AN EXACT MASS FORMULA FOR QUADRATIC FORMS OVER NUMBER FIELDS

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Abstract. In this paper we give an explicit formula for the mass of a quadratic form in $n \geq 3$ variables with respect to a maximal lattice over an arbitrary number field k, and use this to find the mass of many $\mathfrak{a}$-maximal lattices. We make the minor technical assumption that locally the determinant of the form is a unit up to a square if n is odd. The corresponding formula for k totally real was recently computed by Shimura [Shi].

§0 Summary

Our goal is to give an exact formula for the mass of the genus of a quadratic form φ on a maximal lattice defined over an arbitrary number field k. In §2 we explain how knowledge of the Tamagawa number of the special orthogonal group G^φ gives rise to a mass formula. Such a formula expresses the mass as a product of local factors over all places v of k, so our problem is reduced to computing each of these. For the non-archimedian places, these factors were recently computed by Shimura [Shi]. We state his result in §3 and for completeness include a translation between our language and his. In §4 we compute the archimedian factors, treating separately the 3 cases: v real, φ definite; v real, φ indefinite; and v complex. To define the factors in the last two cases, we choose a symmetric space $\mathfrak{Z}_v$ on which G_v^φ acts and a non-zero G_v^φ-invariant volume form $\omega_{\mathfrak{Z}}$. Finally, in §5 we compute the mass of φ with respect to a maximal lattice. We note that this formula agrees with Shimura's when k is totally real. In §6 we conclude by using the local similitude groups to show that this agrees with the mass of many genera of $\mathfrak{a}$-maximal lattices. Our results depend on several technical lemmas which we include as an appendix.

§1 Introduction

We begin with a quadratic space (V, φ) over an algebraic number field k. By this we mean a k-vector space V together with a non-degenerate quadratic form $\varphi : V \longrightarrow k$. Let O_k denote the ring of integers of k and let O_v denote the local ring of integers at each place v of k. We consider (V, φ) as well as its localizations (V_v, φ_v) given by linear extension of scalars to k_v. Given a lattice $\Lambda \subset (V, \varphi)$, we have the associated local lattice $\Lambda_\mathfrak{p} = \Lambda \otimes_{O_k} O_\mathfrak{p} \subset (V_\mathfrak{p}, \varphi_\mathfrak{p})$ at each non-archimedian

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place $\mathfrak{p}$ of k. We occasionally write (Λ, φ) for the restriction of the form φ to Λ, and $(\Lambda_\mathfrak{p}, \varphi_\mathfrak{p})$ for the restriction of $\varphi_\mathfrak{p}$ to $\Lambda_\mathfrak{p}$.

With (V, φ) as above, we let $G^\varphi = G(\varphi)$ be the special orthogonal group of (V, φ) by which we mean the group of determinant 1 invertible linear transformations of V preserving φ. We also define G^φ_v to be the special orthogonal group of (V_v, φ_v). Then we have a natural G^φ-action on (V, φ), and a natural G^φ_v-action on (V_v, φ_v). We say that two lattices $\Lambda, \Lambda' \subseteq (V, \varphi)$ are **globally equivalent** if there exists $g \in G^\varphi$ such that $\Lambda' = g\Lambda$, and **locally equivalent** if for each non-archimedian place v, there exists $g_v \in G^\varphi_v$ such that $\Lambda'_v = g_v \Lambda_v$. We define the **genus** of (Λ, φ) to be the set of all lattices locally equivalent to (Λ, φ), and say that the **classes** of (Λ, φ) are the global equivalence classes of (Λ, φ) in its genus.

Let $G^\varphi_\mathbf{A}$ be the adelization of G^φ, and let $G^\varphi_\mathbf{a}$ and $G^\varphi_\mathbf{h}$ be the product of G^φ_v over the archimedian and non-archimedian places respectively. Then there is a natural $G^\varphi_\mathbf{A}$-action on the space of lattices $\Lambda \subseteq (V, \varphi)$. To see this, take $g = (g_v) \in G^\varphi_\mathbf{A}$ and define $g\Lambda$ to be the lattice $\Lambda'' \subseteq (V, \varphi)$ such that $\Lambda''_v = g_v \Lambda_v$ for all non-archimedian places v. The stabilizer of a lattice (Λ, φ) defines a subgroup $D \in G^\varphi_\mathbf{A}$ such that $D \subset G^\varphi_\mathbf{a}$ and $D \cap G^\varphi_\mathbf{h}$ is open and compact, and by fixing a lattice (Λ, φ) we may parametrize the classes $\mathfrak{Cl}$ of Λ by the elements of $G^\varphi \backslash G^\varphi_\mathbf{A} / D$ using $a \mapsto \Lambda^a := a\Lambda$. We denote by Γ^a the group of **automorphisms** of (Λ^a, φ), defined as those $g \in G^\varphi$ leaving Λ^a invariant. From an adelic perspective,we see that $\Gamma^a = G^\varphi \cap aDa^{-1}$.

We say that a lattice $\Lambda \subseteq (V, \varphi)$ is **maximal** if $\varphi(\Lambda) \subseteq O_k$ and Λ is not properly contained in some lattice Λ' with $\varphi(\Lambda') \subseteq O_k$. There is a similar notion of an **$\mathfrak{a}$-maximal lattice** for any ideal $\mathfrak{a}$, given by replacing O_k by $\mathfrak{a}$. It turns out that for any ideal $\mathfrak{a}$, all of the $\mathfrak{a}$-maximal lattices in (V, φ) are locally equivalent (see [Shi2, Lemma 5.9]), so it makes sense to speak about the genus of $\mathfrak{a}$-maximal lattices.

If (Λ, φ) is a totally definite lattice over a totally real number field k, then we define the mass of its genus to be

$$\mathrm{Mass}(\Lambda, \varphi) = \sum_{a \in \mathfrak{Cl}} [\Gamma^a : 1]^{-1}.$$

If (Λ, φ) is not totally definite (e.g. when k is not totally real) then Γ^a will be an infinite group, but we would still like to somehow keep track of its size. To do this, we allow Γ^a to act on some symmetric space $\mathfrak{Z}$ and choose a measure on $\mathfrak{Z}$ invariant under this action. We then define the mass in terms of the measures of the quotients $\Gamma^a \backslash \mathfrak{Z}$. So in general, we define the **mass** of (Λ, φ) to be

$$\mathrm{Mass}(\Lambda, \varphi) = \sum_{a \in \mathfrak{Cl}} \nu(\Gamma^a), \tag{1.1}$$

where

$$\nu(\Gamma^a) = \begin{cases} [\Gamma^a : 1]^{-1} & \text{if } G_\mathbf{a} \text{ is compact,} \\ [\Gamma^a \cap \{\pm 1\} : 1]^{-1} \mathrm{vol}(\Gamma^a \backslash \mathfrak{Z}) & \text{otherwise.} \end{cases}$$

Our main interest in this paper will be to find an exact formula for the mass of the genus of maximal lattices over an arbitrary number field k when $n \geq 3$. Our approach is to use the Tamagawa number of G^φ to extend Shimura's computation of the mass of a maximal lattice over a totally real number field to a general number field k. Then by interpreting the mass in terms of the volume of the non-archimedian stabilizer of (Λ, φ), we use the local group of similitudes of φ to show that the mass is unchanged as we vary over certain genera of $\mathfrak{a}$-maximal lattices.

This exact mass formula essentially expresses the mass as a product of even integer values of the Dedekind zeta function of k, a power of the index of Λ in its dual lattice, and some gamma function factors. If $\dim_k(V)$ is even, a special value of the L-function of a certain quadratic extension of k also appears.

Summary of Notation

Throughout this paper we take k to be a number field, O_k its ring of integers, and D_k the discriminant of $k/\mathbb{Q}$. We denote by v a valuation (or place) of k. We also let $\mathbf{a}$ and $\mathbf{h}$ denote the archimedian and non-archimedian places of k respectively. Suppose $\mathfrak{p}$ is a prime ideal in O_k lying over the prime p in $\mathbb{Z}$, and $x \in k$. We let $|x|_\mathfrak{p}$ denote the usual $\mathfrak{p}$-adic absolute value of x defined by $|x|_\mathfrak{p} = q^{-\mathrm{ord}_\mathfrak{p}(x)}$, where we take $q = q_\mathfrak{p} = [O_\mathfrak{p} : \mathfrak{p}]$.

We follow the convention that if we have an object R defined at a certain valuation v, we denote it by R_v. If R_v is defined at each of the archimedian valuations, we also write

$$R_\mathbf{a} = \prod_{v \in \mathbf{a}} R_v.$$

For an algebraic group G defined over k, we denote the adelization of G by $G_\mathbf{A}$.

If R is an arbitrary set, we denote by R_n^m the $m \times n$ matrices with coefficients in R. We write the transpose of a matrix A as ${}^tA$. If x is a matrix, then we let x_{ij} denote the entry of x in the i^{th} row and j^{th} column. Conversely given numbers x_{ij}, we let (x_{ij}) denote the matrix whose entries satisfy $(x_{ij})_{ij} = x_{ij}$. We abbreviate the diagonal matrix

$$\begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & a_{nn} \end{pmatrix}$$

by $\mathrm{diag}[a_{11}, \cdots, a_{nn}]$, and denote the $n \times n$ identity matrix by 1_n. Given an arbitrary $n \times n$ matrix A and an integer l with $1 \leq l \leq n$, we define $\det_l(A)$ to be the determinant of the upper left $l \times l$ submatrix of A. If A is a matrix of functions, we define the matrix of 1-forms $dA = (dA_{ij})$. Given two $n \times n$ matrices A and B over $\mathbb{R}$, we say that $A > B$ if the matrix $A - B$ is positive definite, and we set

$$S_+^n = \{A \in \mathbb{R}_n^n \mid {}^tA = A > 0\}.$$

We let (V, φ) denote a non-degenerate quadratic space of dimension n over k, and take $V_v, \Lambda_\mathfrak{p}, G^\varphi, G_v^\varphi, G_\mathbf{A}^\varphi$ as defined in the introduction. If we choose a basis $\{v_1, \cdots, v_n\}$ for V, we may express the bilinear form $\varphi(v, w)$ associated to φ as the matrix $\psi = [\varphi(v_i, v_j)]_{1 \leq i,j \leq n}$. We also let $G^-(\varphi)$ denote the set of invertible linear transformations of V which preserve the form φ and have determinant -1.

For convenience, we define the symbols

$$X = k_n^n, \quad T = \ \{\text{Symmetric } n \times n \text{ matrices with coefficients in } k\}\ ,$$

and their local counterparts T_v, and X_v at a valuation v by replacing k by k_v in the above definition.

We set $i = \sqrt{-1} \in \mathbb{C}$. For $x \in \mathbb{R}$ we let $\lfloor x \rfloor$ be the greatest integer $\leq x$. Also, when there is no danger of confusion, we freely use the letters i, j, k, l as indices. Our equations and statements are numbered first by section, then by order within each section, with the appendix labeled by A (e.g. Lemma A2).

§2 The Tamagawa Number and Local Factors

The main fact that we use in what follows is that the Tamagawa number τ of the special orthogonal group $G = G^{\varphi}$ over any number field k is given by

$$\tau(G) = 2 \qquad \text{if } n \geq 3, \tag{2.1}$$

where $n = \dim_k(V)$. To define this, we first choose a measure $(dx)_{\mathbf{A}}$ on $k_{\mathbf{A}}$ normalized so that

$$\int_{k \backslash k_{\mathbf{A}}} (dx)_{\mathbf{A}} = 1. \tag{2.2}$$

We then define the **Tamagawa number** of G to be

$$\tau(G) = \int_{G \backslash G_{\mathbf{A}}} |\omega_G|_{\mathbf{A}}, \tag{2.3}$$

where ω_G is a non-zero left G-invariant differential form on G of highest degree and $|\omega_G|_{\mathbf{A}}$ is the volume element defined with respect to $(dx)_{\mathbf{A}}$. By the product formula we see $|c\omega_G|_{\mathbf{A}} = |\omega_G|_{\mathbf{A}}$ for $c \in k^{\times}$, and since ω_G is chosen from a 1 dimensional space, this specifies a left G-invariant measure on $G_{\mathbf{A}}$ which is independent of our choice of ω_G. We call the measure associated to ω_G the **Tamagawa measure** on $G_{\mathbf{A}}$. (For a more detailed introduction, see [Tam], [Vos], or [Weil].)

From now on when speaking of an invariant object, we always understand this to mean it is left invariant. For clarity we also define a **volume form** to be a nowhere zero diferential form of highest degree.

For our computations, it will be useful to define another measure $(d'x)_{\mathbf{A}}$ by the restricted product $(d'x)_{\mathbf{A}} = \prod'_v (d'x)_v$ with local measures

$$(d'x)_v = \begin{cases} \text{Haar measure on } k_v \text{ normalized by } \int_{O_{\mathfrak{p}}} (d'x)_v = 1 & \text{if } k_v = k_{\mathfrak{p}}, \\ \text{Lebesgue measure on } \mathbb{R} & \text{if } k_v = \mathbb{R}, \\ i dz \wedge d\bar{z} = 2 \times \text{Lebesgue measure on } \mathbb{R}^2 & \text{if } k_v = \mathbb{C}. \end{cases}$$

This gives $\int_{k \backslash k_{\mathbf{A}}} (d'x)_{\mathbf{A}} = |D_k|^{1/2}$, so in terms of $(d'x)_{\mathbf{A}}$ we have

$$\begin{aligned} \tau(G) &= |D_k|^{\frac{-\dim_k(G)}{2}} \int_{G \backslash G_{\mathbf{A}}} |\omega_G|'_{\mathbf{A}} \\ &= |D_k|^{\frac{-n(n-1)}{4}} \int_{G \backslash G_{\mathbf{A}}} |\omega_G|'_{\mathbf{A}}, \end{aligned} \tag{2.4}$$

where $|\omega_G|'_{\mathbf{A}}$ is the volume element derived from ω_G using $(d'x)_{\mathbf{A}}$ instead of $(dx)_{\mathbf{A}}$.

We now give a general procedure for constructing a suitable invariant volume form $\widetilde{\omega_G}$ on G. By choosing a global basis $\{v_1, \cdots, v_n\}$ for (V, φ) we can represent the bilinear form associated to φ as a matrix ψ. This gives a natural map

$$X = (k)_n^n \xrightarrow{\mathcal{F}} T \qquad x \longmapsto {}^t x \psi x, \tag{2.5}$$

whose fibre over the matrix $\psi \in T$ is the full orthogonal group of φ. Given the volume forms

$$\omega_X = \bigwedge_{i,j} dx_{ij}, \qquad \omega_T = \bigwedge_{i \leq j} dt_{ij} \tag{2.6}$$

on X and T respectively, we can find a differential form ω on X such that

$$\omega_X = \mathcal{F}^*(\omega_T) \wedge \omega. \tag{2.7}$$

By pulling ω back to the fibre and then restricting to the identity component, we get a form $\widetilde{\omega_G}$ on G. From Lemma A6, we see that $\widetilde{\omega_G}$ is a non-zero G-invariant volume form, and is independent of our choice of ω. We will use this construction many times in our calculation, and consistently identify $G = G^\varphi = G^\psi$ as well as the image of Λ under this identification.

For each place v of k, we define the local representation density

$$\beta_v(\psi) = \beta_v(\Lambda, \psi) = \frac{1}{2} \lim_{U \to \psi_v} \frac{\int_{U'} dX}{\int_U dT}, \tag{2.8}$$

where $dX = \prod_{i,j} (dx_{ij})_v$ and $dT = \prod_{i \leq j} (dt_{ij})_v$ are the measures associated to ω_X and ω_T in these coordinates,

$$U' = \begin{cases} \mathcal{F}^{-1}(U) & \text{if } v \in \mathbf{a}, \\ \mathcal{F}^{-1}(U) \cap \{x \in X_v \mid x\Lambda_v = \Lambda_v\} & \text{if } v \in \mathbf{h}, \end{cases}$$

and U is an open neighborhood of ψ_v in T_v. From the construction of $\widetilde{\omega_G}$ above, we can easily see that $\int_{D_v} \widetilde{\omega_G} = \beta_v(\Lambda, \psi)$ where $D \subset G_{\mathbf{A}}$ is the stabilizer of Λ (see [Tam, §6, pp119–120]). In our calculations the lattice Λ will be fixed, so we will often supress Λ and write $\beta_v(\psi)$.

Remark. Notice that both the volume form $\widetilde{\omega_G}$ and the local densities $\beta_v(\psi)$ depend not only on (V, φ) and v, but also on our given choice of basis for (V, φ).

Any choice of volume form ω_G can be used to define an archimedean measure $\tau_{\mathbf{a}}$ on $G_{\mathbf{a}}$ by $\prod_{v \in \mathbf{a}} |\omega_G|'_v$. By choosing $\omega_G = \widetilde{\omega_G}$ as above and expressing (2.4) in terms of local measures, one can prove:

Theorem 2.1. *Let Λ be a lattice in (V, φ), and suppose ψ a matrix representing φ in some global basis for V. Then*

$$\sum_{a \in \mathfrak{Cl}} \tau_{\mathbf{a}}(\Gamma^a \backslash G_{\mathbf{a}}) = \tau(G)\, |D_k|^{\frac{n(n-1)}{4}} \prod_{v \in \mathbf{h}} \beta_v(\Lambda, \psi)^{-1},$$

with $\tau_{\mathbf{a}}$ and $\beta_v(\Lambda, \psi)$ as above, and Γ^a as defined in §1.

Proof. This is proved in [Cas, pp380-382] when $k = \mathbb{Q}$, but the argument there works for any number field k. In his notation, $\beta_v(\Lambda, \psi) = \lambda_v = \tau_v(O^+(\Lambda_v))$ and (due to a typographical error) the right side of (4.19) on p382 should read $2\lambda_\infty^{-1} \prod_{p \neq \infty} \lambda_p^{-1}$. See also [Tam, §6, pp119–120] and [Vos, §15, pp87–88]. □

To simplify our calculations, we use the invertible matrix $\sigma_v \in (k_v)_n^n$ to change basis locally at every place v, so that ψ_v has the standard form

$$\phi_v = {}^t\sigma_v \psi_v \sigma_v = \begin{cases} \begin{bmatrix} 0 & 0 & 2^{-1}1_r \\ 0 & \theta_{\mathfrak{p}} & 0 \\ 2^{-1}1_r & 0 & 0 \end{bmatrix} & \text{if } k_v = k_{\mathfrak{p}}, \\ \begin{bmatrix} 1_q & 0 \\ 0 & -1_r \end{bmatrix} & \text{if } k_v = \mathbb{R}, \\ 1_n & \text{if } k_v = \mathbb{C}, \end{cases} \tag{2.9}$$

with $q, r \in \mathbb{N}$ satisfying either $q + r = n$ and $q \geq r$, or $\dim(\theta_{\mathfrak{p}}) + 2r = n$ and $\theta_{\mathfrak{p}}$ is some anisotropic symmetric matrix with $\dim(\theta_{\mathfrak{p}}) \leq 4$. Since we take Λ to be a maximal lattice, by [Shi2, Lemma 5.6], we can locally choose a free $O_{\mathfrak{p}}$-basis for $\Lambda_{\mathfrak{p}}$ so that $(\Lambda_{\mathfrak{p}}, \varphi_{\mathfrak{p}})$ is represented by the matrix $\phi_{\mathfrak{p}}$ above, and we choose the matrices $\sigma_{\mathfrak{p}}$ so this is true. The following lemma describes how the local factors change under such a change of basis.

Lemma 2.2. *Let v be a place of k and suppose that ψ and ψ' in $(k_v)_n^n$ are related by $\psi' = {}^tA\psi A$ for some invertible $n \times n$ matrix A. Then*

$$\beta_v(\Lambda, \psi') = |\det(A)|_v^{n+1}\, \beta_v(A\Lambda, \psi).$$

Proof. For $n \times n$ matrices $A \in X$ and $t \in T$ we let $[A] : T \longrightarrow T$ denote the map $[A](t) = {}^tAtA$, which corresponds to change of basis by A for the quadratic form associated to t.

Fix an open set U about ψ' in T, and let $V = [A^{-1}](U)$ be the corresponding neighborhood of ψ. Then one can easily check

$$\frac{\mathrm{vol}_X(\mathcal{F}_{\psi'}^{-1}(U))}{\mathrm{vol}_T(U)} \cdot \frac{\mathrm{vol}_T(U)}{\mathrm{vol}_T([A^{-1}](U))} = \frac{\mathrm{vol}_X(A^{-1}\mathcal{F}_{\psi}^{-1}(V)A)}{\mathrm{vol}_T(V)} = \frac{\mathrm{vol}_X(\mathcal{F}_{\psi}^{-1}(V))}{\mathrm{vol}_T(V)},$$

where $\mathcal{F}$ is as in (2.5) and the last equality follows from both parts of Lemma A2.

By passing to the limit as $U \to \psi'$, we have

$$\beta_v(\Lambda, \psi') = \lim_{U \to \psi'} \frac{\mathrm{vol}_T([A^{-1}](U))}{\mathrm{vol}_T(U)} \beta_v(A\Lambda, \psi).$$

This ratio of volumes is given by computing the pull-back of the volume form ω_T under the map $[A]$. We claim that

$$[A]^*(\omega_T) = \det(A)^{n+1}\omega_T,$$

which is to say

$$\bigwedge_{i\leq j} d({}^tAtA)_{ij} = \det(A)^{n+1} \bigwedge_{i\leq j} dt_{ij}. \tag{2.10}$$

Since $[AB] = [B][A]$, we already know (2.10) is true if we replace $\det(A)^{n+1}$ by some multiplicative character on $GL_n(k_v)$. By construction $c(A)$ is a polynomial in the entries of A, and since the only continuous characters on GL_n are powers of the determinant, we easily verify (2.10) by checking the scalar matrices $A = \lambda \cdot 1_n$.

With this we have

$$\lim_{U\to\psi'} \frac{\mathrm{vol}_T([A]^{-1}(U))}{\mathrm{vol}_T(U)} = |\det(A)|_v^{n+1},$$

which proves our lemma. $\square$

§3 The Non-Archimedian Local Factors

The non-archmedian local factors appearing in the mass formula for a maximal lattice Λ have been calculated by Shimura in [Shi], under the condition that locally the determinant of φ is a unit up to a square if n is odd. We now show how his local factors relate to the local factors $\beta_\mathfrak{p}(\Lambda_\mathfrak{p}, \phi_\mathfrak{p})$ appearing in our mass formula.

Fix a basis $\{v_1, \cdots, v_n\}$ for $V_\mathfrak{p}$, let ϕ be the invertible $n \times n$ matrix defined over $k_\mathfrak{p}$ which represents $(V_\mathfrak{p}, \varphi_\mathfrak{p})$ in this basis, and let $\Lambda_\mathfrak{p}$ be a lattice in $(V_\mathfrak{p}, \varphi_\mathfrak{p})$. We define $\beta_\mathfrak{p}(\phi)$ as in §2 to be the limit of the ratio of volumes

$$\beta_\mathfrak{p}(\phi) = \beta_\mathfrak{p}(\Lambda_\mathfrak{p}, \phi) = \frac{1}{2} \lim_{U\to\phi} \frac{\int_{U'} dX}{\int_U dT}, \tag{3.1}$$

where U' is a neighborhood in $X_\mathfrak{p}$ determined by $\Lambda_\mathfrak{p}$ and an open neighborhood U of ϕ in $T_\mathfrak{p}$. We may also write U' as $U'(\phi)$ to emphasize its dependence on the matrix ϕ. Since we are working over a $\mathfrak{p}$-adic field, we have a natural choice of neighborhoods U_i to use for this limit, namely $U_i = \phi + P_i$ where $P_i = (\mathfrak{p}^i)_n^n \cap T_\mathfrak{p}$.

Lemma 3.1. *Let $\Lambda_\mathfrak{p}$ and ϕ be as above, and let $c \in k_\mathfrak{p}^\times$. Then we have*

$$\beta_\mathfrak{p}(\Lambda_\mathfrak{p}, \phi) = |c|_\mathfrak{p}^{\frac{n(n+1)}{2}} \beta_\mathfrak{p}(\Lambda_\mathfrak{p}, c\phi) = |\det(c \cdot 1_n)|_\mathfrak{p}^{\frac{(n+1)}{2}} \beta_\mathfrak{p}(\Lambda_\mathfrak{p}, c\phi).$$

Proof. Since $U \to \phi$, it suffices to compute the limit (3.1) for $U = U_i$. Consider the pre-images

$$U_i'(\phi) = \{x \in X_\mathfrak{p} \mid {}^tx\phi x \in \phi + P_i \text{ and } x\Lambda_\mathfrak{p} = \Lambda_\mathfrak{p}\},$$

and notice $U_i'(\phi) = U'_{i+\mathrm{ord}_\mathfrak{p}(c)}(c\phi)$. Using this we have

$$\begin{aligned}
\beta_\mathfrak{p}(\phi) &= \frac{1}{2} \lim_{i\to\infty} \frac{\int_{U_i'(\phi)} dX}{\int_{U_i} dT} \\
&= \frac{1}{2} \lim_{i\to\infty} \frac{\int_{U'_{i+\mathrm{ord}_\mathfrak{p}(c)}(c\phi)} dX}{\int_{U_i} dT} \\
&= |c|_\mathfrak{p}^{\frac{n(n+1)}{2}} \frac{1}{2} \lim_{i\to\infty} \frac{\int_{U'_{i+\mathrm{ord}_\mathfrak{p}(c)}(c\phi)} dX}{\int_{U_{i+\mathrm{ord}_\mathfrak{p}(c)}} dT} \\
&= |c|_\mathfrak{p}^{\frac{n(n+1)}{2}} \beta_\mathfrak{p}(c\phi),
\end{aligned}$$

which completes the proof. $\square$

Lemma 3.2. *Let $\Lambda_\mathfrak{p}$ and ϕ be as above, and suppose that for our choice of basis we have $\Lambda_\mathfrak{p} = \sum_{i=1}^n O_\mathfrak{p} v_i$. Then $\beta_\mathfrak{p}(\Lambda_\mathfrak{p}, \phi) = \frac{1}{2} e_\mathfrak{p}(\phi)$, where $e_\mathfrak{p}(\phi)$ is as in* [Shi, §8].

Proof. In [Shi, §8] $e_\mathfrak{p}(\phi)$ is given by

$$e_\mathfrak{p}(\phi) = \lim_{i\to\infty} q^{\frac{-n(n-1)}{2}} N_i',$$

where $N_i' = \#\{x \in (O_\mathfrak{p}/\mathfrak{p}^i O_\mathfrak{p})_n^n \mid {}^t x\phi x \equiv \phi \mod P_i\}$. However, U_i is a sum of cosets mod P_i and one can check that U_i' is a sum of cosets mod $(\mathfrak{p})_n^n$, so by counting them we have

$$\beta_\mathfrak{p}(\psi) = \frac{1}{2} \lim_{i\to\infty} \frac{\int_{U_i'} dX}{\int_{U_i} dT} = \frac{1}{2} \lim_{i\to\infty} \frac{\left(\frac{1}{q^i}\right)^{n^2} N_i'}{\left(\frac{1}{q^i}\right)^{\frac{n(n+1)}{2}}} = \frac{1}{2} e_\mathfrak{p}(\psi),$$

which proves the lemma. $\square$

We are interested in computing $\beta_\mathfrak{p}(\phi_\mathfrak{p})$ with respect to a maximal lattice $\Lambda_\mathfrak{p}$ in $(V_\mathfrak{p}, \varphi_\mathfrak{p})$, with $\phi_\mathfrak{p}$ as in §2. By Lemmas 3.1 and 3.2 we know

$$\beta_\mathfrak{p}(\phi_\mathfrak{p}) = |\det(2 \cdot 1_n)|_\mathfrak{p}^{\frac{n+1}{2}} \frac{e_\mathfrak{p}(2\phi_\mathfrak{p})}{2}, \tag{3.2}$$

and by combining this with [Shi; Theorem 8.6(3), Prop. 3.9, (3.1.9)], we obtain

$$\beta_\mathfrak{p}(\phi_\mathfrak{p}) = |\det(2 \cdot 1_n)|_\mathfrak{p}^{\frac{n+1}{2}} q^{\kappa_\mathfrak{p} n} [\widetilde{\Lambda_\mathfrak{p}} : \Lambda_\mathfrak{p}] \xi, \tag{3.3}$$

where $q = \#(O_\mathfrak{p}/\mathfrak{p}O_\mathfrak{p})$, κ is defined by $2O_\mathfrak{p} = \mathfrak{p}^\kappa$,

$$\xi = \begin{cases} (1-q^{-m}) \prod_{i=1}^{m-1} (1-q^{-2i}) & \text{if } t = 0, \\ \prod_{i=1}^{m} (1-q^{-2i}) & \text{if } t = 1, \\ (1+q^{-m}) \prod_{i=1}^{m-1} (1-q^{-2i}) & \text{if } t = 2, \mathfrak{p} \text{ is unramified in } K, \\ & \quad \text{and } \widetilde{\Lambda_\mathfrak{p}} = \Lambda_\mathfrak{p}, \\ 2(1+q)(1+q^{1-m})^{-1} \prod_{i=1}^{m-1} (1-q^{-2i}) & \text{if } t = 2, \mathfrak{p} \text{ is unramified in } K, \\ & \quad \text{and } \widetilde{\Lambda_\mathfrak{p}} \neq \Lambda_\mathfrak{p}, \\ 2 \prod_{i=1}^{m-1} (1-q^{-2i}) & \text{if } t = 2, \text{and } \mathfrak{p} \text{ is ramified in } K, \\ 2(1+q) \prod_{i=1}^{m-1} (1-q^{-2i}) & \text{if } t = 3, \\ 2(1+q)(1-q^{1-m})^{-1} \prod_{i=1}^{m-1} (1-q^{-2i}) & \text{if } t = 4, \end{cases}$$

with $t = \dim(\theta_\mathfrak{p})$, $m = \lfloor n/2 \rfloor$, $K = k(\sqrt{(-1)^{n/2}\det(\varphi)})$, and $\widetilde{\Lambda_\mathfrak{p}} = \{x \in V_\mathfrak{p} \mid 2\varphi_\mathfrak{p}(x, \Lambda_\mathfrak{p}) \in O_\mathfrak{p}\}$. For future reference we explicitly state [Shi, (3.1.9)], which says

$$[\widetilde{\Lambda_\mathfrak{p}} : \Lambda_\mathfrak{p}] = |\det(2\phi_\mathfrak{p})|_\mathfrak{p}^{-1}. \tag{3.4}$$

§4 Archimedian Local Factors

In this section we explicitly compute the volume form ω_G on $G_v = G_v^{\phi_v}$ described in §2 when v is archimedian, and relate ω_G to a natural volume form $\omega_{\mathfrak{Z}}$ on the symmetric space $\mathfrak{Z}_v$. The relationship between ω_G and $\omega_{\mathfrak{Z}}$ is established by constructing a non-zero C_v-invariant volume form ω_C on the fibre C_v of G_v over some chosen point $p_v \in \mathfrak{Z}_v$, and then evaluating $\int_{C_v} \omega_C$. This allows us to connect the associated measures on G_v and $\mathfrak{Z}_v$. We note that when v is real and φ is definite, the situation is much simpler since $\mathfrak{Z}_v = \{1_n\}$ and $C_v = G_v$.

For our calculations we would like to write down ω_G in some set of coordinates on G, and we choose the coordinates given by the strictly lower triangular matrix entries of the natural embedding $G \hookrightarrow (k_v)_n^n$. These give coordinates on an open subset of G whose compliment has measure zero, and the associated coordinate 1-forms give a basis for the cotangent space. The matrix $g^{-1}dg$ is a G-invariant matrix of 1-forms under left multiplication, and so the form

$$\gamma_n = \bigwedge_{i>k} (g^{-1}dg)_{ik} \tag{4.1}$$

gives a G-invariant volume form on G. Since the space of such forms is 1 dimensional, any G-invariant volume form will be a constant multiple of γ_n.

Calculation 4.1. *Suppose v is archimedian. Then in the coordinates given by $G_v \hookrightarrow (k_v)_n^n$, the volume form ω_G described in §2 can we written as*

$$\omega_G = \pm \frac{1}{2^n} \gamma_n = \pm \frac{1}{2^n} \prod_{l=1}^{n} \det{}_l(x)^{-1} \bigwedge_{i>k} dx_{ik}.$$

Proof. To compute ω_G it suffices to compute any non-zero monomial Θ in $\mathcal{F}^*(\omega_T)$, since if $\Theta = f(x) \bigwedge_{(i,k)\in I} dx_{ik}$ for some indexing set I and $\omega = f(x)^{-1} \bigwedge_{(i,k)\notin I} dx_{ik}$ is its complimentary monomial, then $\mathcal{F}^*(\omega_T) \wedge \omega = \Theta \wedge \omega = \omega_X$. We choose to calculate the monomial $\Theta = f(x) \bigwedge_{i\leq k} dx_{ik}$. Since we are only interested in finding ω_G up to sign, it will be enough to compute ω_G for $\phi_v = 1_n$.

From (2.5) we have $t = \mathcal{F}(x) = {}^t x x$ and so $\mathcal{F}^*(dt) = {}^t(dx)x + {}^t x(dx)$. Therefore

$$\begin{aligned} \mathcal{F}^*(\omega_T) &= \bigwedge_{i\leq k} \left(\sum_j dx_{ji}x_{jk} + x_{ji}dx_{jk} \right) \\ &= \Theta + \text{other terms}. \end{aligned} \tag{4.2}$$

We compute Θ by induction on the column bound k_0, showing that

$$\bigwedge_{i\leq k\leq k_0} \left(\sum_j dx_{ji}x_{jk} + x_{ji}dx_{jk} \right) = 2^{k_0} \bigwedge_{i\leq k\leq k_0} \sum_j x_{ji}dx_{jk} + \Psi \tag{4.3}$$

where Ψ is a sum of terms each of which has some dx_{ik} factor with $i > k$.

The case $k_0 = 1$ is clear since the left side is just $2x_{11}dx_{11}$. If $k_0 > 1$ we have

$$\begin{aligned}
(4.4)\qquad & \bigwedge_{i\le k\le k_0} \left(\sum_j dx_{ji}x_{jk} + x_{ji}dx_{jk} \right) \\
&= \bigwedge_{i\le k\le k_0-1} \left(\sum_j dx_{ji}x_{jk} + x_{ji}dx_{jk} \right) \wedge \bigwedge_{i\le k=k_0} \left(\sum_j dx_{ji}x_{jk_0} + x_{ji}dx_{jk_0} \right) \\
&= \left(2^{k_0-1} \bigwedge_{i\le k\le k_0-1} \sum_j x_{ji}dx_{jk} + \Psi \right) \wedge \bigwedge_{i\le k=k_0} \left(\sum_j dx_{ji}x_{jk_0} + x_{ji}dx_{jk_0} \right)
\end{aligned}$$

We now analyze the term $\Xi = \bigwedge_{i\le k_0} \left(\sum_j dx_{ji}x_{jk_0} + x_{ji}dx_{jk_0} \right)$ appearing at the end of (4.4). The only terms of Ξ contributing non-zero terms to Θ come from the column k_0. This is because all of the dx_{jk} terms with $k \le k_0 - 1$ already appear in each term of $\bigwedge_{i\le k\le k_0-1} \sum_j x_{ji}dx_{jk}$ contributing to Θ, and so the wedge product of the two is zero. Also, since the entries of dx are linearly independent, such factors dx_{jk_0} must satisfy $j \le k_0$ to contribute to Θ. So Ξ in (4.4) can be replaced by

$$\begin{aligned}
(4.5)\qquad & \bigwedge_{i<k_0} \left(\sum_j x_{ji}dx_{jk_0} \right) \wedge \left(\sum_j dx_{jk_0}x_{jk_0} + x_{jk_0}dx_{jk_0} \right) \\
&= 2 \bigwedge_{i\le k_0} \left(\sum_j x_{ji}dx_{jk_0} \right),
\end{aligned}$$

which proves (4.3).

By combining (4.3) with $k_0 = n$ and Lemma A3, we see that

$$\Theta = 2^n \bigwedge_{i\le k} ({}^t x dx)_{ik} = 2^n \prod_{l=1}^{n} \det{}_l(x) \bigwedge_{i\le k} dx_{ik} + \text{ other terms,} \tag{4.6}$$

which shows that

$$\omega_G = \frac{1}{2^n} \prod_{l=1}^{n} \det{}_l(x)^{-1} \bigwedge_{i>k} dx_{ik} \tag{4.7}$$

satisfies (2.7). $\square$

§4.1 Computation for $k_v = \mathbb{R}$ with φ definite

If v is real and φ_v is definite, then $G_v = SO_n(\mathbb{R})$. Since $SO_n(\mathbb{R})$ is compact, $\tau_v(G_v)$ is finite. We now find the measure $\tau_{\mathbb{R}}$ of $SO_n(\mathbb{R})$ with respect to ω_G. From Calculation 4.1 and Lemma A3 we see that (up to sign) on G_v

$$\omega_G \sim \bigwedge_{i>k} ({}^t g dg)_{ik} \sim \bigwedge_{i>k} (g^{-1}dg)_{ik},$$

and this together with the volume computation in [Vos, (14.6), p85] relative to volume form $\bigwedge_{i>k}(g^{-1}dg)_{ik}$ gives

$$\tau_{\mathbb{R}}(G_v) = \frac{1}{2}\pi^{\frac{n(n+1)}{4}} \left(\prod_{l=1}^{n} \Gamma(l/2) \right)^{-1}. \tag{4.1.1}$$

§4.2 Computation for $k_v = \mathbb{R}$ with φ indefinite

If v is real and φ_v is indefinite, then we take $\phi_v = \mathrm{diag}[1_q, -1_r]$ as in (2.9), $G_\mathbb{R} = SO(q,r)$, and define the (symmetric) space $\mathfrak{Z}_\mathbb{R}$ by

$$\mathfrak{Z}_\mathbb{R} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^q_r \;\middle|\; x \in \mathbb{R}^r_r, y \in \mathbb{R}^t_r, {}^t x + x > {}^t y y \right\}.$$

To define a $G_\mathbb{R}$-action on $\mathfrak{Z}_\mathbb{R}$, let

$$B(z) = \begin{bmatrix} {}^t x & {}^t y & x \\ 0 & 1_t & y \\ -1_r & 0 & 1_r \end{bmatrix}, \qquad \gamma = \begin{bmatrix} \frac{-1}{\sqrt{2}}_r & 0 & \frac{1}{\sqrt{2}}_r \\ 0 & 1_t & 0 \\ \frac{1}{\sqrt{2}}_r & 0 & \frac{1}{\sqrt{2}}_r \end{bmatrix},$$

$$\mathfrak{Y} = \{Y \in GL_n(\mathbb{R}) \mid {}^t Y \phi_v^{-1} Y = \mathrm{diag}[A, -B] \text{ with } A \in S^q_+, B \in S^r_+\},$$

and induce a $G_\mathbb{R}$-action on $\mathfrak{Z}_\mathbb{R}$ from the bijection

$$\begin{aligned} \mathfrak{Z}_\mathbb{R} \times GL_q(\mathbb{R}) \times GL_r(\mathbb{R}) &\xrightarrow{\sim} \mathfrak{Y} \\ (z, \lambda, \mu) &\longmapsto B(z) \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}, \end{aligned} \tag{4.2.1}$$

by allowing $\alpha \in G_\mathbb{R}$ to act on $\mathfrak{Y}$ by left multiplication. (See [Shi2, §6] for details.) Explicitly, (4.2.1) gives the action $z \mapsto \alpha z$ on $\mathfrak{Z}_\mathbb{R}$ by

$$\alpha B(z) = B(\alpha z) \begin{bmatrix} \lambda_\alpha(z) & 0 \\ 0 & \mu_\alpha(z) \end{bmatrix}, \tag{4.2.2}$$

for some matrices $\lambda_\alpha(z)$ and $\mu_\alpha(z)$.

By choosing a distinguished point $p_\mathbb{R} = \begin{bmatrix} 1_r \\ 0^t_r \end{bmatrix} \in \mathfrak{Z}_\mathbb{R}$, we define a map

$$\begin{aligned} F_\mathbb{R} : G_\mathbb{R} &\longrightarrow \mathfrak{Z}_\mathbb{R} \\ \alpha &\longmapsto \alpha p_\mathbb{R}. \end{aligned} \tag{4.2.3}$$

If we write $\alpha \in G_\mathbb{R}$ as

$$\alpha = \begin{bmatrix} a & b & c \\ g & e & f \\ h & l & w \end{bmatrix} \tag{4.2.4}$$

with $a, d \in \mathbb{R}^r_r$ and $e \in \mathbb{R}^t_t$, then the map $F_\mathbb{R}$ sends

$$\alpha \longmapsto \alpha p_\mathbb{R} = \begin{bmatrix} (w-c)(w+c)^{-1} \\ (\sqrt{2})_t f (w+c)^{-1} \end{bmatrix}. \tag{4.2.5}$$

In these coordinates the stabilizer of $p_\mathbb{R}$ is given by

$$C_\mathbb{R} = \{\alpha \in G_\mathbb{R} \mid f = 0^r_t, c = 0^r_r\}, \tag{4.2.6}$$

and the relation ${}^t x \phi_v x = \phi_v$ implies that l and h are also zero. Thus $C_{\mathbb{R}}$ decomposes as

$$\begin{aligned} C_{\mathbb{R}} &\cong [G_{\mathbb{R}}(1_q) \times G_{\mathbb{R}}(1_r)] \cup \left[G^-_{\mathbb{R}}(1_q) \times G^-_{\mathbb{R}}(1_r)\right] \\ \alpha &\mapsto \left(\begin{bmatrix} a & b \\ g & e \end{bmatrix}, w \right). \end{aligned} \tag{4.2.7}$$

We will be working with the $G_{\mathbb{R}}$-invariant volume form $\omega_{\mathfrak{Z}}$ on $\mathfrak{Z}_{\mathbb{R}}$ constructed in [Shi, §4.2], given by the expression

$$\omega_{\mathfrak{Z}} = \delta(z)^{-n/2} \bigwedge_{i,k} dz_{ik}, \tag{4.2.8}$$

where $\delta(z) = \det(\frac{1}{2}({}^t x + x - {}^t y y))$.

Computation of ω_C and $\int_C \omega_C$

We now compute the expression for ω_C on $C_{\mathbb{R}} = \mathrm{Stab}(p_{\mathbb{R}})$ described in §4. For this it is enough, by the last part of Lemma A6, for us to consider forms whose restrictions to the fibre $C_{\mathbb{R}}$ are equal up to sign. We write this equivalence as $\approx$.

From (4.2.5) we have

$$\begin{aligned} F^*_{\mathbb{R}}(dx) &= -(1_r + (w-c)(w+c)^{-1}) dc (w+c)^{-1} \\ &\qquad + (1_r - (w-c)(w+c)^{-1}) dw (w+c)^{-1} \\ &\approx -2_r \, dc \, w^{-1}, \\ F^*_{\mathbb{R}}(dy) &= -(\sqrt{2})_r df (w+c)^{-1} - (\sqrt{2})_r f (w+c)^{-1} d(w+c)(w+c)^{-1} \\ &\approx (\sqrt{2})_r df \, w^{-1}. \end{aligned}$$

Applying Lemma A2 and $\det(w) \approx 1$ to these gives

$$\begin{aligned} \bigwedge_{i,k} F^*_{\mathbb{R}}(dx)_{ik} &\approx 2^{r^2} \bigwedge_{i,k} dc_{ik}, \\ \bigwedge_{i,k} F^*_{\mathbb{R}}(dy)_{ik} &\approx 2^{\frac{rt}{2}} \bigwedge_{i,k} df_{ik}, \end{aligned}$$

which together with the observation $\delta(p_{\mathbb{R}}) = 1$ yields

$$F^*_{\mathfrak{Z}}(\omega_{\mathbb{R}}) \approx 2^{\frac{rn}{2}} \bigwedge_{i,k} dc_{ik} \bigwedge_{i,k} df_{ik}.$$

We recall from Calculation 4.1,

$$\omega_G \approx 2^{-n} \prod_{l=1}^{n} \det{}_l(\alpha)^{-1} \bigwedge_{i>k} d\alpha_{ik}.$$

By the construction of ω_G in §2 and $F^*_{\mathbb{R}}(\omega_{\mathbb{R}})$ as above, and since the matrix $g^{-1}dg$ of §4 is skew symmetric. we see that the volume form ω_C on the fibre is

$$\begin{aligned} \omega_C &\approx 2^{\frac{-rn}{2}} 2^{-n} \prod_{l=1}^{n} \det{}_l(\alpha)^{-1} \bigwedge_{i>k} da_{ik} \bigwedge_{i>k} de_{ik} \bigwedge_{i,k} dg_{ik} \bigwedge_{i>k} dw_{ik} \\ &\approx 2^{\frac{-rn}{2}} \omega_{SO_q(\mathbb{R})} \wedge \omega_{SO_r(\mathbb{R})}. \end{aligned}$$

By comparison with ω_G in §4.1 and the isomorphism (4.2.7), we find that

$$\begin{aligned}\mathrm{vol}_C(C_{\mathbb{R}}) &= \int_{C_{\mathbb{R}}} |\omega_C| \\ &= 2\cdot 2^{\frac{-rn}{2}}\left[\int_{SO_q(\mathbb{R})}\omega_{SO_q(\mathbb{R})}\right]\left[\int_{SO_r(\mathbb{R})}\omega_{SO_r(\mathbb{R})}\right] \\ &= 2\cdot 2^{\frac{-rn}{2}}\frac{1}{2}\pi^{\frac{q(q+1)}{4}}\left(\prod_{k=1}^{q}\Gamma(k/2)\right)^{-1}\frac{1}{2}\pi^{\frac{r(r+1)}{4}}\left(\prod_{k=1}^{r}\Gamma(k/2)\right)^{-1},\end{aligned}$$

which completes our calculation.

§4.3 Computation for $k_v = \mathbb{C}$

If v is complex, then $G_{\mathbb{C}} = SO_n(\mathbb{C})$ and we define the (symmetric) space $\mathfrak{Z}_{\mathbb{C}}$ by

$$\mathfrak{Z}_{\mathbb{C}} = \{z \in \mathbb{R}^n_n \mid {}^t z = -z, {}^t zz < 1\}.$$

To define a $G_{\mathbb{C}}$-action on $\mathfrak{Z}_{\mathbb{C}}$, we first let

$$B(z) = \begin{bmatrix} 1_n & z \\ -z & 1_n \end{bmatrix}, \qquad I = \begin{bmatrix} 1_n & 0 \\ 0 & -1_n \end{bmatrix},$$

$$\mathfrak{X} = \left\{ X \in GL_{2n}(\mathbb{R}) \,\middle|\, {}^t XIX = \begin{bmatrix} A & 0 \\ 0 & -B \end{bmatrix} \text{ with } A, B \in S^n_+ \right\}.$$

One can check that this gives an injection

$$\begin{aligned} \mathfrak{Z}_{\mathbb{C}} \times GL_n(\mathbb{R}) \times GL_n(\mathbb{R}) &\longrightarrow \mathfrak{X} \\ (z,\lambda,\mu) &\longmapsto B(z)\begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}. \end{aligned} \tag{4.3.1}$$

Writing $\alpha = a + bi \in G_{\mathbb{C}}$ with $a, b \in \mathbb{R}^n_n$, we define $\iota(\alpha) = \left[\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right]$ and allow α to act on $x \in \mathfrak{X}$ by left multiplication by $\iota(\alpha)$

$$\alpha x = \iota(\alpha)x.$$

By a direct calculation we see that this gives a well-defined action on the image of (4.3.1) and can be used to define a $G_{\mathbb{C}}$-action on $\mathfrak{Z}_{\mathbb{C}}$ by

$$\alpha B(z) = \iota(\alpha)B(z) = B(\alpha z)\begin{bmatrix} \lambda_\alpha(z) & 0 \\ 0 & \mu_\alpha(z) \end{bmatrix}, \tag{4.3.2}$$

the key observation being that ${}^t\iota(\alpha)I\iota(\alpha) = I$ for $\alpha \in G_{\mathbb{C}}$. The same calculation shows that

$$\lambda_\alpha(z) = \mu_\alpha(z) = (a + bz),$$

which we henceforth denote by $\mu_\alpha(z)$.

By choosing a distinguished point $p_\mathbb{C} = 0_n^n \in \mathfrak{Z}_\mathbb{C}$, we define a map

$$(4.3.3) \qquad \begin{aligned} F_\mathbb{C} : G_\mathbb{C} &\longrightarrow \mathfrak{Z}_\mathbb{C} \\ \alpha &\longmapsto \alpha p_\mathbb{C}. \end{aligned}$$

Writing this map out in real coordinates we see

$$(4.3.4) \qquad \alpha = a + bi \longmapsto -ba^{-1},$$

where $a, b \in \mathbb{R}_n^n$. In these coordinates the stabilizer of $p_\mathbb{C}$ is given by

$$(4.3.5) \qquad C_\mathbb{C} = \mathrm{Stab}(p_\mathbb{C}) = \{\alpha = a + bi \in G_\mathbb{C} \mid b = 0_n^n\} \cong SO_n(\mathbb{R}).$$

We now construct a $G_\mathbb{C}$-invariant volume form on $\mathfrak{Z}_\mathbb{C}$. To do this we need to know how the differentials transform under the map $F_\mathbb{C}$. We begin with a few definitions. For any two points $w, z \in \mathfrak{Z}_\mathbb{C}$ we let

$$(4.3.6) \qquad \xi(w,z) = 1_n - {}^t w z, \qquad \xi(z) = \xi(z,z),$$

$$(4.3.7) \qquad \delta(w,z) = \det(\xi(w,z)), \qquad \delta(z) = \delta(z,z),$$

which satisfies the relation

$$(4.3.8) \qquad {}^t B(w) I B(z) = \begin{bmatrix} \xi(w,z) & z + {}^t w \\ z + {}^t w & -\xi(w,z) \end{bmatrix}.$$

By combining (4.3.8), ${}^t\iota(\alpha) I \iota(\alpha) = I$, and (4.3.2), we have

$$ {}^t\mu_\alpha(w)(\alpha z - \alpha w)\mu_\alpha(z) = z - w,$$

$$ {}^t\mu_\alpha(w)\xi(\alpha w, \alpha z)\mu_\alpha(z) = \xi(w,z).$$

Fixing $w \in \mathfrak{Z}_\mathbb{C}$, we differentiate these with respect to z and evaluate at $z = w$ to obtain

$$\begin{aligned} d(\alpha z) &= {}^t\mu_\alpha(z)^{-1}\, dz\, \mu_\alpha(z)^{-1}, \\ \delta(\alpha z) &= \det(\mu_\alpha(z))^{-2}\delta(z). \end{aligned}$$

By applying Lemma A4 to these two equations, we see that the expression

$$(4.3.9) \qquad \omega_\mathfrak{Z} = \delta(z)^{\frac{1-n}{2}} \bigwedge_{i>k} dz_{ik}$$

gives a non-zero $G_\mathbb{C}$-invariant volume form on $\mathfrak{Z}_\mathbb{C}$.

Computation of ω_C and $\int_C \omega_C$

We now compute the form ω_C on $C_\mathbb{C} = \mathrm{Stab}(p_\mathbb{C})$ described in §4. By the last part of Lemma A6, it is enough to consider forms whose restrictions to the fibre $C_\mathbb{C}$ are equal up to sign. We write this equivalence as $\approx$.

First we compute $F_\mathbb{C}^*(\omega_\mathfrak{Z})$. From (4.3.4) we have

$$\begin{aligned} F_\mathbb{C}^*(dz) &= -db\, a^{-1} - b\, d(a^{-1}) \\ &\approx db\, a^{-1}, \end{aligned}$$

and so

$$\bigwedge_{i>k} F_{\mathbb{C}}^{*}(dz)_{ik} \approx \bigwedge_{i>k} \left(db\, a^{-1}\right)_{ik}.$$

From the relations defining $G_{\mathbb{C}}$, we know that ${}^{t}a \approx a^{-1}$ and the restriction of ${}^{t}a\, db$ to $C_{\mathbb{C}}$ is skew symmetric, therefore so is $a({}^{t}a\, db)a^{-1} = db\, a^{-1}$. Applying Lemma A5 to this gives

$$\bigwedge_{i>k} db_{ik} = \prod_{l=1}^{n-1} \det_l(a) \bigwedge_{i>k} (db\, a^{-1})_{ik}$$

and so

$$F_{\mathbb{C}}^{*}(\omega_3) = \prod_{l=1}^{n-1} \det_l(a)^{-1} \bigwedge_{i>k} db_{ik}$$

since $\delta(p_{\mathbb{C}}) = 1$.

From our choice of local measures in §2, the real volume form $\widetilde{\omega}$ associated to the complex volume form ω is given by $\omega \wedge \overline{\omega}$. Combining this with Calculation 4.1 we have

$$\begin{aligned}
\widetilde{\omega_G} &= 2^{-2n} \prod_{l=1}^{n} \det_l(z)^{-1} \det_l(\bar{z})^{-1} \bigwedge_{i>k} (i dz_{ik} \wedge d\bar{z}_{ik}) \\
&= 2^{\frac{n(n-5)}{2}} \prod_{l=1}^{n} \det_l(z)^{-1} \det_l(\bar{z})^{-1} \bigwedge_{i>k} (da_{ik} \wedge db_{ik}) \\
&\approx 2^{\frac{n(n-5)}{2}} \prod_{l=1}^{n-1} \det_l(a)^{-2} \bigwedge_{i>k} (da_{ik} \wedge db_{ik}).
\end{aligned}$$

By applying the procedure in §2 to $\widetilde{\omega_G}$ and $F_{\mathbb{C}}^{*}(\omega_3)$ above, we see that the (real) volume form ω_C on the fibre is given by

$$\omega_C = 2^{\frac{n(n-5)}{2}} \prod_{l=1}^{n-1} \det_l(a)^{-1} \bigwedge_{i>k} da_{ik}.$$

From §4.1, we know

$$\int_{SO_n(\mathbb{R})} \omega_G = \frac{1}{2}\pi^{\frac{n(n+1)}{4}} \prod_{j=1}^{n} \Gamma(j/2)^{-1},$$

so we have

$$\mathrm{vol}_C(C_{\mathbb{C}}) = \int_{C_{\mathbb{C}}} \omega_C = 2^{\frac{n(n-3)}{2}} \int_{SO_n(\mathbb{R})} \omega_G = 2^{\frac{n(n-3)}{2}} \left(\frac{1}{2}\pi^{\frac{n(n+1)}{4}} \prod_{j=1}^{n} \Gamma(j/2)^{-1} \right),$$

which completes our calculation.

§5 Mass Formula for Maximal Lattices

In this section we compute an exact mass formula for the genus of maximal lattices $\Lambda \subset (V, \varphi)$. We call a lattice $\Lambda \subset (V, \varphi)$ a **maximal lattice** if $\varphi(\Lambda) \subseteq O_k$ and Λ is maximal with this property.

To define the mass of a genus of lattices over an arbitrary number field k, we need to define symmetric spaces $\mathfrak{Z}_v$ for all $v \in \mathbf{a}$. If v is real and φ_v is definite, then we take $\mathfrak{Z}_v$ to be a single point with measure one. If v is real and φ_v is indefinite or v is complex, then we take $\mathfrak{Z}_v$ as in §4.2 or §4.3 respectively. The spaces $\mathfrak{Z}_v$ come equipped with a transitive G_v-action, an invariant volume form $\omega_{\mathfrak{Z}}$, and a distinguished point p_v. For each $v \in \mathbf{a}$, we define a surjective map

$$\begin{aligned} F_v : G_v &\longrightarrow \mathfrak{Z}_v \\ \alpha &\longmapsto \alpha p_v \end{aligned} \tag{5.1}$$

and denote by C_v the fibre of F_v over p_v. We let

$$\mathfrak{Z} = \prod_{v \in \mathbf{a}} \mathfrak{Z}_v, \qquad C = \prod_{v \in \mathbf{a}} C_v, \qquad p = (p_v)_{v \in \mathbf{a}}, \tag{5.2}$$

and let F denote the product map

$$F : G_{\mathbf{a}} \longrightarrow \mathfrak{Z}. \tag{5.3}$$

We observe that the $C = F^{-1}(p)$ is the fibre of F over p.

We define the **mass** of a quadratic form (V, φ) with respect to a lattice Λ to be

$$\mathrm{Mass}(\Lambda, \varphi) = \sum_{a \in \mathfrak{Cl}} \nu(\Gamma^a) \tag{5.4}$$

where

$$\nu(\Gamma^a) = \begin{cases} [\Gamma^a : 1]^{-1} & \text{if } G_{\mathbf{a}} \text{ is compact,} \\ [\Gamma^a \cap \{\pm 1\} : 1]^{-1} \mathrm{vol}(\Gamma^a \backslash \mathfrak{Z}) & \text{otherwise.} \end{cases} \tag{5.5}$$

Theorem 5.1. *Let (V, φ) be a non-degenerate quadratic space of dimension $n \geq 3$ defined over a number field k of degree d over $\mathbb{Q}$. Then the mass of (V, φ) with respect to a maximal lattice $\Lambda \subset (V, \varphi)$ is given by*

$$\begin{aligned} \mathrm{Mass}(\Lambda, \varphi) = 2|D_k|^{\lfloor \frac{(n-1)^2}{4} \rfloor} \left[\prod_{j=1}^{\lfloor \frac{n-1}{2} \rfloor} |D_k|^{\frac{1}{2}} \left(\frac{(2j-1)!}{(2\pi)^{2j}} \right)^d \zeta_k(2j) \right] [\widetilde{\Lambda} : \Lambda]^{\frac{n-1}{2}} \prod_{v | \mathfrak{e}} \lambda_v \\ \prod_{v \in \mathbf{a}} b_v^{\varphi} \prod_{v \text{ complex}} \left(2^{-\frac{(n-1)(n-2)}{2}} \pi^{\frac{n(n+1)}{4}} \prod_{j=1}^{n} \Gamma(j/2)^{-1} \right) \\ \begin{cases} 2^{-\left(\frac{n-1}{2}\right) d} & \text{if } 2 \nmid n, \\ |D_k|^{\frac{1}{2}} \left[(\frac{n}{2} - 1)! (2\pi)^{-\frac{n}{2}} \right]^d L_k(\frac{n}{2}, \chi) & \text{if } 2 | n, \end{cases} \end{aligned}$$

where r_v and $t_v = \dim(\theta_v)$ are defined by the normalization of φ_v in §2,

$$\Gamma_i(s) = \pi^{\frac{i(i-1)}{4}} \prod_{j=0}^{i-1} \Gamma(s-(j/2)),$$
$$\widetilde{\Lambda} = \{x \in V \mid 2\varphi(x,\Lambda) \in O_k\},$$
$$b_v^{\varphi} = 2^{\frac{r_v n}{2}} \pi^{\frac{(n-r_v)r_v}{2}} \Gamma_{r_v}(r_v/2)\Gamma_{r_v}(n/2)^{-1},$$

$\mathfrak{e}$ is the product of all prime ideals for which $\widetilde{\Lambda}_v \neq \Lambda_v$, $\zeta_k(s)$ and $L_k(s,\chi)$ are zeta and L-functions over k, χ is the non-trivial Hecke character associated to the extension K/k where $K = k(\sqrt{(-1)^{n/2}\det(\varphi)})$, and λ_v is defined by

$$\lambda_v = \begin{cases} 1 & \text{if } t_v = 1, \\ 2^{-1}(1+q)^{-1}(1+q^{1-m})(1+q^{-m}) & \text{if } t_v = 2, \mathfrak{p} \text{ is unramified in } K, \\ & \quad \text{and } \widetilde{\Lambda_{\mathfrak{p}}} \neq \Lambda_{\mathfrak{p}}, \\ 2^{-1} & \text{if } t_v = 2, \text{ and } \mathfrak{p} \text{ is ramified in } K, \\ 2^{-1}(1+q)^{-1}(1-q^{-2m}) & \text{if } t_v = 3, \\ 2^{-1}(1+q)^{-1}(1-q^{1-m})(1-q^{-m}) & \text{if } t_v = 4, \end{cases}$$

where q is the norm of the prime ideal at $v \in \mathbf{h}$ and $m = \lfloor \frac{n}{2} \rfloor$.

Proof. By Lemma A7 applied to F, for each class $a \in \mathfrak{Cl}$ we have

$$\tau_{\mathbf{a}}(\Gamma^a \backslash G_{\mathbf{a}}) = \mathrm{vol}_C((\Gamma^a \cap S)\backslash C_{\mathbf{a}})\mathrm{vol}_{\mathfrak{Z}}(\Gamma^a \backslash \mathfrak{Z}),$$

where $S = \{g \in G_{\mathbf{a}} \mid gz = z \text{ for every } z \in \mathfrak{Z}\}$. By Lemma A1, $S = \{(\pm 1)_{v, v\in\mathbf{a}}\}$ so this becomes

$$\tau_{\mathbf{a}}(\Gamma^a \backslash G_{\mathbf{a}})\mathrm{vol}_C(C_{\mathbf{a}})^{-1} = [\Gamma^a \cap \{\pm 1\} : 1]^{-1}\mathrm{vol}_{\mathfrak{Z}}(\Gamma^a \backslash \mathfrak{Z}). \tag{5.6}$$

This together with Theorem 2.1 and (2.1) gives

$$\begin{aligned} \mathrm{Mass}(\Lambda,\varphi) &= 2|D_k|^{\frac{n(n-1)}{4}} \mathrm{vol}_C(C_{\mathbf{a}})^{-1} \prod_{v\in\mathbf{h}} \beta_v(\Lambda,\psi)^{-1} \\ &= 2|D_k|^{\frac{n(n-1)}{4}} \prod_{v\in\mathbf{a}} \mathrm{vol}_C(C_v)^{-1} \prod_{v\in\mathbf{h}} \beta_v(\Lambda,\psi)^{-1}. \end{aligned} \tag{5.7}$$

From here, we complete the proof in 3 parts:

Part 1: First we prove the case where φ_v is positive definite at all $v \in \mathbf{a}$. In this case $C_v = G_v$ for all $v \in \mathbf{a}$, so by (5.7) we have

$$\mathrm{Mass}(\Lambda,\varphi) = 2|D_k|^{\frac{n(n-1)}{4}} \prod_{v\in\mathbf{a}} \beta_v(\psi)^{-1} \prod_{v\in\mathbf{h}} \beta_v(\Lambda,\psi)^{-1}.$$

By (2.9), $\phi_v = {}^t\sigma_v \psi \sigma_v$ and $|\det(\sigma_v)|_v = \left(\frac{|\det(\phi_v)|_v}{|\det(\psi)|_v}\right)^{\frac{1}{2}}$. Combining this with Lemma 3.2 we have

$$2|D_k|^{\frac{n(n-1)}{4}} \prod_{v\in\mathbf{a}} \left(|\det(\psi)|_v^{\frac{-(n+1)}{2}} |\det(\phi_v)|_v^{\frac{n+1}{2}} \beta_v(\phi_v)^{-1} \right)$$
$$\prod_{v\in\mathbf{h}} \left(|\det(\psi)|_v^{\frac{-(n+1)}{2}} |\det(\phi_v)|_v^{\frac{n+1}{2}} \beta_v(\Lambda_v,\phi_v)^{-1} \right),$$

which by the product formula and $\det(\phi_v) = \pm 1$ for all $v \in \mathbf{a}$, gives

$$2|D_k|^{\frac{n(n-1)}{4}} \prod_{v \in \mathbf{a}} \beta_v(\phi_v)^{-1} \prod_{v \in \mathbf{h}} \left(|\det(\phi_v)|_v^{\frac{n+1}{2}} \beta_v(\Lambda_v, \phi_v)^{-1} \right).$$

Substituting (3.3) and (4.1.1), using (3.4), and noticing $\prod_{v|2} 2^{\kappa_v} = 2^n$, we get

$$2|D_k|^{\frac{n(n-1)}{4}} \left(2\pi^{\frac{-n(n+1)}{4}} \prod_{j=1}^{n} \Gamma(j/2) \right)^d [\widetilde{\Lambda} : \Lambda]^{\frac{n-1}{2}}$$
$$\left(2^{-nd} \prod_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} \zeta_k(2i) \prod_{v|\mathfrak{e}} \lambda_v \right) \begin{cases} 1 & \text{if } 2 \nmid n, \\ L_k(\frac{n}{2}, \chi) & \text{if } 2|n. \end{cases}$$

By rearranging terms we obtain

$$2|D_k|^{\frac{n(n-1)}{4}} \left(2^{-(n-1)d} \right) \left[\left(\pi^{\frac{-n(n+1)}{4}} \prod_{j=1}^{n} \Gamma(j/2) \right)^d \prod_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} \zeta_k(2i) \right]$$
$$[\widetilde{\Lambda} : \Lambda]^{\frac{n-1}{2}} \prod_{v|\mathfrak{e}} \lambda_v \begin{cases} 1 & \text{if } 2 \nmid n, \\ L_k(\frac{n}{2}, \chi) & \text{if } 2|n, \end{cases}$$

$$= 2|D_k|^{\frac{n(n-1)}{4}} \left(2^{-(n-1)d} \right) \left[\prod_{j=1}^{\lfloor \frac{n-1}{2} \rfloor} \left(\frac{(2j-1)!}{(2\pi)^{2j}} \right)^d \zeta_k(2j) \right]$$
$$[\widetilde{\Lambda} : \Lambda]^{\frac{n-1}{2}} \prod_{v|\mathfrak{e}} \lambda_v \begin{cases} 2^{\frac{n-1}{2}d} & \text{if } 2 \nmid n, \\ \left[2^{n-1} (\frac{n}{2} - 1)! (2\pi)^{-\frac{n}{2}} \right]^d L_k(\frac{n}{2}, \chi) & \text{if } 2|n, \end{cases}$$

$$= 2|D_k|^{\frac{n(n-1)}{4}} \left[\prod_{j=1}^{\lfloor \frac{n-1}{2} \rfloor} \left(\frac{(2j-1)!}{(2\pi)^{2j}} \right)^d \zeta_k(2j) \right] [\widetilde{\Lambda} : \Lambda]^{\frac{n-1}{2}} \prod_{v|\mathfrak{e}} \lambda_v$$
$$\begin{cases} 2^{-(\frac{n-1}{2})d} & \text{if } 2 \nmid n, \\ \left[(\frac{n}{2} - 1)! (2\pi)^{-\frac{n}{2}} \right]^d L_k(\frac{n}{2}, \chi) & \text{if } 2|n, \end{cases}$$

$$= 2|D_k|^{\lfloor \frac{(n-1)^2}{4} \rfloor} \left[\prod_{j=1}^{\lfloor \frac{n-1}{2} \rfloor} D_k^{\frac{1}{2}} \left(\frac{(2j-1)!}{(2\pi)^{2j}} \right)^d \zeta_k(2j) \right] [\widetilde{\Lambda} : \Lambda]^{\frac{n-1}{2}} \prod_{v|\mathfrak{e}} \lambda_v$$
$$\begin{cases} 2^{-(\frac{n-1}{2})d} & \text{if } 2 \nmid n, \\ D_k^{\frac{1}{2}} \left[(\frac{n}{2} - 1)! (2\pi)^{-\frac{n}{2}} \right]^d L_k(\frac{n}{2}, \chi) & \text{if } 2|n. \end{cases}$$

Part 2: Now suppose that all $v \in \mathbf{a}$ are real, but perhaps φ_v is indefinite at some v. Take

$$b_v^{\varphi} = 2^{\frac{r_v n}{2}} \pi^{\frac{(n-r_v) r_v}{2}} \Gamma_{r_v}(r_v/2) \Gamma_{r_v}(n/2)^{-1}$$

as above where r_v is defined by the normalization of φ_v in (2.9). For each indefinite v, we add an additional factor of b_v^φ from the formula in part 1, which is seen by observing

$$\mathrm{vol}_C(C_v)^{-1} = \left(2\pi^{\frac{-n(n+1)}{4}} \prod_{j=1}^{n} \Gamma(j/2)\right) b_v^\varphi$$

and that $b_v^\varphi = 1$ if v is definite. Combined with the previous formula this proves the case where all $v \in \mathbf{a}$ are real.

Part 3: Finally, consider arbitrary $v \in \mathbf{a}$. We define $r_v = 0$ for v complex, and so for such v we have $b_v^\varphi = 1$. Since each complex place replaces two real places in the totally real formula, we again have a correction factor. The relevant calculation to check for v complex is

$$\mathrm{vol}_C(C_v)^{-1} = \left(2\pi^{\frac{-n(n+1)}{4}} \prod_{j=1}^{n} \Gamma(j/2)\right)^2 \left(2^{-\frac{(n-1)(n-2)}{2}} \pi^{\frac{n(n+1)}{4}} \prod_{j=1}^{n} \Gamma(j/2)^{-1}\right) b_v^\varphi.$$

This together with Part 2 proves the theorem. $\square$

One interesting application of Theorem 5.1 is to the case of a maximal indefinite quadratic form (Λ, φ) in $n \geq 3$ variables. In this case our formula explicitly computes the volume of the quotient $\Gamma^a \backslash \mathfrak{Z}$.

Corollary 5.2. *Let (Λ, φ) be a maximal indefinite quadratic form in $n \geq 3$ variables and let D the subgroup of $G_{\mathbf{A}}$ stabilizing Λ. Then*

$$\mathrm{vol}(\Gamma^a \backslash \mathfrak{Z}) = \varepsilon\, [k_{\mathbf{A}}^\times : k^\times \sigma(D)]\, \mathrm{Mass}(\Lambda, \varphi)$$

where σ is the spinor norm map $G_{\mathbf{A}}^\varphi \longrightarrow k_{\mathbf{A}}^\times/(k_{\mathbf{A}}^\times)^2$ (see [Shi, (2.1.1)]) and ε is either 1 or 2 depending on whether $\dim(V)$ *is odd or even. If k has class number one, then*

$$\mathrm{vol}(\Gamma^a \backslash \mathfrak{Z}) = \varepsilon\, \mathrm{Mass}(\Lambda, \varphi).$$

Proof. Since $n \geq 3$ and φ is indefinite, the classes and the spinor genera in the genus of Λ coincide. From this and [Shi, Lemma 2.3(4)] we know that the number of classes is $[k_{\mathbf{A}}^\times : k^\times \sigma(D)]$. We also know that $\nu(\Gamma^a)$ is independent of the class a by [Shi, Thrm 5.10(1)]. Finally, $-1 \in \Gamma^a$ exactly when $\det(-1_n) = 1$ which happens exactly when $2|\dim(V)$. This proves the first assertion.

For the second part, from [Shi, Lemma 2.5] we know that $k_{\mathbf{A}}^\times / k^\times \sigma(D)$ is a quotient of the ideal class group of k. Thus if the class number of k is one, then $[k_{\mathbf{A}}^\times : k^\times \sigma(D)] = 1$. $\square$

§6 Mass formula for $\mathfrak{a}$-maximal lattices

In this section we use the local similitude groups $\widetilde{G}_{\mathfrak{p}}^\varphi$ to show that the mass of the genus of $\mathfrak{a}$-maximal lattices is the same for many ideals $\mathfrak{a}$. We do this by noticing that $\mathrm{Mass}(\Lambda, \varphi)$ depends only on the volume of non-archimedian stabilizer $D_{\mathbf{h}}$ of (Λ, φ), and then showing that the action of $\widetilde{G}_{\mathfrak{p}}^\varphi$ preserves these volumes.

Let $\widetilde{G}_\mathfrak{p}^\varphi = \{\widetilde{g} \in GL_n(k_\mathfrak{p}) \mid {}^t\widetilde{g}\,\varphi_\mathfrak{p}\widetilde{g} = \xi(\widetilde{g})\varphi_\mathfrak{p} \text{ for some } \xi(\widetilde{g}) \in k_\mathfrak{p}^\times\}$ be the local group of similitudes of φ, and let $\Xi_\mathfrak{p}(\varphi) = \{\xi(\widetilde{g}) \mid \widetilde{g} \in G_\mathfrak{p}^\varphi\}$ be the set of similitude multipliers of $G_\mathfrak{p}^\varphi$. We also recall the local decomposition

$$(V_\mathfrak{p}, \varphi_\mathfrak{p}) = (H_{2r}, \eta_{2r}) \bigoplus (W_\mathfrak{p}, \theta_\mathfrak{p})$$

where $(H_{2r}, \eta_{2r}) \cong \bigoplus_{i=1}^r (F_\mathfrak{p}^2, \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right])$ and $(W_\mathfrak{p}, \theta_\mathfrak{p})$ is anisotropic of dimension $t_\mathfrak{p}$. By [OM, 63:19, p170] we know that $t_\mathfrak{p} \leq 4$. One can easily compute $\Xi_\mathfrak{p}(\eta_{2r}) = F_\mathfrak{p}^\times$, and so $\Xi_\mathfrak{p}(\varphi) = \Xi_\mathfrak{p}(\eta_{2r}) \cap \Xi_\mathfrak{p}(\theta_\mathfrak{p}) = \Xi_\mathfrak{p}(\theta_\mathfrak{p})$.

Lemma 6.1. *Suppose Λ and Λ' are two lattices in the quadratic space (V, φ) over k with stabilizers $D, D' \subset G_\mathbf{A}^\varphi$ respectively. Then*

$$\frac{\mathrm{Mass}(\Lambda, \varphi)}{\mathrm{Mass}(\Lambda', \varphi)} = \frac{\mathrm{vol}(D'_\mathbf{h})}{\mathrm{vol}(D_\mathbf{h})},$$

where the local volumes are defined by $\mathrm{vol}(D_v) = \int_{D_v} \widetilde{\omega_G}$.

Proof. This follows by combining (5.7) with the remarks after (2.8). □

Lemma 6.2. *Suppose $D_\mathfrak{p}$ is an open compact subgroup of $G_\mathfrak{p}^\varphi$ and $\alpha \in \widetilde{G}_\mathfrak{p}^\varphi$, then* $\mathrm{vol}(D_\mathfrak{p}) = \mathrm{vol}(\alpha^{-1} D_\mathfrak{p}\, \alpha)$.

Proof. This is equivalent to showing that the volume form ω_G on G_v is invariant under conjugation by α. To see this holds, following the procedure of §2 we can realize G as a fibre of the map $\mathcal{F} : \widetilde{G}_v \to k_v^\times$ given by $\mathcal{F}(\widetilde{g}) = \xi(\widetilde{g})$, which gives $\omega_{\widetilde{G}} = \omega_G \wedge \mathcal{F}^*(\frac{d\xi}{\xi})$. Since $\widetilde{G}_v$ is unimodular and $\frac{d\xi}{\xi}$ is clearly invariant under conjugation, we see that ω_G is also invariant. □

Theorem 6.3. *Suppose (Λ', φ) is a non-degenerate $\mathfrak{a}$-maximal lattice of dimension $n \geq 3$ defined over a number field k. Then*

$$\mathrm{Mass}(\Lambda', \varphi) = \mathrm{Mass}(\Lambda, \varphi)$$

where (Λ, φ) is a maximal lattice, and $\mathfrak{a}_\mathfrak{p}$ satisfies the following conditions:

If n is odd, then $\mathfrak{a}_\mathfrak{p}$ is a square,

If n is even and $t_\mathfrak{p} = 2$, then $\mathfrak{a}_\mathfrak{p}$ is a norm from $K_\mathfrak{p} = k_\mathfrak{p}\left(\sqrt{(-1)^{\frac{n}{2}} d_\mathfrak{p}}\right)$.

This mass is explicitly given in Theorem 5.1.

Proof. By Lemmas 6.1 and 6.2, it suffices to show for all primes $\mathfrak{p}$ that $\Lambda'_\mathfrak{p} = \widetilde{g}\Lambda_\mathfrak{p}$ for some $\widetilde{g} \in \widetilde{G}_\mathfrak{p}^\varphi$, and by [Shi2, Lemma 5.9, p33] we know this is true for any two $\mathfrak{a}$-maximal lattices. By comparing their values under φ we see that

$$\Lambda_\mathfrak{p} \text{ is } O_\mathfrak{p}\text{-maximal} \iff \widetilde{g}\Lambda_\mathfrak{p} \text{ is } \xi(\widetilde{g})O_\mathfrak{p}\text{-maximal},$$

so the proof reduces to characterizing the set $\mathrm{ord}_\mathfrak{p}(\Xi_\mathfrak{p}(\theta_\mathfrak{p}))$. We do this by using the local models for $(W_\mathfrak{p}, \theta_\mathfrak{p})$ in [Shi, §3.2].

If $t_\mathfrak{p}$ is odd then we can never find a similitude of odd valuation, since if $ord_\mathfrak{p}(\xi(\widetilde{g}))$ is odd then taking determinants gives $ord_\mathfrak{p}(\det(\widetilde{g})^2) = \mathrm{ord}_\mathfrak{p}(\xi(\widetilde{g})^{t_\mathfrak{p}})$ which is odd. Conversely, if $\pi_\mathfrak{p}$ is a uniformizer in $k_\mathfrak{p}$ then we can construct $\pi_\mathfrak{p}^2$ in $\Xi(\Theta_p)$ by using $\widetilde{g} = \mathrm{diag}[\pi_p, \cdots, \pi_p]$.

If $t_\mathfrak{p} = 0$, then $(V_\mathfrak{p}, \varphi_\mathfrak{p})$ is a direct sum of hyperbolic planes and $\Xi_\mathfrak{p}(\theta_\mathfrak{p}) = k_\mathfrak{p}^\times$.

If $t_\mathfrak{p} = 2$, then $(W_\mathfrak{p}, \theta_\mathfrak{p}) \cong (K_\mathfrak{p}, cN_{K_\mathfrak{p}/k_\mathfrak{p}}(x))$ where $K_\mathfrak{p} = k_\mathfrak{p}(\sqrt{-\det(\varphi)})$ and $c \in k^\times$. Therefore $K_\mathfrak{p}^\times \subseteq \widetilde{G}_\mathfrak{p}^\theta$ and so $\Xi_\mathfrak{p}(\theta_\mathfrak{p}) \supseteq N_{K_\mathfrak{p}/F_\mathfrak{p}}(K_\mathfrak{p}^\times)$.

If $t_\mathfrak{p} = 4$, then $(W_\mathfrak{p}, \theta_\mathfrak{p}) \cong (B_\mathfrak{p}, N_{B_\mathfrak{p}/k_\mathfrak{p}}(x))$ where $B_\mathfrak{p}$ is a division quaternion algebra over $k_\mathfrak{p}$. Since $B^\times \subseteq \widetilde{G}_\mathfrak{p}^\theta$ and $N_{B_\mathfrak{p}/k_\mathfrak{p}}(B_\mathfrak{p}^\times) = k_\mathfrak{p}^\times$, we have $\Xi_\mathfrak{p}(\theta_\mathfrak{p}) = k_\mathfrak{p}^\times$. □

Remark. In terms of the invariants $(n_\mathfrak{p}, d_\mathfrak{p}, c_\mathfrak{p})$ for the local quadratic space $(V_\mathfrak{p}, \varphi_\mathfrak{p})$, the condition $t_v = 1$ is equivalent to $c_\mathfrak{p} = \left(\frac{(-1)^{n/2}, (-1)^{n/2} d_\mathfrak{p}}{\mathfrak{p}}\right)$ when n is odd, and $t_v = 2$ is equivalent to $[K_\mathfrak{p} : k_\mathfrak{p}] = 2$ when n is even and $K_\mathfrak{p}$ is as above.

Appendix

It will be convenient to know a few lemmas about matrices of differentials. If we take $x = (x_{ij})$ to be a matrix of functions, then we define the matrix dx to be the matrix (dx_{ij}) of differentials of x.

Lemma A1. *Let $\mathfrak{Z}_v$ be a symmetric space of the type described in §4.2 or §4.3. Then $\{g \in G_v \mid gz = z \text{ for every } z \in \mathfrak{Z}_v\} = \{\pm 1_n\}$.*

Proof. This is the analagous statement of [Shi2, Prop. 6.4(5)] for orthogonal groups, and has the same proof with obvious modifications. □

Lemma A2. *Let dx and dx' be two $r \times t$ matrices of linearly independent differentials, and suppose $dx' = a(dx)$ for some $r \times r$ constant matrix a. Then*

$$\bigwedge_{i,k} dx'_{ik} = \det(a)^t \bigwedge_{i,k} dx_{ik}.$$

Similarly, if $dx' = (dx)a'$ for some $t \times t$ constant matrix a', then

$$\bigwedge_{i,k} dx'_{ik} = \det(a')^r \bigwedge_{i,k} dx_{ik}.$$

Proof. This is well known, and follows from the action of a (resp. a') on a column (resp. row) vector. □

Lemma A3. *Let dx and dx' be two $n \times n$ matrices of linearly independent differentials and suppose $dx' = a(dx)$ for some $n \times n$ constant matrix a. Then*

$$\bigwedge_{i \le k} dx'_{ik} = \prod_{l=1}^{n} \det{}_l(a) \bigwedge_{i \le k} dx_{ik} + \sum \begin{pmatrix} \textit{terms containing at least one} \\ \textit{factor } dx_{ik} \textit{ with } i > k \end{pmatrix}.$$

Proof. It will be enough to analyze the columns $k \ge k_0$, proving inductively that for each $1 \le k_0 \le n$ we have

$$\text{(A3.1)} \qquad \bigwedge_{\substack{i \le k \\ k \ge k_0}} dx'_{ik} = \prod_{l=k_0}^{n-1} \det{}_l(a) \bigwedge_{\substack{i \le k \\ k \ge k_0}} dx_{ik} + \Omega,$$

where Ω is a sum of terms each containing at least one factor dx_{ik} with $i > k$.

If $k_0 = n$ then

$$\begin{aligned}\bigwedge_{i\leq k_0} dx'_{ik_0} &= \bigwedge_{i\leq k_0} \sum_j a_{ij} dx_{jk_0} \\ &= \bigwedge_{i\leq k_0} \sum_{\sigma\in S_n} a_{i\sigma(i)} dx'_{\sigma(i)k_0} \\ &= \det(a) \bigwedge_{i\leq k_0} dx_{ik_0}\end{aligned}$$

since the only non-zero terms in the wedge product come from permutations of the row index i.

Proceeding inductively, we consider the row k_0 and assume (A3.1) holds for all $k > k_0$. Then

(A3.2)
$$\begin{aligned}\bigwedge_{\substack{i\leq k\\ k\geq k_0}} dx'_{ik} &= \bigwedge_{i\leq k_0} dx'_{ik_0} \wedge \bigwedge_{\substack{i\leq k\\ k\geq k_0+1}} dx'_{ik} \\ &= \left(\bigwedge_{i\leq k_0} \sum_j a_{ij} dx_{jk_0}\right) \wedge \left(\prod_{l=k_0+1}^{n-1} \det_l(a) \bigwedge_{\substack{i\leq k\\ k\geq k_0+1}} dx_{ik} + \Omega\right).\end{aligned}$$

The terms dx_{jk_0} of $\bigwedge_{i\leq k_0}\sum_j a_{ij}dx_{jk_0}$ with $j > k_0$ cannot contribute to the term $\bigwedge_{i\leq k, k\geq k_0} dx_{ik}$ since the entries of dx are linearly independent. Therefore the only terms which contribute to it are the dx_{jk_0} with $j \leq k_0$ and these can be written as the following sum over permutations on the row index i:

$$\begin{aligned}\bigwedge_{i\leq k_0} \sum_{j\leq k_0} a_{ij} dx_{jk_0} &= \bigwedge_{i\leq k_0} \sum_{\sigma\in S_{k_0}} a_{i\sigma(i)} dx'_{\sigma(i)k_0} \\ &= \det_{k_0}(a) \bigwedge_{i\leq k_0} dx_{ik_0}.\end{aligned}$$

Combining this with (A3.2), we prove (A3.1). Our lemma then follows from (A3.1) by taking $k_0 = 1$. □

Lemma A4. *Let dx and dx' be two skew-symmetric $n\times n$ matrices of differentials whose upper triangular coordinates are linearly independent, and suppose $dx' = {}^t a(dx)a$ for some $n\times n$ constant matrix a. Then*

$$\bigwedge_{i>k} dx'_{ik} = \det(a)^{n-1} \bigwedge_{i>k} dx_{ik}.$$

Proof. This is proved in the same way as Lemma 3.2, the only difference being that the computation for scalar matrices here gives $\det(a)^{n-1}$. □

Lemma A5. *Let dx and dx' be two skew-symmetric $n \times n$ matrices of differentials whose upper triangular coordinates are linearly independent, and suppose $dx' = (dx)a$ for some $n \times n$ constant matrix a. Then*

$$\bigwedge_{i>k} dx'_{ik} = \prod_{l=1}^{n-1} \det_l(a) \bigwedge_{i>k} dx_{ik}.$$

Proof. We prove by induction that

$$\bigwedge_{\substack{k<i \\ i\geq i_0}} dx'_{ik} = \prod_{l=i_0}^{n-1} \det_l(a) \bigwedge_{\substack{k<i \\ i\geq i_0}} dx_{ik} \tag{A5.1}$$

for all $1 \leq i_0 \leq n$.

In the case $i_0 = n$, the non-zero terms of $\bigwedge_{k<n} \sum_j dx_{nj} a_{jk}$ come from choosing one term $dx_{nj} a_{jk}$ for each k with no repetition among the j indices. Thus the j index is a permutation of the k index, and we have

$$\bigwedge_{k<n} \sum_{\sigma \in S_{n-1}} dx_{n\sigma(k)} a_{\sigma(k)k} = \det_{n-1}(a) \bigwedge_{k<n} \sum_j dx_{nk}.$$

Now suppose $i_0 < n$. By induction we have

$$\begin{aligned}
\bigwedge_{\substack{k<i \\ i\geq i_0}} \sum_j dx_{ij} a_{jk} &= \left(\bigwedge_{k<i_0} \sum_j dx_{i_0 j} a_{jk} \right) \wedge \left(\bigwedge_{\substack{k<i \\ i\geq i_0+1}} \sum_j dx_{ij} a_{jk} \right) \\
&= \left(\bigwedge_{k<i_0} \sum_j dx_{i_0 j} a_{jk} \right) \wedge \left(\prod_{l=i_0+1}^{n-1} \det_l(a) \bigwedge_{\substack{k<i \\ i\geq i_0+1}} dx_{ik} \right).
\end{aligned}$$

By skew-symmetry of dx, we see that all of the terms in $\bigwedge_{k<i_0} \sum_j dx_{i_0 j} a_{jk}$ with $j \geq i_0$ would give zero when wedged together with $\bigwedge_{k<i, i\geq i_0+1} dx_{ik}$. Thus the only terms that contribute have the form

$$\sum_{\sigma \in S_{i_0}} dx_{i_0 \sigma(k)} a_{jk} = \det_{i_0-1}(a) \bigwedge_{k<i_0} dx_{i_0 k},$$

which together with the above proves (A5.1). Our result follows from (A5.1) by taking $i_0 = 1$. □

We now state two basic lemmas about volume forms on manifolds.

Lemma A6. *Let $F : X \to Y$ be a map of C^∞-manifolds of dimensions n and m respectively, with rank$(F) = m$. Suppose that X is a group acting on Y and the map F commutes with this action. Choose $p \in Y$ and let $C = F^{-1}(p)$ be the fibre over p. Given X-invariant volume forms ω_X and ω_Y on X and Y respectively, we can define a unique volume form ω_C on C by choosing $\omega \in (\bigwedge^{n-m})^*(X)$ such that*

$$\omega \wedge F^*(\omega_Y) = \omega_X \tag{A6.1}$$

and taking ω_C to be the restriction $\omega|_C$ of ω to C. Further, ω_C is C-invariant and when computing ω_C it suffices to take forms on X with coefficients in the fibre C over p.

Proof. In this situation, the forms on X are determined by their definition on any neighborhood, so it is sufficient to check locally on X.

Choose a point $q \in F^{-1}(p) \subset X$. Taking $y_1, \cdots, y_m$ to be a set of coordinates on Y in some neighborhood of p, we can pull these back to give coordinates $x_1, \cdots, x_m$ on some neighborhood of q in X. Since $F^{-1}(p)$ is a regular submanifold of X, we can extend these to give a complete set of coordinates $x_1, \cdots, x_n$ on a possibly smaller neighborhood of q. In these coordinates we have

$$\omega_X = f(x) \bigwedge_{i=1}^{n} dx_i, \tag{A6.2}$$

$$F^*(\omega_Y) = f_1(x) \bigwedge_{i=1}^{m} dx_i. \tag{A6.3}$$

From this we see that any ω on X satisfying (A6.1) must have the form

$$\omega = \frac{f(x)}{f_1(x)} \bigwedge_{i=m+1}^{n} dx_i + \sum \begin{pmatrix} \text{terms containing at least one} \\ \text{factor from } \{dx_1, \cdots, dx_m\} \end{pmatrix}. \tag{A6.4}$$

Such an ω exists and is a volume form since both ω_X and ω_Y are nowhere vanishing. Uniqueness of ω_C follows since $x_1, \cdots, x_m$ are constant on C, so all terms of (A6.4) except the first term vanish on C.

To see the C-invariance of ω_C, let $c_0 \in C$ act on (A6.1). This gives

$$c_0^* \wedge F^*(\omega_Y) = \omega_X.$$

But by uniqueness of ω_C we have the second part of

$$c_0^*(\omega_C) = c_0^*(\omega)|_C = \omega_C,$$

so ω_C is C-invariant.

The final assertion is easy, and can be checked in the coordinates $x_1, \cdots, x_n$ above. We write $f_1(x) = f_2(x) + f_2'(x)$ where $f_2'(x)$ has coefficients all of which are zero on C, and observe that the $f_2'(x)$ term disappears whether we restrict coefficients before or after choosing ω. $\square$

Lemma A7. *Suppose we are in the setting of Lemma A6, and take some Fuchsian subgroup $\Gamma \subseteq X$. We let μ_C, μ_X, and μ_Y denote the measures associated to ω_C, ω_X, and ω_Y respectively. Then*

$$\mu_X(\Gamma \backslash X) = \mu_Y(\Gamma \backslash Y)\mu_C((\Gamma \cap S)\backslash C),$$

where $S = \{x \in X \mid xy = y$ for every $y \in Y\}$.

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