Research Description

Automorphic forms and Hilbert's eleventh problem

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My research focuses on answering arithmetic questions about quadratic forms, and their connection to automorphic forms. These questions are pursued by studying both analytic and algebraic features of the associated automorphic forms.

Completed Work

Local Properties of Quadratic Forms

Paper [Ha1] establishes an exact mass formula for orthogonal groups with respect to a maximal lattice L over an arbitrary number field. This extends a result of Shimura [Sh4] which holds over totally real number fields, and is proved by reinterpreting his result in terms of the Tamagawa number and computing the relevant archimedian local factors. When a lattice is alone in its genus, this allows one to compute the volumes of the associated symmetric space quotients $\Gamma\backslash Z$ where Z is the symmetric space associated to the orthogonal group and Γ is the stabilizer of L. A formula of this kind is also useful in studying self-dual lattices, which are necessarily maximal. Using results of [Sh3], the mass of a general lattice can be obtained from this formula by computing the appropriate group indices, though in general this may be quite complicated.

Paper [GHY] is joint work with W. T. Gan and J. Yu in which we provide a more conceptual proof of Shimura's mass formulas [Sh3, Sh4, Sh5] for a maximal totally definite quadratic/hermitian lattice L over a totally real number field F of degree d. We observe that the stabilizer of a maximal lattice at a finite place is always a maximal parahoric subgroup of the associated orthogonal/unitary group G, so by [Gr] we may associate to it canonical local measures $|\omega_v|$ on G_v and associate to G a motive M_G of Artin-Tate type. Using these local measures, we may construct a Tamagawa measure μ on $G(\mathbb{A})$ for which by [GrG] we know

$$\int_{G(k)\backslash G(\mathbb{A})} \mu = \frac{1}{2^{ld}} L(M_G)\tau(G),$$

where $L(M_G)$ denotes the L-function associated to M_G evaluated at $s = 0$, $\tau(G)$ is the Tamagawa number of G, and l is the absolute rank of G over F. By Bruhat-Tits theory we may then relate this measure to the more natural measure μ' giving the stabilizer $Stab(L) \subset G(\mathbb{A})$ of L volume 1, for which the mass has the correspondingly simple expression

$$Mass(L) = \int_{G(k)\backslash G(\mathbb{A})} \mu'.$$

The local constants of proportionality relating μ_v and μ'_v are explicitly computed in terms of the number of positive roots over $\bar{F}$ and order over the residue field of certain reductive quotients of both G_v and the integral model H_v defined in [Gr, §4].

Global properties of Quadratic Forms

Paper [Ha2] describes some practical conditions for representability of a number m by a positive definite integral quadratic form Q in $n = 3$ or 4 variables. This allows one to practically compute which numbers are represented by Q, provided that when $n = 3$ we restrict ourselves to numbers within a fixed square class $t\mathbb{Z}^2$ which is not of exceptional-type. (The restriction to $m \in t\mathbb{Z}^2$ is due to the ineffective lower bound $L(1, \chi_t) \geq C_\varepsilon t^{-\varepsilon}$, which is intimately related to the possible existence of a Siegel zero for Dirichlet L-functions.)

This is proved by analyzing the theta function $\Theta_Q(z)$ as a sum of an Eisenstein series $E(z)$ and a cusp form $f(z)$, and computing explicit lower and upper bounds for the growth of the Eisenstein and cuspidal coefficients respectively. While the final results are for $\mathbb{Q}$, the necessary local computations here are done (at all primes) over a totally real number field in preparation for future work. The local factors are understood and computed using an explicit reduction procedure, which lends itself to quick computations.

I have also written extensive PARI/GP and Magma code implementing this algorithm, which reduces this problem to a routine (though often lengthy) computation. This is done in conjunction with the freely available program HECKE which computes the Fourier coefficients of a basis of eigenforms [St]. As an example of its usefulness, it is used to answer the long-standing conjecture that for $Q = x^2 + 3y^2 + 5z^2 + 7w^2$ the only locally represented numbers not represented by Q are 2 and 22. This result and the supporting code are important because they make effective and practical (for small discriminants) a result long-known to hold for sufficiently large m, allowing for concrete numerical applications (to [Bh] for example).

Paper [Ha3] answers the question of representability of a number by an integral definite ternary quadratic form Q within an exceptional-type square class $t\mathbb{Z}^2$. As a consequence we describe the nature of a local-global principle based on the spinor genus, showing that under certain circumstances there may be infinitely many numbers which are (locally) represented by the spinor genus of Q but are not (globally) represented by Q itself. (This was also independently established by Schulze-Pillot in [SP] using different methods.)

This result is also proved by analyzing $\Theta_Q(z)$, however within an exceptional-type square class $t\mathbb{Z}^2$, we have $\Theta_Q(z) = E(z) + H(z) + f(z)$ where both $H(z)$ and $f(z)$ are cuspidal but only $f(z)$ has cuspidal Shimura lift. The main term here comes from the sum $E(z) + H(z)$ and its size is controlled by a certain quadratic character ψ_t. By studying when the main term fails to beat the error term and using the non-negativity of the Fourier coefficients of $\Theta_Q(z)$, we obtain information about the coefficients of the weight 2 Shimura lift $g(z) := Shi(f, t)$. Finally, by considering the Galois representations associated to the Hecke eigenform components of $g(z)$, we determine that $g(z)$ is a sum of CM forms and pairs of sums or differences of cusp forms and their twists by ψ_t. This answers a question of Schulze-Pillot [SP] about the structure of $g(z)$, and gives the local-global principle above when translated in terms of $f(z)$.

Future Research Plans

In the following problems, we assume that the quadratic form being considered is integer-valued and totally positive definite.

Quadratic forms representing a given set of integers

For convenience, we will say that a quadratic form which represents all positive integers is **universal**. Knowledge of all universal quaternary quadratic forms with even cross terms is essential in the proof of the 15-theorem [Con, Sch, Bh], which states that all quadratic forms with even cross terms which represent the first 15 numbers necessarily represent all positive integers.

Using a similar approach to enumerate all quaternary quadratic forms (with no restriction on the cross terms), Bhargava has found a list of 6,436 forms which contains all possibly universal quaternary forms. However, contrary to the case of the 15-theorem, the arithmetic techniques used to prove universality of those forms only applies to about 4,000 of these possibilities. By applying the results of [Ha2], for each of these remaining forms Q, one can determine an explicit lower bound B such that all integers $m > B$ are represented by Q (provided m satisfies certain easily computable congruence conditions which ensure m is locally represented by Q). By checking the representability of all integers $m \leq B$, we can determine whether Q is universal. In this way, we can determine which of these remaining forms is universal, thereby establishing a version of the 15-theorem which applies to all quadratic forms without any restriction on their cross terms.

This approach was successfully carried out in [Ha2] to show that the form $Q_1 = x^2+3y^2+5z^2+7w^2$ represents all integers ≥ 0 except for 2 and 22. To do this quickly, we compute the representations of the auxilliary ternary form $Q_2 = x^2 + 3y^2 + 5z^2$, and use this to check representability by Q_1 by subtracting off several large values of $7x^2$. This idea reduces the computing time from about 73 million years to just over an hour, most of which is spent computing representations by Q_2. While there are about 2,500 forms involved in the generalization of the 15-theorem, they all arise as extensions of only about 20 ternary forms. By applying the idea above, it should be practical to check the representability of the remaining forms.

The methods of [Bh] apply equally well to show that for any subset $\mathbb{S} \subseteq \mathbb{N}$ there is a finite subset $\mathbb{S}_{\mathrm{f}} \subseteq \mathbb{S}$ which if represented by a quadratic form Q, guarantees that Q represents all of $\mathbb{S}$. Assuming that $\mathbb{S}$ contains enough small numbers (e.g., the set of primes), one can use the same procedure (as when $\mathbb{S} = \mathbb{N}$) to identify the finite subset $\mathbb{S}_{\mathrm{f}}$, thus proving a version of the 15-theorem for $\mathbb{S}$.

Representing integers by a quadratic form over a totally real F

In 1929 Tartakowski [T] proved by analytic methods that any given quadratic form Q in $n \geq 5$ variables represents every sufficiently large $m \in \mathbb{Z} \geq 0$ provided m is represented mod D for every D (i.e., provided m is locally represented by Q). This approach was brilliantly refined by Kloosterman [K] who was able to establish a similar result when $n = 4$, however the main term in this case is slightly more delicate and may require that m has a priori bounded divisibility at finitely many (anisotropic) primes. By avoiding finitely many exceptional-type square classes, a similar result was established when $n = 3$ in [Du-SP]. Given these results, two questions naturally come to mind:

Question A: Precisely how large m must be to ensure its representability by Q?

Question B: What numbers m are represented by Q when it has only 3 variables?

(When Q is a binary form the problem is purely arithmetical and so the question is less natural from this perspective.) In [Ha2] and [Ha3] we answer these two questions when Q is a positive definite quadratic form over $\mathbb{Z}$.

Given the success of this approach over $\mathbb{Z}$, it is natural to ask whether one can extend these techniques to the ring of integers $\mathcal{O}_F$ of a totally real number field F. In the language of modular forms, the passage from $\mathbb{Z}$ to $\mathcal{O}_F$ causes the theta function $\Theta_Q(z)$ to become a Hilbert modular form for some congruence subgroup of $SL_2(\mathcal{O}_F)$. Hilbert modular forms have been extensively studied, and one can hope to apply similar techniques to extend the results of [Ha2, Ha3] to this case. The main technical difference between the modular and Hilbert modular cases is that the latter involves an infinite group of (totally positive) units, and that the Fourier coefficients of a cusp form $f(z)$ are supported on the cone of totally positive numbers and have some transformation properties with respect to this unit group.

With this application in mind, the local computations in [Ha2] were done in the setting of a totally real number field. The other key ingredients in our approach are:

a) The Shimura lift – This has been generalized to the Hilbert modular setting by Shimura [Sh1, Sh2].
b) The existence of an associated Galois representation for even weight forms – This follows from the work of Taylor, Blasius, and Rogowski [Ta1, Ta2, B–R].
c) Subconvexity estimates for Hecke eigenvalues – This has been done at the good places by Brylinski and Labesse [Br–L].
d) Subconvexity estimates for the square-free Fourier coefficients – This was recently done by Cogdell, Piatetski-Shapiro, and Sarnak [C–PS–S].

Given these developments, it seems quite promising that this approach will lead to a complete answer to Question 1 when $n \geq 3$, thereby resolving the analytic part of the question posed by Hilbert in his eleventh problem. Beyond this, one can ask more generally about representations of one quadratic form by another. This problem has many similar features, and involves a detailed study of the associated Siegel modular theta functions, providing an interesting avenue for future work.

Analytic properties of L-functions of classical groups

Aside from their applications to quadratic forms, automorphic forms are interesting their own right. In particular, I am interested in establishing analytic properties of L-functions associated to classical groups via certain integral representations. In [Sh3], by a careful analysis of the doubling method, Shimura has shown (under certain assumptions) that the standard L-function of a cusp form on a unitary group is meromorphic and can precisely characterize its possible set of poles. I am interested in extending these methods to characterize the analyticity of other automorphic L-functions on classical groups.

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