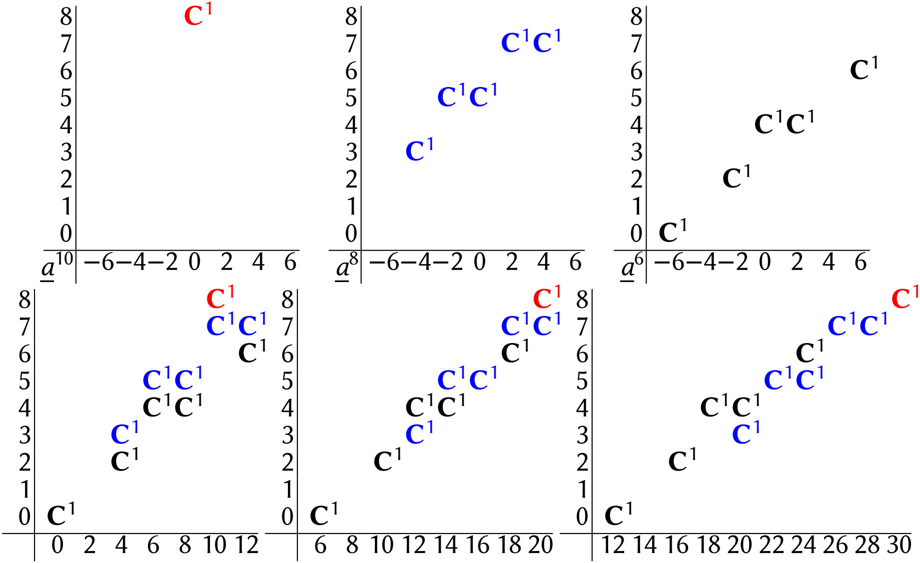
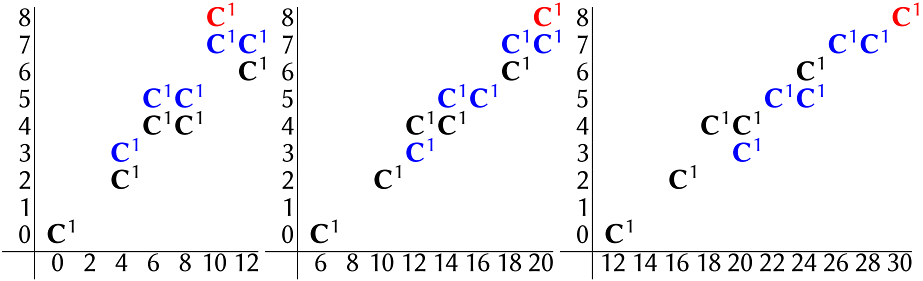
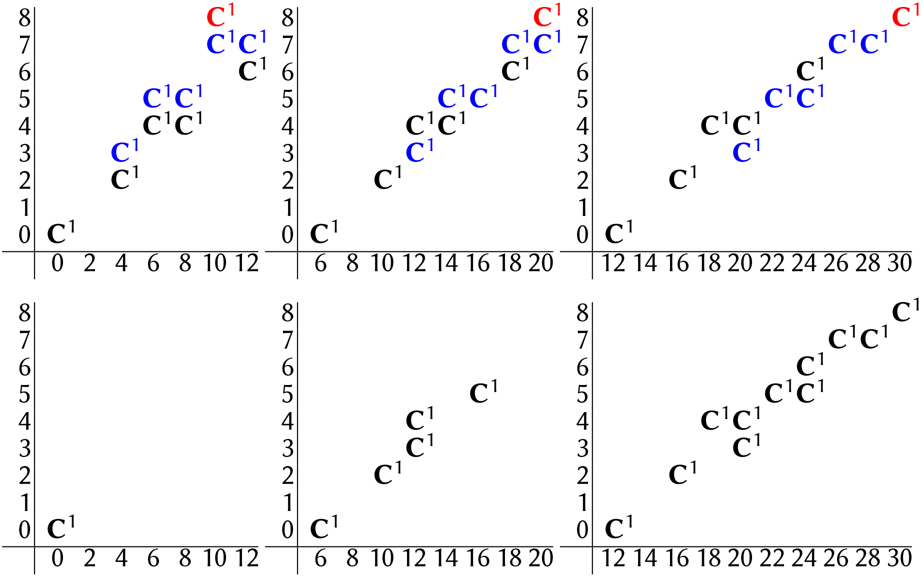



8 C1 C1 C1
7 C1C1 C1C1 C1C1
6 C1 C1 C1
5 C1C1 C1C1 C1C1
4 C1C1 C1C1 C1C1
3 C1 C1 C1
2 C1 C1 C1
1
0 C1 C1 C1
0 2 4 6 8 10 12
6 8 10 12 14 16 18 20
12 14 16 18 20 22 24 26 28 30

Comparing $\overline{\mathrm{H}}(L)$ and $\overline{\mathrm{H}}_N(L)$



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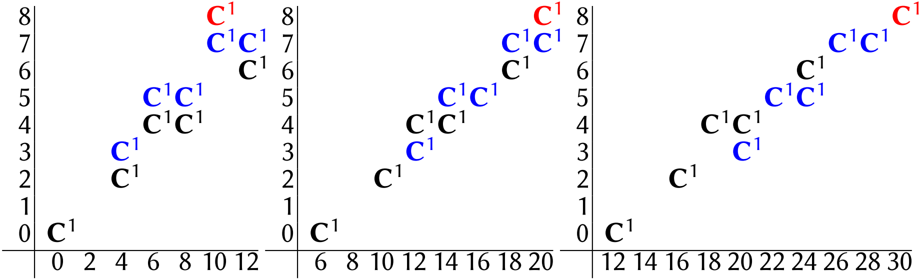
Dunfield–Gukov–Rasmussen - “The Superpolynomial for Knot Homologies” conjectured that there is a differential $d_N\colon \overline{\mathrm{H}}(L) \to \overline{\mathrm{H}}(L)$ of degree $\underline{a}^{-2}\underline{q}^{2N}\underline{t}^{-1}$ whose $(\underline{q}, \underline{t})$-bigraded homology, after setting $\underline{a} = \underline{q}^N$, is $\overline{\mathrm{H}}_N(L)$.



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8						$\mathbf{C}^1$	
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6							$\mathbf{C}^1$
5				$\mathbf{C}^1$	$\mathbf{C}^1$		
4				$\mathbf{C}^1$	$\mathbf{C}^1$		
3			$\mathbf{C}^1$				
2			$\mathbf{C}^1$				
1							
0	$\mathbf{C}^1$						

	6	8	10	12	14	16	18	20
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2			$\mathbf{C}^1$					
1								
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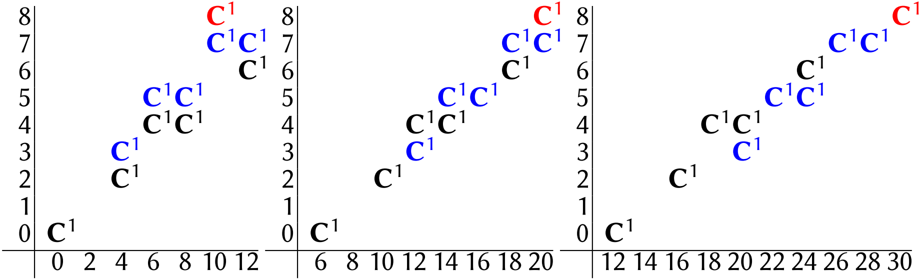

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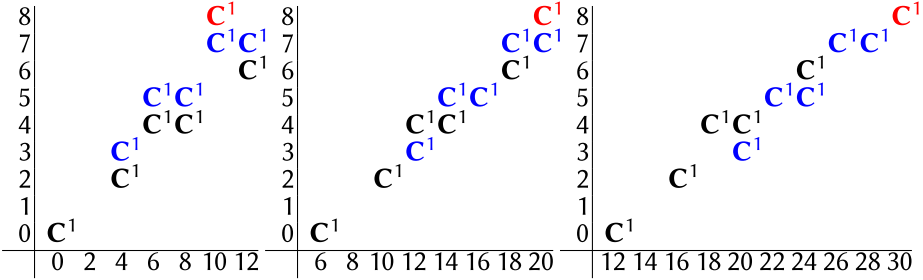

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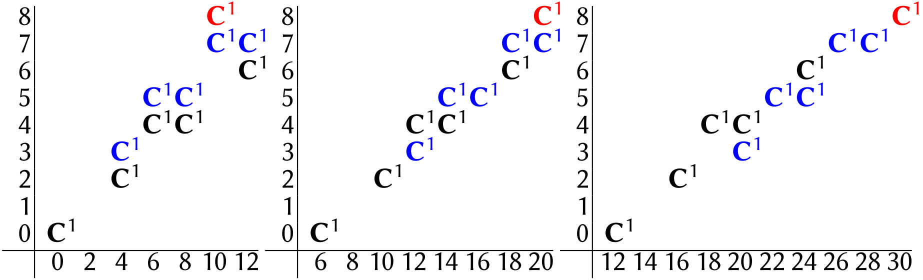

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Corollary: for a knot K, $\overline{\mathrm{H}}_N(K)$ is the grading-collapse of $\overline{\mathrm{H}}(K)$ for $N \gg 0$.

Comparing $\overline{\mathrm{H}}(L)$ and $\overline{\mathrm{H}}_N(L)$

Grading collapse $\underline{a} = \underline{q}^3$ of $\overline{\mathrm{H}}(T_{4,5})$ versus $\overline{\mathrm{H}}_3(T_{4,5})$.

	24	26	28	30	32	34	36	38	40	42	44	46	48	50	52	54
15																1
14														1	1	1
13													1	1	1	
12												1	11	1		
11										1	2	2	1			
10									1	1	11	1				
9									2	2	1					
8							1	1	1	1						
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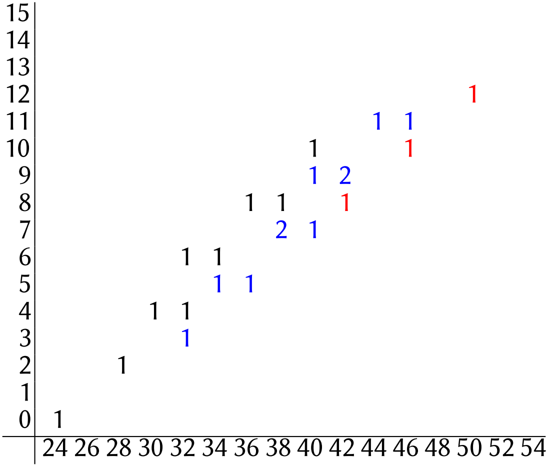

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10									1			1				
9									1	2						
8							1	1		1						
7								2	1							
6					1	1										
5						1	1									
4				1	1											
3					1											
2			1													
1																
0	1															

Expected: $\overline{\mathrm{H}}(T_{n,n+1}) \Rightarrow \overline{\mathrm{H}}_N(T_{n,n+1})$ is nontrivial if and only if $N < n$.

Background on knot Floer homology

Dunfield–Gukov–Rasmussen also conjecture $\exists\, d_0 \colon \overline{\mathrm{H}}(L) \to \overline{\mathrm{H}}(L)$ whose $(\underline{q}, \underline{t})$-bigraded homology is the reduced knot Floer homology of L.

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$$\Delta(T_{3,4}) = \begin{pmatrix} & & & & +1 & & & & \\ & -\underline{q}^{-4} & -\underline{q}^{-2} & & -1 & & -\underline{q}^{2} & -\underline{q}^{4} & \\ \underline{q}^{-6} & & & +\underline{q}^{-2} & +1 & +\underline{q}^{2} & & & +\underline{q}^{6} \end{pmatrix} = \underline{q}^{-6} - \underline{q}^{-4} + 1 - \underline{q}^{4} + \underline{q}^{6}$$

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$$\mathrm{HFK}(T_{3,4}) = \begin{array}{r|ccccccc} 0 & & & & & & & \mathbf{C}^1 \\ -1 & & & & & & \mathbf{C}^1 & \\ -2 & & & & \mathbf{C}^1 & & & \\ -3 & & & & & & & \\ -4 & & & & & & & \\ -5 & & \mathbf{C}^1 & & & & & \\ -6 & \mathbf{C}^1 & & & & & & \\ \hline & -6 & -4 & -2 & 0 & 2 & 4 & 6 \end{array}$$

Comparing $\overline{\mathrm{H}}(K)$ and $\mathrm{HFK}(K)$

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	−6	−4	−2	0	2	4	6
8				1			
7					1	1	
6							1
5			1	1			
4				1	1		
3		1					
2			1				
1							
0	1						

	−6	−4	−2	0	2	4	6
0							1
−1						1	
−2				1			
−3							
−4							
−5		1					
−6	1						

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2			1				
1							
0	1						

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0							1
-1						1	
-2				1			
-3							
-4							
-5		1					
-6	1						

If we instead set $\underline{a} = \underline{t}^{-1}$, we get

	-6	-4	-2	0	2	4	6
0							1
-1					1	1	
-2				11	1		
-3			1	1			
-4			1				
-5		1					
-6	1						

Comparing $\overline{\mathrm{H}}(K)$ and $\mathrm{HFK}(K)$

DGR's conjectural d_0 differential is of degree $\underline{a}^{-2}\underline{t}^{-3}$, and the $(\underline{q}, \underline{t})$-bigraded homology, after setting $\underline{a} = \underline{t}^{-1}$, is conjectured to be HFK.

Recall: d_N is of degree $\underline{a}^{-2}\underline{q}^{2N}\underline{t}^{-1}$ with grading collapse $\underline{a} = \underline{q}^N$ to get $\overline{\mathrm{H}}_N$.

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To better understand this, let's express these using Gorsky's change of variables $a = \underline{a}^2\underline{t} \qquad q = \underline{q}^2 \qquad t = \underline{q}^{-2}\underline{t}^{-2}$ from q, t-Catalan combinatorics.

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$$d_N \text{ is of degree } a^{-1}q^N t^0 \qquad d_0 \text{ is of degree } a^{-1}q^1 t^1$$

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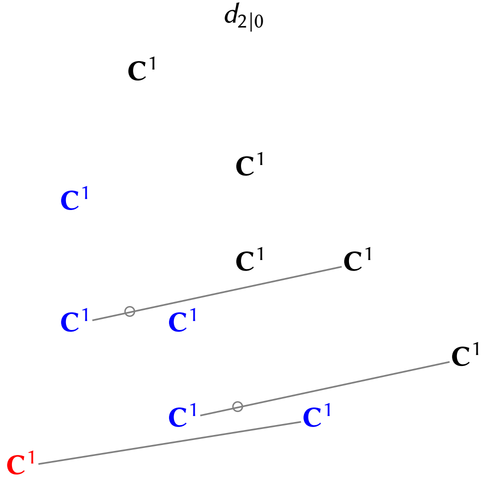
Gorsky–Gukov–Stošić '13 conjecture that there is a family of anticommuting differentials $d_{N|M}$ for all $\{(N, M) \in \mathbf{N}^2 \mid (N, M) \neq (0, 0)\}$ on $\overline{\mathrm{H}}(K)$ such that DGR's $d_N = d_{N|0}$ and $d_0 = d_{1|1}$.

$$d_{N|M} \text{ is of degree } a^{-1}q^N t^M$$

whose $(\underline{q}, \underline{t})$-bigraded homology with $\underline{a} = \underline{q}^{N-M}\underline{t}^{-M}$ is "$\mathfrak{gl}(N|M)$ homology", a conjectural categorification of the $\mathfrak{gl}(N|M)$ link polynomial = $\overline{P}\big|_{\underline{a}=\underline{q}^{N-M}}$.

The $d_{N|M}$ differentials

Differentials on $\overline{\mathrm{H}}(T_{3,4})$:



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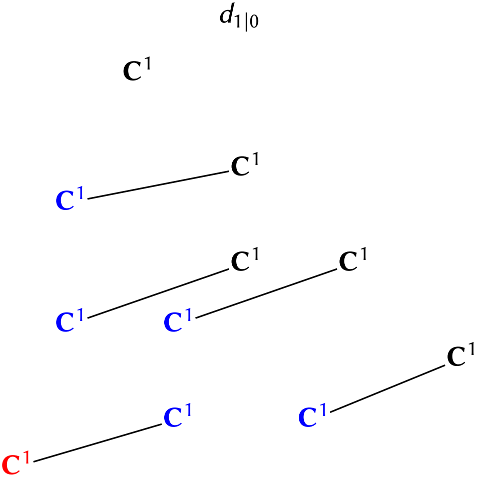
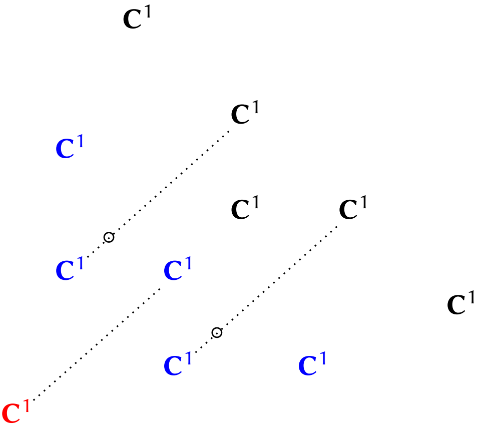


The $d_{N|M}$ differentials

Differentials on $\overline{\mathrm{H}}(T_{3,4})$:

$d_{1|1}$



The $d_{N|M}$ differentials

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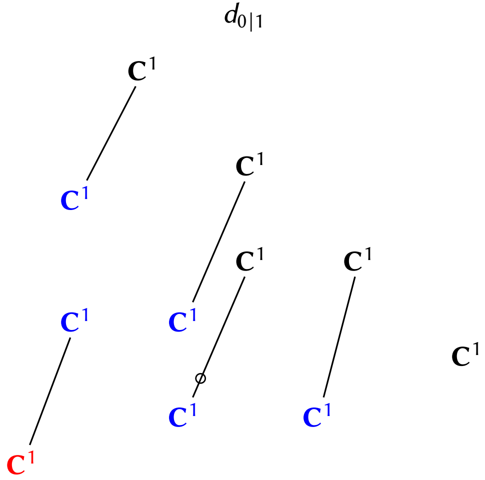
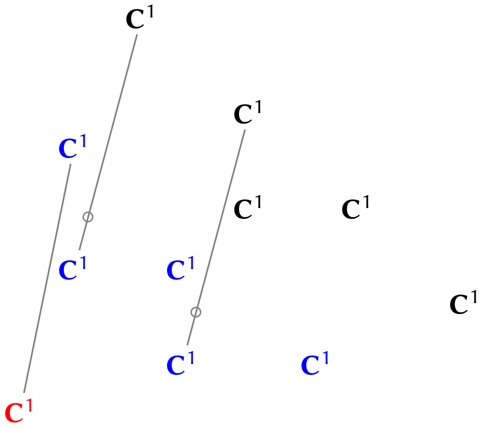


The $d_{N|M}$ differentials

Differentials on $\overline{\mathrm{H}}(T_{3,4})$:

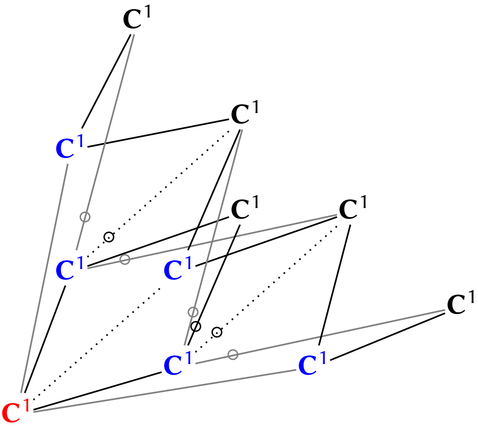
$d_{0|2}$



The $d_{N|M}$ differentials

Differentials on $\overline{\mathrm{H}}(T_{3,4})$:

all differentials



q, t-symmetry and q, t-unimodality

Recall that the bottom a-layer of $\overline{\mathrm{H}}(T_{n,n+1})$ is the q, t-Catalan number. For example, $(\underline{at})^{-12}\,\overline{\mathscr{P}}(T_{4,5})$ is

a^3

1						

a^2

1						
1	1					
1	1	1				
	1	1	1			

a^1

1						
1	1					
1	2	1				
	2	2	1			
	1	2	2	1		
			1	1	1	

a^0

1						
	1					
	1	1				
	1	1	1			
		1	1	1		
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						1

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1						
1	1					
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	2	2	1			
	1	2	2	1		
			1	1	1	

a^0

1						
	1					
	1	1				
	1	1	1			
		1	1	1		
			1	1	1	
						1

Along each antidiagonal, the numbers we see are palindromic and unimodal.

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1			
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a^1

1					
1	1				
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	2	2	1		
	1	2	2	1	
			1	1	1

a^0

1						
	1					
	1	1				
	1	1	1			
		1	1	1		
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a^1

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1	1					
1	2	1				
	2	2	1			
	1	2	2	1		
			1	1	1	

a^0

1						
	1					
	1	1				
	1	1	1			
		1	1	1		
			1	1	1	
						1

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Gorsky–Hogancamp–Mellit "Tautological classes and symmetry in Khovanov–Rozansky homology" construct an action of $\mathfrak{sl}_2 = \mathbf{C}\langle E, F, H\rangle$ on $\overline{\mathrm{H}}(K)$ such that $\deg(E) = qt^{-1}$ and $\deg(F) = q^{-1}t$.

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1						
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1						
1	1					
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	2	2	1			
	1	2	2	1		
			1	1	1	

a^0

1						
	1					
	1	1				
	1	1	1			
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						1

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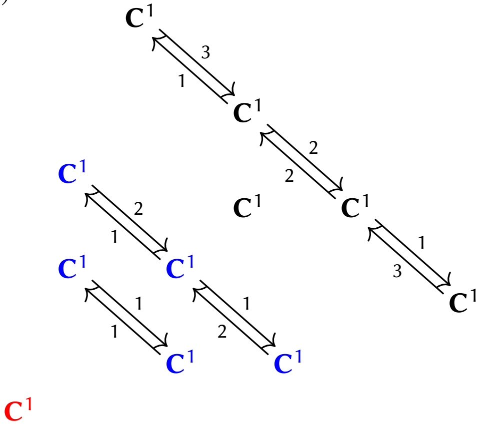
Corollary: q, t-symmetry and q, t-unimodality of $\overline{\mathrm{H}}(K)$.

q, t-symmetry and q, t-unimodality

E and F on $\overline{\mathrm{H}}(T_{3,4})$:



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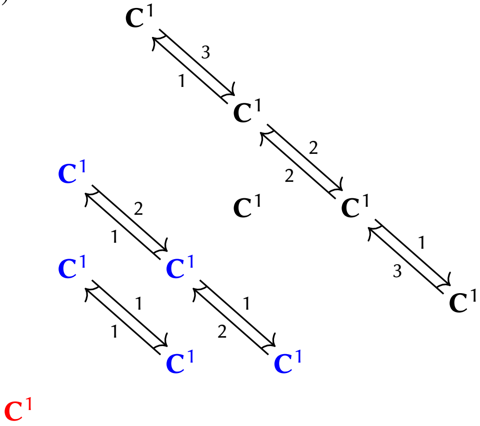

Gorsky–Chandler - "Structures in HOMFLY-PT homology" conjectures that $d_{N|M}$ and E, F, H should satisfy

$$[E, d_{N|M}] = M\, d_{N+1|M-1} \quad [F, d_{N|M}] = N\, d_{N-1|M+1} \quad [H, d_{N|M}] = (N-M)\, d_{N|M}$$

$\mathfrak{sl}_2$ action and $d_{N|M}$

A very highly structured space!