q, t-Catalan combinatorics and $T_{n,n+1}$

Joshua Wang

June 2026

Background on triply-graded homology

The HOMFLYPT polynomial is a rational function $\overline{P}(L) \in \mathbf{Z}(\underline{q}, \underline{a})$ for each oriented link L satisfying the skein relation

$$\underline{a}\overline{P}\Big($$  $$\Big) - \underline{a}^{-1}\overline{P}\Big($$  $$\Big) = (\underline{q} - \underline{q}^{-1})\overline{P}\Big($$  $$\Big)$$

and normalization $\overline{P}(\text{unknot}) = 1$.

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Torus knot examples:

$$\overline{P}(T_{2,3}) = \begin{pmatrix} -\underline{a}^4(1) \\ +\underline{a}^2(\underline{q}^{-2} + \underline{q}^2) \end{pmatrix} \qquad \overline{P}(T_{2,5}) = \begin{pmatrix} -\underline{a}^6(\underline{q}^{-2} + \underline{q}^2) \\ +\underline{a}^4(\underline{q}^{-4} + 1 + \underline{q}^4) \end{pmatrix}$$

$$\overline{P}(T_{3,4}) = \begin{pmatrix} \underline{a}^{10}\,(1) \\ -\underline{a}^8\,(\underline{q}^{-4} + \underline{q}^{-2} + 1 + \underline{q}^2 + \underline{q}^4) \\ +\underline{a}^6\,(\underline{q}^{-6} + \underline{q}^{-2} + 1 + \underline{q}^2 + \underline{q}^6) \end{pmatrix}$$

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$\overline{\mathrm{H}}(T_{3,4})$ viewed in "$\underline{a}$-layers" with $\underline{q}$ horizontal and $\underline{t}$ vertical:

8				$\mathbf{C}^1$			
7							
6							
5							
4							
3							
2							
1							
0							
$\underline{a}^{10}$	-6	-4	-2	0	2	4	6

8							
7					$\mathbf{C}^1$	$\mathbf{C}^1$	
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5			$\mathbf{C}^1$	$\mathbf{C}^1$			
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2							
1							
0							
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$$\overline{\mathscr{P}}(T_{3,4}) = \underline{a}^{10}(\underline{t}^8) + \underline{a}^8(\underline{q}^{-4}\underline{t}^3 + \underline{q}^{-2}\underline{t}^5 + \underline{t}^5 + \underline{q}^2\underline{t}^7 + \underline{q}^4\underline{t}^7) + \underline{a}^6(\underline{q}^{-6} + \underline{q}^{-2}\underline{t}^2 + \underline{t}^4 + \underline{q}^2\underline{t}^4 + \underline{q}^6\underline{t}^6)$$

$$\overline{P}(T_{3,4}) = \underline{a}^{10}(1) - \underline{a}^8(\underline{q}^{-4} + \underline{q}^{-2} + 1 + \underline{q}^2 + \underline{q}^4) + \underline{a}^6(\underline{q}^{-6} + \underline{q}^{-2} + 1 + \underline{q}^2 + \underline{q}^6)$$

q, t-Catalan combinatorics

Gorsky '10 conjectured a relationship between $\overline{\mathrm{H}}(T_{n,n+1})$ and q, t-Catalan combinatorics, later proved by Hogancamp '17.

Gorsky - "q, t-Catalan numbers and knot homology"
Hogancamp - "Khovanov-Rozansky homology and higher Catalan sequences"

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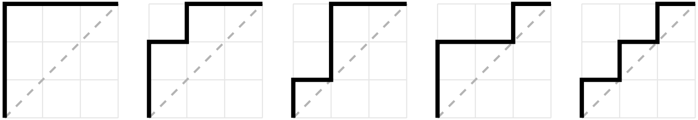
Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ count Dyck paths: paths from $(0, 0)$ to (n, n) staying above the diagonal $y = x$.



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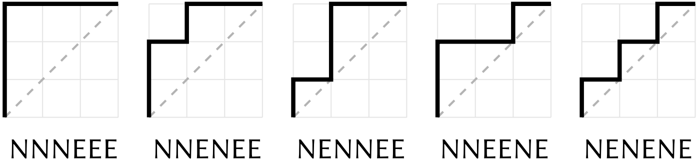

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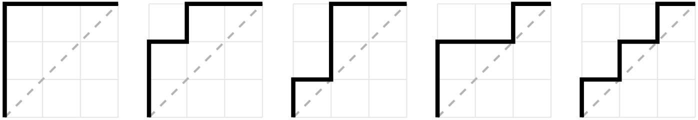

NNNEEE NNENEE NENNEE NNEENE NENENE

OEIS A000108: $C_1, C_2, \ldots = 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, \ldots$

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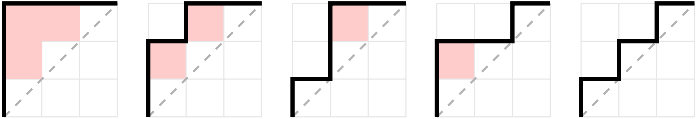
For a Dyck path π, let area(π) be the number of cells between π and the diagonal (drawn in red below).



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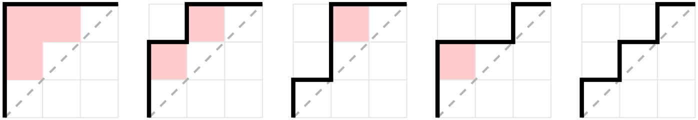
The cells above π form a Young diagram.



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For a Dyck path π, let area(π) be the number of cells between π and the diagonal (drawn in red below).

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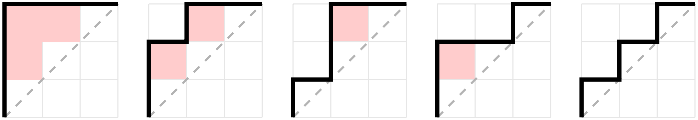

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For a Dyck path π, let $\text{area}(\pi)$ be the number of cells between π and the diagonal (drawn in red below).

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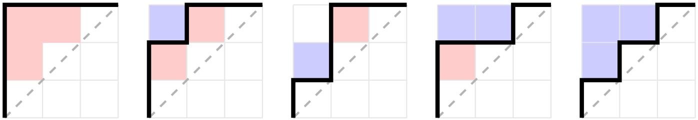

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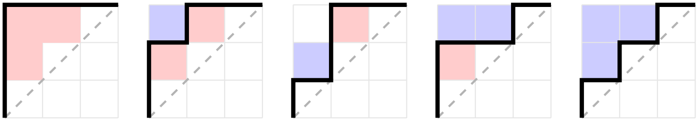
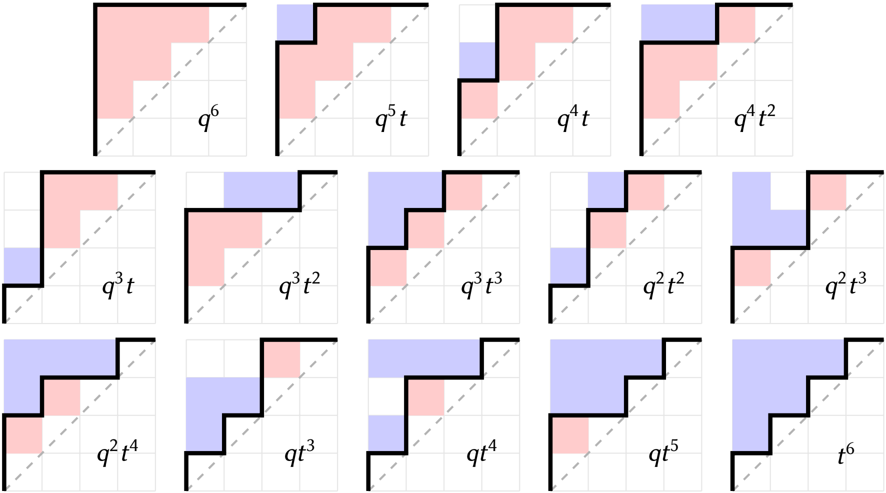


The q, t-Catalan number $C_n(q, t) \in \mathbf{Z}[q, t]$ is the weighted count

$$C_n(q, t) = \sum_{\text{Dyck paths } \pi} q^{\text{area}(\pi)} t^{\text{dinv}(\pi)} \qquad C_3(q, t) = q^3 + q^2 t + qt + qt^2 + t^3$$

q, t-Catalan combinatorics



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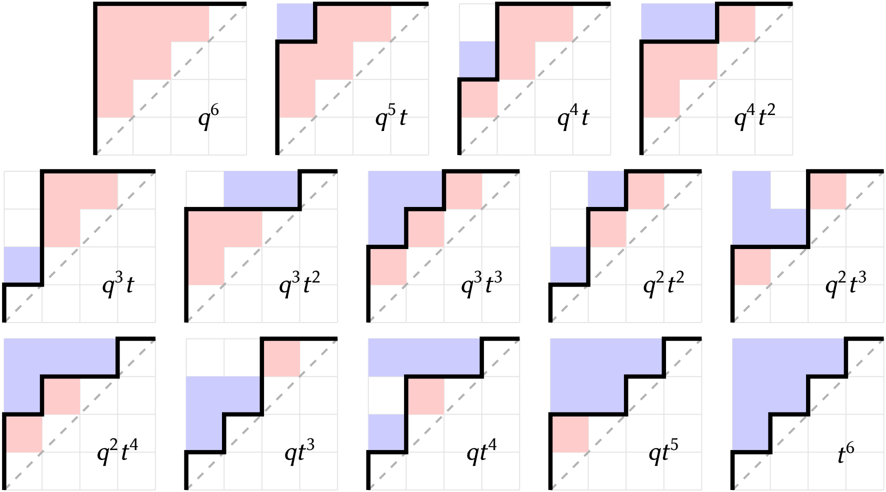

Originally defined by Garsia–Haiman '96 differently. This version is due to Haiman '00. Also see Haglund - "The q, t-Catalan Numbers and the Space of Diagonal Harmonics"

Notation from A. Garsia's "The Macdonald Polynomial Web Page":

$$C_3(q,t) = \begin{array}{|cccc} 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & & & 1 \\ \hline \end{array} = \begin{array}{cccc} t^3 & & & \\ & qt^2 & & \\ & qt & q^2t & \\ & & & q^3 \end{array}$$

$$C_4(q,t) = \begin{array}{|ccccccc} 1 & & & & & & \\ & 1 & & & & & \\ & 1 & 1 & & & & \\ & 1 & 1 & 1 & & & \\ & & 1 & 1 & 1 & & \\ & & & 1 & 1 & 1 & \\ & & & & & & 1 \\ \hline \end{array}$$

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$$C_5(q,t) = \left|\begin{array}{ccccccccccc} 1 & & & & & & & & & & \\ & 1 & & & & & & & & & \\ & 1 & 1 & & & & & & & & \\ & 1 & 1 & 1 & & & & & & & \\ & 1 & 2 & 1 & 1 & & & & & & \\ & & 1 & 2 & 1 & 1 & & & & & \\ & & 1 & 2 & 2 & 1 & 1 & & & & \\ & & & 1 & 2 & 2 & 1 & 1 & & & \\ & & & & 1 & 1 & 2 & 1 & 1 & & \\ & & & & & & 1 & 1 & 1 & 1 & \\ & & & & & & & & & & 1 \\ \hline \end{array}\right.$$

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Theorem (Haiman '98, Garsia–Haglund '01): $C_n(q,t)$ is q,t-symmetric and q,t-unimodal.

$\overline{\mathrm{H}}(T_{n,n+1})$ and q,t-Catalan combinatorics

Gorsky noticed that the bottom $\underline{a}$-layer of $\overline{\mathrm{H}}(T_{3,4})$ looks like $C_3(q,t)$.

8				$\mathbf{C}^1$			
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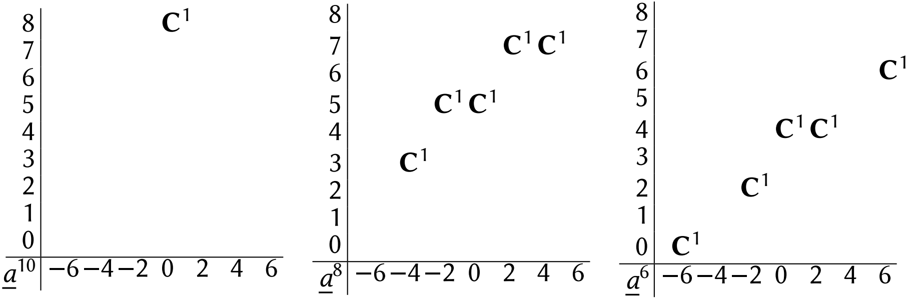

With $a = \underline{a}^2\underline{t}$ $\quad q = \underline{q}^2$ $\quad t = \underline{q}^{-2}\underline{t}^{-2}$, we have

$$\overline{\mathscr{P}}(T_{3,4}) = \underline{a}^{10}(\underline{t}^8) + \underline{a}^8(\underline{q}^{-4}\underline{t}^3 + \underline{q}^{-2}\underline{t}^5 + \underline{t}^5 + \underline{q}^2\underline{t}^7 + \underline{q}^4\underline{t}^7) + \underline{a}^6(\underline{q}^{-6} + \underline{q}^{-2}\underline{t}^2 + \underline{t}^4 + \underline{q}^2\underline{t}^4 + \underline{q}^6\underline{t}^6)$$

$$= \underline{a}^6\underline{t}^6\left[a^2(1) + a(t^2 + t + qt + q + q^2) + (t^3 + qt^2 + qt + q^2t + q^3)\right]$$

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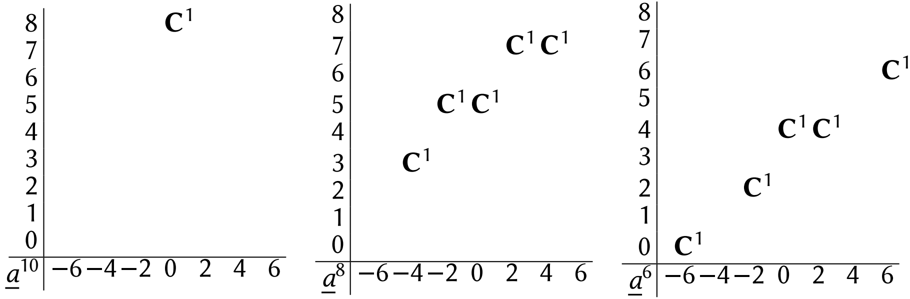

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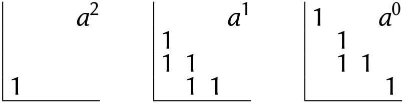
The q,t-polynomial for each a-layer:



$(\underline{at})^{-6}\,\overline{\mathscr{P}}(T_{3,4})$ is

			a^2
1			

			a^1
1			
1	1		
	1	1	

1			a^0
	1		
	1	1	
			1

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a^2

1			

a^1

1			
1	1		
	1	1	

a^0

1			
	1		
	1	1	
			1

$(\underline{at})^{-12}\,\overline{\mathscr{P}}(T_{4,5})$ is

a^3

1						

a^2

1						
1	1					
1	1	1				
	1	1	1			

a^1

1						
1	1					
1	2	1				
	2	2	1			
	1	2	2	1		
			1	1	1	

a^0

1						
	1					
	1	1				
	1	1	1			
		1	1	1		
			1	1	1	
						1

$\overline{\mathrm{H}}(T_{n,n+1})$ and q, t-Catalan combinatorics

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a^2

1			

a^1

1			
1	1		
	1	1	

a^0

1			
	1		
	1	1	
			1

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a^3

1						

a^2

1						
1	1					
1	1	1				
	1	1	1			

a^1

1						
1	1					
1	2	1				
	2	2	1			
	1	2	2	1		
			1	1	1	

a^0

1						
	1					
	1	1				
	1	1	1			
		1	1	1		
			1	1	1	
						1

In these two examples, the bottom a-layers are exactly $C_3(q,t)$ and $C_4(q,t)$.

$\overline{\mathrm{H}}(T_{n,n+1})$ and q,t-Catalan combinatorics

$(\underline{at})^{-6}\,\overline{\mathscr{P}}(T_{3,4})$ is

a^2

1			

a^1

1			
1	1		
	1	1	

a^0

1			
	1		
	1	1	
			1

$(\underline{at})^{-12}\,\overline{\mathscr{P}}(T_{4,5})$ is

a^3

1						

a^2

1						
1	1					
1	1	1				
	1	1	1			

a^1

1						
1	1					
1	2	1				
	2	2	1			
	1	2	2	1		
			1	1	1	

a^0

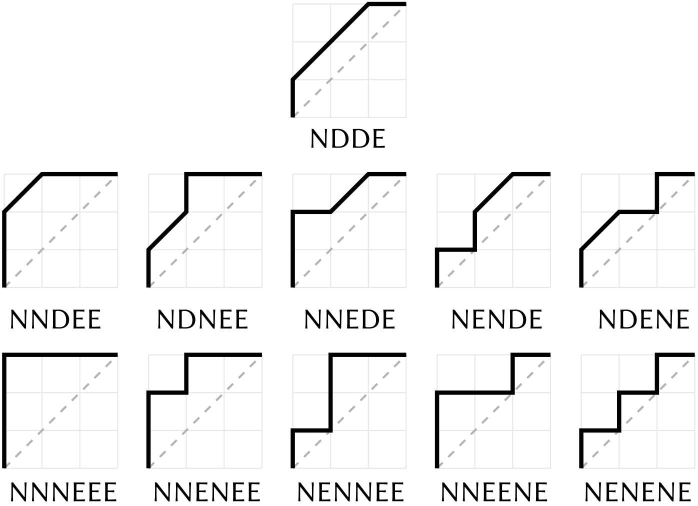
1						
	1					
	1	1				
	1	1	1			
		1	1	1		
			1	1	1	
						1

In these two examples, the bottom a-layers are exactly $C_3(q,t)$ and $C_4(q,t)$.

Gorsky conjectured that the bottom a-layer of $\overline{\mathrm{H}}(T_{n,n+1})$ is always q,t-Catalan number $C_n(q,t)$!

$\overline{\mathrm{H}}(T_{n,n+1})$ and q, t-Catalan combinatorics

Gorsky also conjectured that the higher a-degree q, t-polynomials are weighted counts of Schröder paths (the little q, t-Schröder numbers)



Schröder–Hipparchus numbers

If we set $q = 1 = t$, then $(\underline{at})^{-6}\,\overline{\mathscr{P}}(T_{3,4})\big|_{q=1=t} = a^2 + 5a + 5$

$$
\begin{array}{|cccc}
 & & & a^2 \\
 & & & \\
 & & & \\
1 & & & \\
\hline
\end{array}
\qquad
\begin{array}{|cccc}
 & & & a^1 \\
1 & & & \\
1 & 1 & & \\
 & 1 & 1 & \\
\hline
\end{array}
\qquad
\begin{array}{|cccc}
1 & & & a^0 \\
 & 1 & & \\
 & 1 & 1 & \\
 & & & 1 \\
\hline
\end{array}
$$

Schröder–Hipparchus numbers

If we set $q = 1 = t$, then $(\underline{at})^{-6}\,\overline{\mathscr{P}}(T_{3,4})\big|_{q=1=t} = a^2 + 5a + 5$

			a^2
1			

			a^1
1			
1	1		
	1	1	

1			a^0
	1		
	1	1	
			1

and $(\underline{at})^{-12}\,\overline{\mathrm{H}}(T_{4,5})\big|_{q=1=t} = a^3 + 9a^2 + 21a + 14$

						a^3
1						

						a^2
1						
1	1					
1	1	1				
	1	1	1			

						a^1
1						
1	1					
1	2	1				
	2	2	1			
	1	2	2	1		
			1	1	1	

1						a^0
	1					
	1	1				
	1	1	1			
		1	1	1		
			1	1	1	
						1

Schröder–Hipparchus numbers

If we set $q = 1 = t$, then $(\underline{at})^{-6}\,\overline{\mathscr{P}}(T_{3,4})\big|_{q=1=t} = a^2 + 5a + 5$

a^2:

1			

a^1:

1			
1	1		
	1	1	

a^0:

1			
	1		
	1	1	
			1

and $(\underline{at})^{-12}\,\overline{\mathrm{H}}(T_{4,5})\big|_{q=1=t} = a^3 + 9a^2 + 21a + 14$

a^3:

1						

a^2:

1						
1	1					
1	1	1				
	1	1	1			

a^1:

1						
1	1					
1	2	1				
	2	2	1			
	1	2	2	1		
			1	1	1	

a^0:

1						
	1					
	1	1				
	1	1	1			
		1	1	1		
			1	1	1	
						1

Gorsky's conjecture specializes to:

the coefficient of a^d in $(\underline{at})^{-n(n-1)}\overline{\mathscr{P}}(T_{n,n+1})\big|_{q=1=t}$

Schröder–Hipparchus numbers

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a^2

1			

a^1

1			
1	1		
	1	1	

a^0

1			
	1		
	1	1	
			1

and $(\underline{at})^{-12}\, \overline{\mathrm{H}}(T_{4,5})\big|_{q=1=t} = a^3 + 9a^2 + 21a + 14$

a^3

1						

a^2

1						
1	1					
1	1	1				
	1	1	1			

a^1

1						
1	1					
1	2	1				
	2	2	1			
	1	2	2	1		
			1	1	1	

a^0

1						
	1					
	1	1				
	1	1	1			
		1	1	1		
			1	1	1	
						1

Gorsky's conjecture specializes to:

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$=$ the number of Schröder paths in $[0, n]^2$ with d diagonal steps

Schröder–Hipparchus numbers

If we set $q = 1 = t$, then $(\underline{at})^{-6}\,\overline{\mathscr{P}}(T_{3,4})\big|_{q=1=t} = a^2 + 5a + 5$

$$
\begin{array}{|cccc}
 & & & a^2 \\
 & & & \\
 & & & \\
1 & & & \\
\hline
\end{array}
\qquad
\begin{array}{|cccc}
 & & & a^1 \\
1 & & & \\
1 & 1 & & \\
 & 1 & 1 & \\
\hline
\end{array}
\qquad
\begin{array}{|cccc}
1 & & & a^0 \\
 & 1 & & \\
 & 1 & 1 & \\
 & & & 1 \\
\hline
\end{array}
$$

and $(\underline{at})^{-12}\,\overline{\mathrm{H}}(T_{4,5})\big|_{q=1=t} = a^3 + 9a^2 + 21a + 14$

$$
\begin{array}{|ccccccc}
 & & & & & & a^3 \\
 & & & & & & \\
 & & & & & & \\
 & & & & & & \\
 & & & & & & \\
 & & & & & & \\
1 & & & & & & \\
\hline
\end{array}
\qquad
\begin{array}{|ccccccc}
 & & & & & & a^2 \\
 & & & & & & \\
 & & & & & & \\
1 & & & & & & \\
1 & 1 & & & & & \\
1 & 1 & 1 & & & & \\
 & 1 & 1 & 1 & & & \\
\hline
\end{array}
\qquad
\begin{array}{|ccccccc}
 & & & & & & a^1 \\
1 & & & & & & \\
1 & 1 & & & & & \\
1 & 2 & 1 & & & & \\
 & 2 & 2 & 1 & & & \\
 & 1 & 2 & 2 & 1 & & \\
 & & & 1 & 1 & 1 & \\
\hline
\end{array}
\qquad
\begin{array}{|ccccccc}
1 & & & & & & a^0 \\
 & 1 & & & & & \\
 & 1 & 1 & & & & \\
 & 1 & 1 & 1 & & & \\
 & & 1 & 1 & 1 & & \\
 & & & 1 & 1 & 1 & \\
 & & & & & & 1 \\
\hline
\end{array}
$$

Gorsky's conjecture specializes to:

the coefficient of a^d in $(\underline{at})^{-n(n-1)}\overline{\mathscr{P}}(T_{n,n+1})\big|_{q=1=t}$

= the number of Schröder paths in $[0, n]^2$ with d diagonal steps

= the number of d-dimensional facets of the associahedron K_{n+1}

Schröder–Hipparchus numbers

Total rank of a^d-layer of $(\underline{at})^{-n(n-1)}\overline{\mathrm{H}}(T_{n,n+1})$:

n \ d	7	6	5	4	3	2	1	0	$\dim \overline{\mathrm{H}}(T_{n,n+1})$
1								1	1
2							1	2	3
3						1	5	5	11
4					1	9	21	14	45
5				1	14	56	84	42	197
6			1	20	120	300	330	132	903
7		1	27	225	825	1485	1287	429	4279
8	1	35	385	1925	5005	7007	5005	1430	20793

Schröder–Hipparchus numbers

Total rank of a^d-layer of $(\underline{at})^{-n(n-1)}\overline{\mathrm{H}}(T_{n,n+1})$:

$n \backslash d$	7	6	5	4	3	2	1	0	$\dim \overline{\mathrm{H}}(T_{n,n+1})$
1								1	1
2							1	2	3
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5				1	14	56	84	42	197
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Total rank is the nth Hipparchus number (150 BCE).

Schröder–Hipparchus numbers

Total rank of a^d-layer of $(\underline{at})^{-n(n-1)}\overline{\mathrm{H}}(T_{n,n+1})$:

$n \backslash d$	7	6	5	4	3	2	1	0	$\dim \overline{\mathrm{H}}(T_{n,n+1})$
1								1	1
2							1	2	3
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5				1	14	56	84	42	197
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Total rank is the nth Hipparchus number (150 BCE).

Gorsky's conjecture was proved by Hogancamp '17 by showing that $\overline{\mathscr{P}}(T_{n,n+1})$ and the q, t-Schröder numbers satisfy the same recursion (a complicated recursion on a large superset).