AN EXACT MASS FORMULA FOR QUADRATIC FORMS OVER NUMBER FIELDS

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Abstract. In this paper we give an explicit formula for the mass of a quadratic form in \( n \geq 3 \) variables with respect to a maximal lattice over an arbitrary number field \( k \), and use this to find the mass of many \( \mathfrak{a} \)-maximal lattices. We make the minor technical assumption that locally the determinant of the form is a unit up to a square if \( n \) is odd. The corresponding formula for \( k \) totally real was recently computed by Shimura [Shi].

§0 Summary

Our goal is to give an exact formula for the mass of the genus of a quadratic form \( \varphi \) on a maximal lattice defined over an arbitrary number field \( k \). In §2 we explain how knowledge of the Tamagawa number of the special orthogonal group \( G_\varphi \) gives rise to a mass formula. Such a formula expresses the mass as a product of local factors over all places \( v \) of \( k \), so our problem is reduced to computing each of these. For the non-archimedean places, these factors were recently computed by Shimura [Shi]. We state his result in §3 and for completeness include a translation between our language and his. In §4 we compute the archimedean factors, treating separately the 3 cases: \( v \) real, \( \varphi \) definite; \( v \) real, \( \varphi \) indefinite; and \( v \) complex. To define the factors in the last two cases, we choose a symmetric space \( 3_v \) on which \( G_\varphi \) acts and a non-zero \( G_\varphi \)-invariant volume form \( \omega_3 \). Finally, in §5 we compute the mass of \( \varphi \) with respect to a maximal lattice. We note that this formula agrees with Shimura’s when \( k \) is totally real. In §6 we conclude by using the local similitude groups to show that this agrees with the mass of many genera of \( \mathfrak{a} \)-maximal lattices. Our results depend on several technical lemmas which we include as an appendix.

§1 Introduction

We begin with a quadratic space \((V, \varphi)\) over an algebraic number field \( k \). By this we mean a \( k \)-vector space \( V \) together with a non-degenerate quadratic form \( \varphi : V \rightarrow k \). Let \( O_k \) denote the ring of integers of \( k \) and let \( O_v \) denote the local ring of integers at each place \( v \) of \( k \). We consider \((V, \varphi)\) as well as its localizations \((V_v, \varphi_v)\) given by linear extension of scalars to \( k_v \). Given a lattice \( \Lambda \subset (V, \varphi) \), we have the associated local lattice \( \Lambda_v = \Lambda \otimes_{O_k} O_v \subset (V_v, \varphi_v) \) at each non-archimedean place.

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place \( p \) of \( k \). We occasionally write \((\Lambda, \varphi)\) for the restriction of the form \( \varphi \) to \( \Lambda \), and \((\Lambda_p, \varphi_p)\) for the restriction of \( \varphi_p \) to \( \Lambda_p \).

With \((V, \varphi)\) as above, we let \( G^\varphi = G(\varphi) \) be the special orthogonal group of \((V, \varphi)\) by which we mean the group of determinant 1 invertible linear transformations of \( V \) preserving \( \varphi \). We also define \( G^\varphi_v \) to be the special orthogonal group of \((V_v, \varphi_v)\). Then we have a natural \( G^\varphi \)-action on \((V, \varphi)\), and a natural \( G^\varphi_v \)-action on \((V_v, \varphi_v)\). We say that two lattices \( \Lambda, \Lambda' \subseteq (V, \varphi) \) are globally equivalent if there exists \( g \in G^\varphi \) such that \( \Lambda' = g\Lambda \), and locally equivalent if for each non-archimedian place \( v \), there exists \( g_v \in G^\varphi_v \) such that \( \Lambda'_v = g_v\Lambda_v \). We define the genus of \((\Lambda, \varphi)\) to be the set of all lattices locally equivalent to \((\Lambda, \varphi)\), and say that the classes of \((\Lambda, \varphi)\) are the global equivalence classes of \((\Lambda, \varphi)\) in its genus.

Let \( G^\varphi_A \) be the adelization of \( G^\varphi \), and let \( G^\varphi_A^0 \) and \( G^\varphi_h \) be the product of \( G^\varphi_v \) over the archimedean and non-archimedean places respectively. Then there is a natural \( G^\varphi_A \)-action on the space of lattices \( \Lambda \subseteq (V, \varphi) \). To see this, take \( g = (g_v) \in G^\varphi_A \) and define \( g\Lambda \) to be the lattice \( \Lambda'' \subseteq (V, \varphi) \) such that \( \Lambda''_v = g_v\Lambda_v \) for all non-archimedian places \( v \). The stabilizer of a lattice \((\Lambda, \varphi)\) defines a subgroup \( D \in G^\varphi_A \) such that \( D \subseteq G^\varphi_A^0 \) and \( D \cap G^\varphi_h \) is open and compact, and by fixing a lattice \((\Lambda, \varphi)\) we may parametrize the classes \( \mathcal{C}_1 \) of \( \Lambda \) by the elements of \( G^\varphi \backslash G^\varphi_A / D \) using \( a \mapsto \Lambda^a := a\Lambda \).

We denote by \( \Gamma^a \) the group of automorphisms of \((\Lambda^a, \varphi)\), defined as those \( g \in G^\varphi \) leaving \( \Lambda^a \) invariant. From an adelic perspective, we see that \( \Gamma^a = G^\varphi \cap aDa^{-1} \).

We say that a lattice \( \Lambda \subseteq (V, \varphi) \) is maximal if \( \varphi(\Lambda) \subseteq O_k \) and \( \Lambda \) is not properly contained in some lattice \( \Lambda' \) with \( \varphi(\Lambda') \subseteq O_k \). There is a similar notion of an \( \mathfrak{a} \)-maximal lattice for any ideal \( \mathfrak{a} \), given by replacing \( O_k \) by \( \mathfrak{a} \). It turns out that for any ideal \( \mathfrak{a} \), all of the \( \mathfrak{a} \)-maximal lattices in \((V, \varphi)\) are locally equivalent (see [Shi2, Lemma 5.9]), so it makes sense to speak about the genus of \( \mathfrak{a} \)-maximal lattices.

If \((\Lambda, \varphi)\) is a totally definite lattice over a totally real number field \( k \), then we define the mass of its genus to be

\[
\text{Mass}(\Lambda, \varphi) = \sum_{\mathfrak{a} \in \mathcal{C}_1} \left[ \Gamma^a : 1 \right]^{-1}.
\]

If \((\Lambda, \varphi)\) is not totally definite (e.g. when \( k \) is not totally real) then \( \Gamma^a \) will be an infinite group, but we would still like to somehow keep track of its size. To do this, we allow \( \Gamma^a \) to act on some symmetric space \( \mathfrak{H} \) and choose a measure on \( \mathfrak{H} \) invariant under this action. We then define the mass in terms of the measures of the quotients \( \Gamma^a \backslash \mathfrak{H} \). So in general, we define the mass of \((\Lambda, \varphi)\) to be

\[
(1.1) \quad \text{Mass}(\Lambda, \varphi) = \sum_{\mathfrak{a} \in \mathcal{C}_1} \nu(\Gamma^a),
\]

where

\[
\nu(\Gamma^a) = \begin{cases} 
\left[ \Gamma^a : 1 \right]^{-1} & \text{if } G_\mathfrak{a} \text{ is compact}, \\
\left[ \Gamma^a \cap \{ \pm 1 \} : 1 \right]^{-1}\text{vol}(\Gamma^a \backslash \mathfrak{H}) & \text{otherwise}.
\end{cases}
\]

Our main interest in this paper will be to find an exact formula for the mass of the genus of maximal lattices over an arbitrary number field \( k \) when \( n \geq 3 \). Our approach is to use the Tamagawa number of \( G^\varphi \) to extend Shimura’s computation of the mass of a maximal lattice over a totally real number field to a general number
field \( k \). Then by interpreting the mass in terms of the volume of the non-archimedian stabilizer of \( (\Lambda, \varphi) \), we use the local group of similitudes of \( \varphi \) to show that the mass is unchanged as we vary over certain genera of \( \alpha \)-maximal lattices.

This exact mass formula essentially expresses the mass as a product of even integer values of the Dedekind zeta function of \( k \), a power of the index of \( \Lambda \) in its dual lattice, and some gamma function factors. If \( \dim_k(V) \) is even, a special value of the \( L \)-function of a certain quadratic extension of \( k \) also appears.

**Summary of Notation**

Throughout this paper we take \( k \) to be a number field, \( O_k \) its ring of integers, and \( D_k \) the discriminant of \( k/\mathbb{Q} \). We denote by \( v \) a valuation (or place) of \( k \). We also let \( \mathbf{a} \) and \( \mathbf{h} \) denote the archimedean and non-archimedean places of \( k \) respectively. Suppose \( \mathfrak{p} \) is a prime ideal in \( O_k \) lying over the prime \( p \) in \( \mathbb{Z} \), and \( x \in k \). We let \( |x|_p \) denote the usual \( p \)-adic absolute value of \( x \) defined by \( |x|_p = q^{-\text{ord}_p(x)} \), where we take \( q = q_p = [O_p : \mathfrak{p}] \).

We follow the convention that if we have an object \( R \) defined at a certain valuation \( v \), we denote it by \( R_v \). If \( R_v \) is defined at each of the archimedean valuations, we also write

\[
R_{\mathbf{a}} = \prod_{v \in \mathbf{a}} R_v.
\]

For an algebraic group \( G \) defined over \( k \), we denote the adelization of \( G \) by \( G_{\mathbf{A}} \).

If \( R \) is an arbitrary set, we denote by \( R^m_n \) the \( m \times n \) matrices with coefficients in \( R \). We write the transpose of a matrix \( A \) as \( ^tA \). If \( x \) is a matrix, then we let \( x_{ij} \) denote the entry of \( x \) in the \( i \)th row and \( j \)th column. Conversely given numbers \( x_{ij} \), we let \( (x_{ij}) \) denote the matrix whose entries satisfy \( (x_{ij})_{ij} = x_{ij} \). We abbreviate the diagonal matrix

\[
\begin{pmatrix}
a_{11} & 0 & \cdots & 0 \\
0 & a_{22} & \cdots & 0 \\
\vdots & \vdots & \ddots & 0 \\
0 & 0 & 0 & a_{nn}
\end{pmatrix}
\]

by \( \text{diag}[a_{11}, \cdots, a_{nn}] \), and denote the \( n \times n \) identity matrix by \( 1_n \). Given an arbitrary \( n \times n \) matrix \( A \) and an integer \( l \) with \( 1 \leq l \leq n \), we define \( \det_l(A) \) to be the determinant of the upper left \( l \times l \) submatrix of \( A \). If \( A \) is a matrix of functions, we define the matrix of 1-forms \( dA = (dA_{ij}) \). Given two \( n \times n \) matrices \( A \) and \( B \) over \( \mathbb{R} \), we say that \( A > B \) if the matrix \( A - B \) is positive definite, and we set

\[
S^n_+ = \{ A \in \mathbb{R}^n | ^tA = A > 0 \}.
\]

We let \( (V, \varphi) \) denote a non-degenerate quadratic space of dimension \( n \) over \( k \), and take \( V_v, \Lambda_p, G^\varphi, G^\varphi_p, G^\varphi_{\mathbf{A}} \) as defined in the introduction. If we choose a basis \( \{v_1, \cdots, v_n\} \) for \( V \), we may express the bilinear form \( \varphi(v, w) \) associated to \( \varphi \) as the matrix \( \psi = [\varphi(v_i, v_j)]_{1 \leq i, j \leq n} \). We also let \( G^\varphi - (V, \varphi) \) denote the set of invertible linear transformations of \( V \) which preserve the form \( \varphi \) and have determinant \( -1 \).

For convenience, we define the symbols

\[
X = k^n, \quad T = \{ \text{Symmetric } n \times n \text{ matrices with coefficients in } k \},
\]

and their local counterparts \( T_v \), and \( X_v \) at a valuation \( v \) by replacing \( k \) by \( k_v \) in the above definition.
We set $i = \sqrt{-1} \in \mathbb{C}$. For $x \in \mathbb{R}$ we let $|x|$ be the greatest integer $\leq x$. Also, when there is no danger of confusion, we freely use the letters $i,j,k,l$ as indices. Our equations and statements are numbered first by section, then by order within each section, with the appendix labeled by A (e.g. Lemma A2).

§2 THE TAMAGAWA NUMBER AND LOCAL FACTORS

The main fact that we use in what follows is that the Tamagawa number $\tau$ of the special orthogonal group $G = G^r$ over any number field $k$ is given by

\begin{equation}
\tau(G) = 2 \quad \text{if } n \geq 3,
\end{equation}

where $n = \dim_k(V)$. To define this, we first choose a measure $(dx)_A$ on $k_A$ normalized so that

\begin{equation}
\int_{k \setminus k_A} (dx)_A = 1.
\end{equation}

We then define the Tamagawa number of $G$ to be

\begin{equation}
\tau(G) = \int_{G \setminus G_A} |\omega_G|_A,
\end{equation}

where $\omega_G$ is a non-zero left $G$-invariant differential form on $G$ of highest degree and $|\omega_G|_A$ is the volume element defined with respect to $(dx)_A$. By the product formula we see $|c \omega_G|_A = |\omega_G|_A$ for $c \in k^\times$, and since $\omega_G$ is chosen from a 1 dimensional space, this specifies a left $G$-invariant measure on $G_A$ which is independent of our choice of $\omega_G$. We call the measure associated to $\omega_G$ the Tamagawa measure on $G_A$. (For a more detailed introduction, see [Tam], [Vos], or [Weil].)

From now on when speaking of an invariant object, we always understand this to mean it is left invariant. For clarity we also define a volume form to be a nowhere zero differential form of highest degree.

For our computations, it will be useful to define another measure $(d'x)_A$ by the restricted product $(d'x)_A = \prod_v (d'x)_v$ with local measures

\begin{equation}
(d'x)_v = \begin{cases}
\text{Haar measure on } k_v \text{ normalized by } \int_{G_v} (d'x)_v = 1 & \text{if } k_v = k_p, \\
\text{Lebesgue measure on } \mathbb{R} & \text{if } k_v = \mathbb{R}, \\
\text{idz} \wedge d\bar{z} = 2 \times \text{Lebesgue measure on } \mathbb{R}^2 & \text{if } k_v = \mathbb{C}.
\end{cases}
\end{equation}

This gives $\int_{k \setminus k_A} (d'x)_A = |D_k|^{1/2}$, so in terms of $(d'x)_A$ we have

\begin{equation}
\tau(G) = |D_k|^{-\frac{\dim(G)}{2}} \int_{G \setminus G_A} |\omega_G|'_A = |D_k|^{-\frac{n(n-1)}{4}} \int_{G \setminus G_A} |\omega_G|'_A,
\end{equation}

where $|\omega_G|'_A$ is the volume element derived from $\omega_G$ using $(d'x)_A$ instead of $(dx)_A$. 
We now give a general procedure for constructing a suitable invariant volume form $\omega_G$ on $G$. By choosing a global basis $\{v_1, \ldots, v_n\}$ for $(V, \varphi)$ we can represent the bilinear form associated to $\varphi$ as a matrix $\psi$. This gives a natural map

\[ X = (k)_n \overset{\mathcal{F}}{\longrightarrow} T \]

\[ x \mapsto {}^t x \psi x, \]

whose fibre over the matrix $\psi \in T$ is the full orthogonal group of $\varphi$. Given the volume forms

\[ \omega_X = \bigwedge_{i,j} dx_{ij}, \quad \omega_T = \bigwedge_{i \leq j} dt_{ij} \]

on $X$ and $T$ respectively, we can find a differential form $\omega$ on $X$ such that

\[ \omega_X = \mathcal{F}^*(\omega_T) \wedge \omega. \]

By pulling $\omega$ back to the fibre and then restricting to the identity component, we get a form $\widetilde{\omega}_G$ on $G$. From Lemma A6, we see that $\widetilde{\omega}_G$ is a non-zero $G$-invariant volume form, and is independent of our choice of $\omega$. We will use this construction many times in our calculation, and consistently identify $G = G^\varphi = G^\psi$ as well as the image of $\Lambda$ under this identification.

For each place $v$ of $k$, we define the local representation density

\[ \beta_v(\psi) = \beta_v(\Lambda, \psi) = \frac{1}{2} \lim_{U \to v} \frac{\int_U dX}{\int_U dT}, \]

where $dX = \prod_{i,j} (dx_{ij})_v$, and $dT = \prod_{i \leq j} (dt_{ij})_v$ are the measures associated to $\omega_X$ and $\omega_T$ in these coordinates,

\[ U' = \begin{cases} \mathcal{F}^{-1}(U) & \text{if } v \in a, \\ \mathcal{F}^{-1}(U) \cap \{ x \in X_v \mid x \Lambda_v = \Lambda_v \} & \text{if } v \in h, \end{cases} \]

and $U$ is an open neighborhood of $\psi_v$ in $T_v$. From the construction of $\widetilde{\omega}_G$ above, we can easily see that $\int_D \omega_G = \beta_v(\Lambda, \psi)$ where $D \subset G_\Lambda$ is the stabilizer of $\Lambda$ (see [Tam, §6, pp119–120]). In our calculations the lattice $\Lambda$ will be fixed, so we will often suppress $\Lambda$ and write $\beta_v(\psi)$.

**Remark.** Notice that both the volume form $\omega_G$ and the local densities $\beta_v(\psi)$ depend not only on $(V, \varphi)$ and $v$, but also on our given choice of basis for $(V, \varphi)$.

Any choice of volume form $\omega_G$ can be used to define an archimedian measure $\tau_a$ on $G_a$ by $\prod_{v \in a} |\omega_G|_v$. By choosing $\omega_G = \widetilde{\omega}_G$ as above and expressing (2.4) in terms of local measures, one can prove:

**Theorem 2.1.** Let $\Lambda$ be a lattice in $(V, \varphi)$, and suppose $\psi$ a matrix representing $\varphi$ in some global basis for $V$. Then

\[ \sum_{a \in \mathcal{A}} \tau_a(G_a) = \tau(G) |D_k|^{\frac{n(n-1)}{2}} \prod_{v \in h} \beta_v(\Lambda, \psi)^{-1}, \]
with $\tau_n$ and $\beta_n(\Lambda, \psi)$ as above, and $\Gamma^n$ as defined in \S 1.

Proof. This is proved in [Cas, pp380-382] when $k = \mathbb{Q}$, but the argument there works for any number field $k$. In his notation, $\beta_v(\Lambda, \psi) = \lambda_v = \tau_v(O^+(\Lambda_v))$ and (due to a typographical error) the right side of (4.19) on p382 should read $2\zeta_n^{-1} \prod_{p \in \mathbb{P}} \lambda_p^{-1}$. See also [Tam, \S 6, pp119–120] and [Vos, \S 15, pp87–88].

To simplify our calculations, we use the invertible matrix $\sigma_v \in (k_v)^n$ to change basis locally at every place $v$, so that $\psi_v$ has the standard form

$$
\phi_v = \begin{cases}
0 & 0 & 2^{-1}r \\
0 & \theta_p & 0 \\
2^{-1}r & 0 & 0 \\
1_q & 0 & \\
0 & -1_r \\
1_n & \\
\end{cases}
$$

if $k_v = k_p$,

$$
\begin{cases}
1_q & 0 & \\
0 & -1_r \\
1_n & \\
\end{cases}
$$

if $k_v = \mathbb{R},$

$$
\begin{cases}
1_n & \\
\end{cases}
$$

if $k_v = \mathbb{C},$

with $q, r \in \mathbb{N}$ satisfying either $q + r = n$ and $q \geq r$, or $\dim(\theta_p) + 2r = n$ and $\theta_p$ is some anisotropic symmetric matrix with $\dim(\theta_p) \leq 4$. Since we take $\Lambda$ to be a maximal lattice, by [Shi2, Lemma 5.6], we can locally choose a free $O_p$-basis for $\Lambda_p$ so that $(\Lambda_p, \varphi_p)$ is represented by the matrix $\phi_p$ above, and we choose the matrices $\sigma_p$ so this is true. The following lemma describes how the local factors change under such a change of basis.

Lemma 2.2. Let $v$ be a place of $k$ and suppose that $\psi$ and $\psi'$ in $(k_v)^n$ are related by $\psi' = \psi A$ for some invertible $n \times n$ matrix $A$. Then

$$
\beta_v(\Lambda, \psi') = |\det(A)|^{n+1} \beta_v(\Lambda\Lambda, \psi).
$$

Proof. For $n \times n$ matrices $A \in X$ and $t \in T$ we let $[A] : T \to T$ denote the map $[A](t) = \psi A$, which corresponds to change of basis by $A$ for the quadratic form associated to $t$.

Fix an open set $U$ about $\psi'$ in $T$, and let $V = [A^{-1}](U)$ be the corresponding neighborhood of $\psi$. Then one can easily check

$$
\frac{\vol_X(\mathcal{F}_{\psi'}^{-1}(U))}{\vol_T(U)} \cdot \frac{\vol_T(U)}{\vol_T([A^{-1}](U))} = \frac{\vol_X(\mathcal{F}_{\psi'}^{-1}(V)A)}{\vol_T(V)} = \frac{\vol_X(\mathcal{F}_{\psi'}^{-1}(V))}{\vol_T(V)},
$$

where $\mathcal{F}$ is as in (2.5) and the last equality follows from both parts of Lemma A2.

By passing to the limit as $U \to \psi'$, we have

$$
\beta_v(\Lambda, \psi') = \lim_{U \to \psi'} \frac{\vol_T([A^{-1}](U))}{\vol_T(U)} \beta_v(\Lambda\Lambda, \psi).
$$

This ratio of volumes is given by computing the pull-back of the volume form $\omega_T$ under the map $[A]$. We claim that

$$
[A]^*(\omega_T) = \det(A)^{n+1} \omega_T,
$$

where $[A]^*$ denotes the pull-back of the volume form $\omega_T$ under the map $[A]$.\qed
which is to say
\[(2.10) \quad \bigwedge_{i \leq j} d(AtA)_{ij} = \det(A)^{n+1} \bigwedge_{i \leq j} dt_{ij}.\]

Since \([AB] = [B][A]\), we already know (2.10) is true if we replace \(\det(A)^{n+1}\) by some multiplicative character on \(GL_n(k_v)\). By construction \(c(A)\) is a polynomial in the entries of \(A\), and since the only continuous characters on \(GL_n\) are powers of the determinant, we easily verify (2.10) by checking the scalar matrices \(A = \lambda \cdot 1_n\).

With this we have
\[
\lim_{U \to \psi} \frac{\text{vol}_T([A]^{-1}(U))}{\text{vol}_T(U)} = |\det(A)|_\psi^{n+1},
\]
which proves our lemma. \(\square\)

\section{The Non-Archimedian Local Factors}

The non-archimedian local factors appearing in the mass formula for a maximal lattice \(\Lambda\) have been calculated by Shimura in [Shi], under the condition that locally the determinant of \(\varphi\) is a unit up to a square if \(n\) is odd. We now show how his local factors relate to the local factors \(p(\mathcal{P})\) appearing in our mass formula.

Fix a basis \(\{v_1, \ldots, v_n\}\) for \(V_p\), let \(\phi\) be the invertible \(n \times n\) matrix defined over \(k_p\) which represents \(\mathcal{P}\) in this basis, and let \(\Lambda_p\) be a lattice in \((V_p, \varphi_p)\). We define \(\beta_p(\phi)\) as in §2 to be the limit of the ratio of volumes
\[(3.1) \quad \beta_p(\phi) = \beta_p(\Lambda_p, \phi) = \frac{1}{2} \lim_{i \to \infty} \frac{\int_{U_i} dX}{\int_{U_i} dT},\]
where \(U_i\) is a neighborhood in \(X_p\) determined by \(\mathcal{P}\) and an open neighborhood \(U\) of \(p\) in \(T_p\). We may also write \(U_i\) as \(U_i = U_i + \mathcal{P}\) to emphasize its dependence on the matrix \(\phi\). Since we are working over a \(p\)-adic field, we have a natural choice of neighborhoods \(U_i\) to use for this limit, namely \(U_i = \phi + P_i\) where \(P_i = (p)^{\frac{n(n+1)}{2}} \cap T_p\).

\textbf{Lemma 3.1.} Let \(\Lambda_p\) and \(\phi\) be as above, and let \(c \in k_p^\times\). Then we have
\[
\beta_p(\Lambda_p, \phi) = |c|^{\frac{n(n+1)}{2}} \beta_p(\Lambda_p, c\phi) = |\det(c \cdot 1_n)|_p^{\frac{n(n+1)}{2}} \beta_p(\Lambda_p, c\phi).
\]

\textbf{Proof.} Since \(U \to \phi\), it suffices to compute the limit (3.1) for \(U = U_i\). Consider the pre-images
\[
U_i' = \{ x \in X_p \mid x \phi x \in \phi + P_i \text{ and } x\Lambda_p = \Lambda_p \},
\]
and notice \(U_i'(\phi) = U_i'(c\phi)\). Using this we have
\[
\beta_p(\phi) = \frac{1}{2} \lim_{i \to \infty} \frac{\int_{U_i'} dX}{\int_{U_i} dT}
= \frac{1}{2} \lim_{i \to \infty} \frac{\int_{U_i'} dX}{\int_{U_i} dT}
= \frac{n(n+1)}{2} \lim_{i \to \infty} \frac{\int_{U_i'} dX}{\int_{U_i} dT}
= |c|^{\frac{n(n+1)}{2}} \beta_p(c\phi),
\]
which completes the proof. \(\square\)
Lemma 3.2. Let \( \Lambda_p \) and \( \phi \) be as above, and suppose that for our choice of basis we have \( \Lambda_p = \sum_{i=1}^n O_p v_i \). Then \( \beta_p(\Lambda_p, \phi) = \frac{1}{2} e_p(\phi) \), where \( e_p(\phi) \) is as in [Shi, §8].

Proof. In [Shi, §8] \( e_p(\phi) \) is given by

\[
\beta_p(\psi) = \frac{1}{2} \lim_{i \to \infty} \frac{\int_{U_i'} dX}{\int_{U_i} dT} = \frac{1}{2} \lim_{i \to \infty} \left( \frac{1}{q^2} \right)^{n^2} \frac{N_i'}{n(n+1)} = \frac{1}{2} e_p(\psi),
\]

which proves the lemma. \( \square \)

We are interested in computing \( \beta_p(\phi_p) \) with respect to a maximal lattice \( \Lambda_p \) in \((V_p, \varphi_p)\), with \( \phi_p \) as in §2. By Lemmas 3.1 and 3.2 we know

\[
\beta_p(\phi_p) = \left| \det(2 \cdot 1_n) \right|_p^{\frac{\alpha+1}{2}} e_p(2\phi_p),
\]

and by combining this with [Shi; Theorem 8.6(3), Prop. 3.9, (3.1.9)], we obtain

\[
\beta_p(\phi_p) = \left| \det(2 \cdot 1_n) \right|_p^{\frac{\alpha+1}{2}} q^m \kappa \langle \Delta_p : \Lambda_p \rangle \xi,
\]

where \( q = \#(O_p/pO_p) \), \( \kappa \) is defined by \( 2O_p = p^\kappa \),

\[
\xi = \begin{cases} 
(1 - q^{-m}) \prod_{i=1}^{m-1} (1 - q^{-2i}) & \text{if } t = 0, \\
\prod_{i=1}^m (1 - q^{-2i}) & \text{if } t = 1, \\
(1 + q^{-m}) \prod_{i=1}^{m-1} (1 - q^{-2i}) & \text{if } t = 2, p \text{ is unramified in } K, \\
2(1 - q)(1 + q^{-1-m})^{-1} \prod_{i=1}^{m-1} (1 - q^{-2i}) & \text{if } t = 2, p \text{ is ramified in } K, \\
2 \prod_{i=1}^{m-1} (1 - q^{-2i}) & \text{if } t = 2, \tilde{p} \neq \Lambda_p, \\
2(1 + q) \prod_{i=1}^{m-1} (1 - q^{-2i}) & \text{if } t = 3, \\
2(1 + q)(1 - q^{-1-m})^{-1} \prod_{i=1}^{m-1} (1 - q^{-2i}) & \text{if } t = 4,
\end{cases}
\]

with \( t = \dim(\theta_p) \), \( m = \lfloor n/2 \rfloor \), \( K = k(\sqrt{(-1)^{n/2} \det(\varphi)}) \), and \( \tilde{\Lambda}_p = \{ x \in V_p \mid 2\varphi_p(x, \Lambda_p) \in O_p \} \). For future reference we explicitly state [Shi, (3.1.9)], which says

\[
\langle \tilde{\Lambda}_p : \Lambda_p \rangle = \left| \det(2\phi_p) \right|_p^{-1}.
\]
In this section we explicitly compute the volume form $\omega_G$ on $G_v = G_v^\circ$ described in §2 when $v$ is archimedian, and relate $\omega_G$ to a natural volume form $\omega_\mathfrak{Z}$ on the symmetric space $\mathfrak{Z}_v$. The relationship between $\omega_G$ and $\omega_\mathfrak{Z}$ is established by constructing a non-zero $C_v$-invariant volume form $\omega_C$ on the fibre $C_v$ of $G_v$ over some chosen point $p_v \in \mathfrak{Z}_v$, and then evaluating $\int_C \omega_C$. This allows us to connect the associated measures on $G_v$ and $\mathfrak{Z}_v$. We note that when $v$ is real and $\varphi$ is definite, the situation is much simpler since $\mathfrak{Z}_v = f_1^n g$ and $C_v = G_v$.

For our calculations we would like to write down $\omega_G$ in some set of coordinates on $G$, and we choose the coordinates given by the strictly lower triangular matrix entries of the natural embedding $G \hookrightarrow (k_v)_n$. These give coordinates on an open subset of $G$ whose compliment has measure zero, and the associated coordinate $1$-forms give a basis for the cotangent space. The matrix $g^{-1}dg$ is a $G$-invariant matrix of $1$-forms under left multiplication, and so the form

$$(4.1) \quad \gamma_n = \bigwedge_{i>k} (g^{-1}dg)_{ik}$$

gives a $G$-invariant volume form on $G$. Since the space of such forms is $1$-dimensional, any $G$-invariant volume form will be a constant multiple of $\gamma_n$.

**Calculation 4.1.** Suppose $v$ is archimedian. Then in the coordinates given by $G_v \hookrightarrow (k_v)_n$, the volume form $\omega_G$ described in §2 can be written as

$$\omega_G = \pm \frac{1}{2^n} \gamma_n = \pm \frac{1}{2^n} \prod_{i=1}^n \det_i(x)^{-1} \bigwedge_{i>k} dx_{ik}.$$ 

**Proof.** To compute $\omega_G$ it suffices to compute any non-zero monomial $\Theta$ in $\mathfrak{F}^\ast(\omega_T)$, since if $\Theta = f(x) \bigwedge_{(i,k) \in I} dx_{ik}$ for some indexing set $I$ and $\omega = f(x)^{-1} \bigwedge_{(i,k) \not\in I} dx_{ik}$ is its complimentary monomial, then $\mathfrak{F}^\ast(\omega_T) \wedge \omega = \Theta \wedge \omega = \omega_X$. We choose to calculate the monomial $\Theta = f(x) \bigwedge_{i \leq k} dx_{ik}$. Since we are only interested in finding $\omega_G$ up to sign, it will be enough to compute $\omega_G$ for $\varphi_v = 1_n$.

From (2.5) we have $t = \mathfrak{F}(x) = t_1 x x$ and so $\mathfrak{F}^\ast(dt) = t_1(dx)x + t_1(x)(dx)$. Therefore

$$(4.2) \quad \mathfrak{F}^\ast(\omega_T) = \bigwedge_{i \leq k} \left( \sum_j dx_{ji}x_{jk} + x_{ji}dx_{jk} \right) = \Theta + \text{other terms}.$$ 

We compute $\Theta$ by induction on the column bound $k_0$, showing that

$$(4.3) \quad \bigwedge_{i \leq k \leq k_0} \left( \sum_j dx_{ji}x_{jk} + x_{ji}dx_{jk} \right) = q^{k_0} \bigwedge_{i \leq k \leq k_0} \sum_j x_{ji}dx_{jk} + \Psi$$

where $\Psi$ is a sum of terms each of which has some $dx_{ik}$ factor with $i > k$. 


The case \( k_0 = 1 \) is clear since the left side is just \( 2x_{11} dx_{11} \). If \( k_0 > 1 \) we have

\[
\bigwedge_{i \leq k \leq k_0} \left( \sum_j dx_{ji} x_{jk} + x_{ji} dx_{jk} \right)
\]

\[
= \bigwedge_{i \leq k \leq k_0 - 1} \left( \sum_j dx_{ji} x_{jk} + x_{ji} dx_{jk} \right) \bigwedge_{i \leq k = k_0} \left( \sum_j dx_{ji} x_{jk_{k0}} + x_{ji} dx_{jk_{k0}} \right)
\]

\[
= \left( 2^{k_0 - 1} \bigwedge_{i \leq k \leq k_0 - 1} \sum_j x_{ji} dx_{jk} + \Psi \right) \bigwedge_{i \leq k = k_0} \left( \sum_j dx_{ji} x_{jk_{k0}} + x_{ji} dx_{jk_{k0}} \right)
\]

We now analyze the term \( \Xi = \bigwedge_{i \leq k \leq k_0} \left( \sum_j dx_{ji} x_{jk_{k0}} + x_{ji} dx_{jk_{k0}} \right) \) appearing at the end of (4.4). The only terms of \( \Xi \) contributing non-zero terms to \( \Theta \) come from the column \( k_0 \). This is because all of the \( dx_{jk} \) terms with \( k \leq k_0 - 1 \) already appear in each term of \( \bigwedge_{i \leq k \leq k_0 - 1} \sum_j x_{ji} dx_{jk} \) contributing to \( \Theta \), and so the wedge product of the two is zero. Also, since the entries of \( dx \) are linearly independent, such factors \( dx_{jk_{k0}} \) must satisfy \( j \leq k_0 \) to contribute to \( \Theta \). So \( \Xi \) in (4.4) can be replaced by

\[
\bigwedge_{i < k_0} \left( \sum_j x_{ji} dx_{jk_{k0}} \right) \bigwedge \left( \sum_j dx_{jk_{k0}} x_{jk_{k0}} + x_{jk_{k0}} dx_{jk_{k0}} \right)
\]

\[
= 2 \bigwedge_{i \leq k_0} \left( \sum_j x_{ji} dx_{jk_{k0}} \right),
\]

which proves (4.3).

By combining (4.3) with \( k_0 = n \) and Lemma A3, we see that

\[
\Theta = 2^n \bigwedge_{i \leq k} (t x_{ik}) = 2^n \prod_{l=1}^{n} \det_i(x) \bigwedge_{i \leq k} dx_{ik} + \text{other terms},
\]

which shows that

\[
\omega_G = \frac{1}{2^n} \prod_{l=1}^{n} \det_i(x)^{-1} \bigwedge_{i > k} dx_{ik}
\]

satisfies (2.7). \( \square \)

§4.1 Computation for \( k_v = \mathbb{R} \) with \( \varphi \) definite

If \( v \) is real and \( \varphi \) is definite, then \( G_v = SO_n(\mathbb{R}) \). Since \( SO_n(\mathbb{R}) \) is compact, \( \tau_v(G_v) \) is finite. We now find the measure \( \tau_v \) of \( SO_n(\mathbb{R}) \) with respect to \( \omega_G \). From Calculation 4.1 and Lemma A3 we see that (up to sign) on \( G_v \)

\[
\omega_G \sim \bigwedge_{i > k} (t g_{ik}) \sim \bigwedge_{i > k} (g^{-1} dg)_{ik},
\]

and this together with the volume computation in [Vos, (14.6), p85] relative to volume form \( \bigwedge_{i > k} (g^{-1} dg)_{ik} \) gives

\[
\omega_G = \frac{1}{2} x_{11}^{n(n+1)/2} \left( \prod_{l=1}^{n} \Gamma(l/2) \right)^{-1}.
\]
4.2 Computation for $k_v = \mathbb{R}$ with $\varphi$ indefinite

If $v$ is real and $\varphi_v$ is indefinite, then we take $\phi_v = \text{diag}[1_{q}, -1_{r}]$ as in (2.9), $G_{\mathbb{R}} = SO(q, r)$, and define the (symmetric) space $\mathfrak{Z}_{\mathbb{R}}$ by

$$\mathfrak{Z}_{\mathbb{R}} = \left\{ \left[ \begin{array}{c} x \\ y \end{array} \right] \in \mathbb{R}^q \middle| x \in \mathbb{R}_r, y \in \mathbb{R}_r, \quad ^tx + x > ^tyy \right\}.$$  

To define a $G_{\mathbb{R}}$-action on $\mathfrak{Z}_{\mathbb{R}}$, let

$$B(z) = \left[ \begin{array}{cc} ^tx & ^ty \\ 0 & 1_r \end{array} \right], \quad \gamma = \left[ \begin{array}{ccc} \frac{1}{\sqrt{2}r} & 0 & \frac{1}{\sqrt{2}r} \\ 0 & 1_r & 0 \\ \frac{1}{\sqrt{2}r} & 0 & \frac{1}{\sqrt{2}r} \end{array} \right],$$

$$\mathfrak{Y} = \{ Y \in GL_n(\mathbb{R}) \mid ^tY \phi_v^{-1}Y = \text{diag}[A, B] \text{ with } A \in S^+_q, B \in S^+_r \},$$

and induce a $G_{\mathbb{R}}$-action on $\mathfrak{Z}_{\mathbb{R}}$ from the bijection

$$(4.2.1) \quad \mathfrak{Z}_{\mathbb{R}} \times GL_q(\mathbb{R}) \times GL_r(\mathbb{R}) \xrightarrow{\sim} \mathfrak{Y}$$

by allowing $\alpha \in G_{\mathbb{R}}$ to act on $\mathfrak{Y}$ by left multiplication. (See [Shi2, §6] for details.) Explicitly, (4.2.1) gives the action $z \mapsto \alpha z$ on $\mathfrak{Z}_{\mathbb{R}}$ by

$$(4.2.2) \quad \alpha B(z) = B(\alpha z) \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix},$$

for some matrices $\lambda_\alpha(z)$ and $\mu_\alpha(z)$.

By choosing a distinguished point $p_{\mathbb{R}} = \left[ \begin{array}{c} 1_r \\ 0_r \end{array} \right] \in \mathfrak{Z}_{\mathbb{R}}$, we define a map

$$(4.2.3) \quad F_{\mathbb{R}} : G_{\mathbb{R}} \longrightarrow \mathfrak{Z}_{\mathbb{R}}$$

$$\alpha \longmapsto \alpha p_{\mathbb{R}}.$$  

If we write $\alpha \in G_{\mathbb{R}}$ as

$$(4.2.4) \quad \alpha = \left[ \begin{array}{ccc} a & b & c \\ g & e & f \\ h & l & w \end{array} \right]$$

with $a, d \in \mathbb{R}_r$ and $e \in \mathbb{R}_r^*$, then the map $F_{\mathbb{R}}$ sends

$$(4.2.5) \quad \alpha \longmapsto \alpha p_{\mathbb{R}} = \left[ \begin{array}{c} (w - c)(w + c)^{-1} \\ (\sqrt{2})_l f(w + c)^{-1} \end{array} \right].$$

In these coordinates the stabilizer of $p_{\mathbb{R}}$ is given by

$$(4.2.6) \quad C_{\mathbb{R}} = \{ \alpha \in G_{\mathbb{R}} \mid f = 0_r, c = 0_r^* \},$$
and the relation \( t^* x \phi_n x = \phi_n \) implies that \( l \) and \( h \) are also zero. Thus \( C_\mathbb{R} \) decomposes as

\[
C_\mathbb{R} \cong [G_\mathbb{R}(1_q) \times G_\mathbb{R}(1_r)] \cup [G_\mathbb{R}^{-1}(1_q) \times G_\mathbb{R}^{-1}(1_r)]
\]

(4.2.7)

\[
\alpha \mapsto \left( \begin{array}{cc}
 a & b \\
 g & e
\end{array} \right), w.
\]

We will be working with the \( G_\mathbb{R} \)-invariant volume form \( \omega_3 \) on \( \mathfrak{g}_\mathbb{R} \) constructed in [Shi, §4.2], given by the expression

\[
\omega_3 = \delta(z)^{-n/2} \bigwedge_{i,k} dz_{ik},
\]

where \( \delta(z) = \det(\frac{1}{2}(t^* x + x - t^* y y)). \)

**Computation of \( \omega_C \) and \( \int_C \omega_C \)**

We now compute the expression for \( \omega_C \) on \( C_\mathbb{R} = \text{Stab}(p_\mathbb{R}) \) described in §4. For this it is enough, by the last part of Lemma A6, for us to consider forms whose restrictions to the fibre \( C_\mathbb{R} \) are equal up to sign. We write this equivalence as \( \approx \).

From (4.2.5) we have

\[
F^*_\mathbb{R}(dx) = -(1_r + (w - c)(w + c)^{-1})dc(w + c)^{-1}
+ (1_r - (w - c)(w + c)^{-1})dw(w + c)^{-1}
\approx -2_r dc w^{-1},
\]

\[
F^*_\mathbb{R}(dy) = -(\sqrt{2})_r df(w + c)^{-1} - (\sqrt{2})_r f(w + c)^{-1}d(w + c)(w + c)^{-1}
\approx (\sqrt{2})_r df w^{-1}.
\]

Applying Lemma A2 and \( \det(w) \approx 1 \) to these gives

\[
\bigwedge_{i,k} F^*_\mathbb{R}(dx)_{ik} \approx 2^n \bigwedge_{i,k} dc_{ik},
\]

\[
\bigwedge_{i,k} F^*_\mathbb{R}(dy)_{ik} \approx 2^n \bigwedge_{i,k} df_{ik},
\]

which together with the observation \( \delta(p_\mathbb{R}) = 1 \) yields

\[
F^*_3(\omega_3) \approx 2^n \bigwedge_{i,k} dc_{ik} \bigwedge_{i,k} df_{ik}.
\]

We recall from Calculation 4.1,

\[
\omega_G \approx 2^{-n} \prod_{l=1}^n \det_l(\alpha)^{-1} \bigwedge_{i > k} d\alpha_{ik}.
\]

By the construction of \( \omega_G \) in §2 and \( F^*_3(\omega_3) \) as above, and since the matrix \( g^{-1} dg \) of §4 is skew symmetric, we see that the volume form \( \omega_C \) on the fibre is

\[
\omega_C \approx 2^{-n} 2^n \prod_{l=1}^n \det_l(\alpha)^{-1} \bigwedge_{i > k} da_{ik} \bigwedge_{i > k} de_{ik} \bigwedge_{i,k} dg_{ik} \bigwedge_{i,k} dw_{ik}
\approx 2^{-n} \omega_{SO_q(\mathbb{R})} \wedge \omega_{SO_r(\mathbb{R})}.
\]
By comparison with $\omega_G$ in §4.1 and the isomorphism (4.2.7), we find that

$$\text{vol}_C(C_{\mathbb{R}}) = \int_{C_k} |\omega_C|$$

$$= 2 \cdot 2^{-r} \frac{1}{2\pi} \left( \prod_{k=1}^r \Gamma(k/2) \right)^{-1} \frac{1}{2\pi} \Gamma(r+1) \left( \prod_{k=1}^r \Gamma(k/2) \right)^{-1},$$

which completes our calculation.

§4.3 Computation for $k_v = \mathbb{C}$

If $v$ is complex, then $G_{\mathbb{C}} = SO_n(\mathbb{C})$ and we define the (symmetric) space $\mathfrak{Z}_{\mathbb{C}}$ by

$$\mathfrak{Z}_{\mathbb{C}} = \{ z \in \mathbb{R}_{n}^{n} | {}^t z = -z, {}^t z z < 1 \}.$$

To define a $G_{\mathbb{C}}$-action on $\mathfrak{Z}_{\mathbb{C}}$, we first let

$$B(z) = \begin{bmatrix} 1_n & z \\ -z & 1_n \end{bmatrix}, \quad I = \begin{bmatrix} 1_n & 0 \\ 0 & -1_n \end{bmatrix},$$

$$\mathfrak{X} = \left\{ X \in GL_{2n}(\mathbb{R}) \bigg| {}^t IX = \begin{bmatrix} A & 0 \\ 0 & -B \end{bmatrix} \right\} \text{ with } A, B \in S^n_{+}.$$

One can check that this gives an injection

$$\mathfrak{Z}_{\mathbb{C}} \times GL_n(\mathbb{R}) \times GL_n(\mathbb{R}) \longrightarrow \mathfrak{X}$$

$$\begin{array}{c} (z, \lambda, \mu) \longrightarrow B(z) \begin{bmatrix} \lambda & -B \\ -B & \mu \end{bmatrix}. \end{array} \quad (4.3.1)$$

Writing $\alpha = a + bi \in G_{\mathbb{C}}$ with $a, b \in \mathbb{R}_{n}^{n}$, we define $\iota(\alpha) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ and allow $\alpha$ to act on $x \in \mathfrak{X}$ by left multiplication by $\iota(\alpha)$

$$\alpha x = \iota(\alpha)x.$$

By a direct calculation we see that this gives a well-defined action on the image of (4.3.1) and can be used to define a $G_{\mathbb{C}}$-action on $\mathfrak{Z}_{\mathbb{C}}$ by

$$\alpha B(z) = \iota(\alpha)B(z) = B(\lambda_{\alpha}(z)) \begin{bmatrix} \lambda_{\alpha}(z) & 0 \\ 0 & \mu_{\alpha}(z) \end{bmatrix}, \quad (4.3.2)$$

the key observation being that $\iota(\alpha)I\iota(\alpha) = I$ for $\alpha \in G_{\mathbb{C}}$. The same calculation shows that

$$\lambda_{\alpha}(z) = \mu_{\alpha}(z) = (a + bz),$$

which we henceforth denote by $\mu_{\alpha}(z)$. 
By choosing a distinguished point $p_C = 0_n^n \in Z_C$, we define a map

$$F_C : G_C \longrightarrow Z_C$$

\[ (4.3.3) \]

$$\alpha \longmapsto \alpha p_C. \]$$

Writing this map out in real coordinates we see

\[ (4.3.4) \]

$$\alpha = a + bi \longmapsto -ba^{-1},$$

where $a, b \in \mathbb{R}_n$. In these coordinates the stabilizer of $p_C$ is given by

\[ (4.3.5) \]

$$C_C = \text{Stab}(p_C) = \{ \alpha = a + bi \in G_C \mid b = 0_n^n \} \cong SO_n(\mathbb{R}).$$

We now construct a $G_C$-invariant volume form on $Z_C$. To do this we need to know how the differentials transform under the map $F_C$. We begin with a few definitions. For any two points $w, z \in Z_C$ we let

\[ (4.3.6) \]

$$\xi(w, z) = 1_n - t_wz, \quad \xi(z) = \xi(z, z),$$

\[ (4.3.7) \]

$$\delta(w, z) = \det(\xi(w, z)), \quad \delta(z) = \delta(z, z),$$

which satisfies the relation

\[ (4.3.8) \]

$$t_B(w)IB(z) = \begin{bmatrix} \xi(w, z) & z + t_w \xi(w, z) \\ z + t_w & -\xi(w, z) \end{bmatrix}. $$

By combining (4.3.8), $t^t(\alpha)I t(\alpha) = I$, and (4.3.2), we have

$$t^t(\alpha)(w)(\alpha z - \alpha w)\mu_\alpha(z) = z - w,$$

$$t^t(\alpha)(w)\xi(\alpha w, \alpha z)\mu_\alpha(z) = \xi(w, z).$$

Fixing $w \in Z_C$, we differentiate these with respect to $z$ and evaluate at $z = w$ to obtain

$$d(\alpha z) = t^t(\alpha)(z)^{-1} dz \mu_\alpha(z)^{-1},$$

$$\delta(\alpha z) = \det(\mu_\alpha(z))^{-2} \delta(z).$$

By applying Lemma A4 to these two equations, we see that the expression

\[ (4.3.9) \]

$$\omega_3 = \delta(z)^{\frac{i - n}{2}} \bigwedge_{i > k} dz_{ik}$$

gives a non-zero $G_C$-invariant volume form on $Z_C$.

**Computation of $\omega_C$ and $\int_C \omega_C$**

We now compute the form $\omega_C$ on $C_C = \text{Stab}(p_C)$ described in §4. By the last part of Lemma A6, it is enough to consider forms whose restrictions to the fibre $C_C$ are equal up to sign. We write this equivalence as $\approx$.

First we compute $F_C^*(\omega_3)$. From (4.3.4) we have

$$F_C^*(dz) = -da^{-1} - b da^{-1} \]$$

$$\approx da^{-1},$$
and so

$$\bigwedge_{i > k} F^*_C(dz)_{ik} \approx \bigwedge_{i > k} (db a^{-1})_{ik}.$$  

From the relations defining $G_C$, we know that $'a \approx a^{-1}$ and the restriction of $'a \ db$ to $C_C$ is skew symmetric, therefore so is $a('a \ db)a^{-1} = db a^{-1}$. Applying Lemma A5 to this gives

$$\bigwedge_{i > k} db_{ik} = \prod_{l=1}^{n-1} \det_l(a) \bigwedge_{i > k} (db a^{-1})_{ik}$$

and so

$$F^*_C(\omega_3) = \prod_{l=1}^{n-1} \det_l(a)^{-1} \bigwedge_{i > k} db_{ik}$$

since $\delta(p_C) = 1$.

From our choice of local measures in §2, the real volume form $\tilde{\omega}$ associated to the complex volume form $\omega$ is given by $\omega \wedge \varpi$. Combining this with Calculation 4.1 we have

$$\tilde{\omega}_G = 2^{-2n} \prod_{l=1}^{n} \det_l(z)^{-1} \det_l(\bar{z})^{-1} \bigwedge_{i > k} (idz_{ik} \wedge d\bar{z}_{ik})$$

$$= 2^{\frac{n(n-5)}{2}} \prod_{l=1}^{n} \det_l(z)^{-1} \det_l(\bar{z})^{-1} \bigwedge_{i > k} (da_{ik} \wedge db_{ik})$$

$$\approx 2^{\frac{n(n-5)}{2}} \prod_{l=1}^{n-1} \det_l(a)^{-2} \bigwedge_{i > k} (da_{ik} \wedge db_{ik}).$$

By applying the procedure in §2 to $\tilde{\omega}_G$ and $F^*_C(\omega_3)$ above, we see that the (real) volume form $\omega_C$ on the fibre is given by

$$\omega_C = 2^{\frac{n(n-5)}{2}} \prod_{l=1}^{n-1} \det_l(a)^{-1} \bigwedge_{i > k} da_{ik}.$$  

From §4.1, we know

$$\int_{SO_n(\mathbb{R})} \omega_G = \frac{1}{2\pi} 2^{\frac{n(n+1)}{2}} \frac{n}{j} \Gamma(j/2)^{-1},$$

so we have

$$\text{vol}_C(C_C) = \int_{C_C} \omega_C = 2^{\frac{n(n-3)}{2}} \int_{SO_n(\mathbb{R})} \omega_G = 2^{\frac{n(n-3)}{2}} \left( \frac{1}{2\pi} 2^{\frac{n(n+1)}{2}} \frac{n}{j} \Gamma(j/2)^{-1} \right),$$

which completes our calculation.
In this section we compute an exact mass formula for the genus of maximal lattices \( \Lambda \subset (V, \varphi) \). We call a lattice \( \Lambda \subset (V, \varphi) \) a **maximal lattice** if \( \varphi(\Lambda) \subseteq O_k \) and \( \Lambda \) is maximal with this property.

To define the mass of a genus of lattices over an arbitrary number field \( k \), we need to define symmetric spaces \( Z_v \) for all \( v \in \mathfrak{a} \). If \( v \) is real and \( \varphi_v \) is definite, then we take \( Z_v \) to be a single point with measure one. If \( v \) is real and \( \varphi_v \) is indefinite or \( v \) is complex, then we take \( Z_v \) as in 4.2 or 4.3 respectively. The spaces \( Z_v \) come equipped with a transitive \( G_v \)-action, an invariant volume form \( \omega_v \), and a distinguished point \( p_v \). For each \( v \in \mathfrak{a} \), we define a surjective map

\[
F_v : G_v \rightarrow Z_v
\]

and denote by \( C_v \) the fibre of \( F_v \) over \( p_v \). We let

\[
\mathfrak{z} = \prod_{v \in \mathfrak{a}} Z_v, \quad C = \prod_{v \in \mathfrak{a}} C_v, \quad p = (p_v)_{v \in \mathfrak{a}},
\]

and let \( F \) denote the product map

\[
F : G_\mathfrak{a} \rightarrow \mathfrak{z}.
\]

We observe that the \( C = F^{-1}(p) \) is the fibre of \( F \) over \( p \).

We define the **mass** of a quadratic form \((V, \varphi)\) with respect to a lattice \( \Lambda \) to be

\[
\text{Mass}(\Lambda, \varphi) = \sum_{\alpha \in \mathfrak{t}} \nu(\Gamma^\alpha)
\]

where

\[
\nu(\Gamma^\alpha) = \begin{cases} 
[\Gamma^\alpha : 1]^{-1} & \text{if } G_\mathfrak{a} \text{ is compact}, \\
[\Gamma^\alpha \cap \{\pm 1\} : 1]^{-1} \text{vol}(\Gamma^\alpha \backslash Z) & \text{otherwise}.
\end{cases}
\]

**Theorem 5.1.** Let \((V, \varphi)\) be a non-degenerate quadratic space of dimension \( n \geq 3 \) defined over a number field \( k \) of degree \( d \) over \( \mathbb{Q} \). Then the mass of \((V, \varphi)\) with respect to a maximal lattice \( \Lambda \subset (V, \varphi) \) is given by

\[
\text{Mass}(\Lambda, \varphi) = 2|D_k|^\left(\frac{n-1}{2}\right)^2 \left[\sum_{j=1}^{\frac{n-1}{2}} |D_k|^\frac{j}{2}\left(\frac{(2j-1)!}{(2\pi)^{2j}}\right)^d \zeta_k(2j)\right]\left[\mathcal{A} : \Lambda\right]^{\frac{n-1}{2}} \prod_{v \mid \mathbb{c}} \lambda_v \prod_{v \in \mathfrak{a}} b_v \prod_{v \text{ complex}} \left(2^{\frac{(n-1)(n-2)}{2}} \pi \frac{n(n+1)}{4} \prod_{j=1}^{n} \Gamma(j/2)^{-1}\right) \left\{\begin{array}{ll}
2^{-\frac{(n-1)d}{2}} & \text{if } 2 \nmid n, \\
|D_k|^\frac{j}{2}\left[(2\pi)^{\frac{n}{2}} - \frac{n}{2}\right]^d L_k\left(\frac{n}{2}, \chi\right) & \text{if } 2 \mid n,
\end{array}\right.
\]
where \( r_v \) and \( t_v = \dim(\theta_v) \) are defined by the normalization of \( \varphi_v \) in \( \S 2 \),

\[
\Gamma_t(s) = \pi^{-(s-1)/2} \prod_{j=0}^{i-1} \Gamma(s-(j/2)),
\]

\[
\tilde{\Lambda} = \{ x \in V \mid 2\varphi(x, \Lambda) \in O_k \},
\]

\[
b^c_v = 2 \frac{r_m}{\pi^{(n-r_m)c_v}} \Gamma_r(r_v/2) \Gamma_r(n/2)^{-1},
\]

\( c \) is the product of all prime ideals for which \( \tilde{\Lambda}_v \neq \Lambda_v \), \( \zeta_k(s) \) and \( L_k(s, \chi) \) are zeta and \( L \)-functions over \( k \), \( \chi \) is the non-trivial Hecke character associated to the extension \( K/k \) where \( K = k(\sqrt{-1}^{n/2} \det) \), and \( \lambda_v \) is defined by

\[
\lambda_v = \begin{cases} 
1 & \text{if } t_v = 1, \\
2^{-1}(1+q)^{-1}(1+q^{1-m})(1+q^{-m}) & \text{if } t_v = 2, \text{ } p \text{ is unramified in } K, \\
2^{-1} & \text{if } t_v = 2, \text{ } \tilde{\Lambda}_p \neq \Lambda_p, \\
2^{-1}(1+q)^{-1}(1-q^{2m}) & \text{if } t_v = 3, \\
2^{-1}(1+q)^{-1}(1-q^{1-m})(1-q^{-m}) & \text{if } t_v = 4,
\end{cases}
\]

where \( q \) is the norm of the prime ideal at \( v \in \mathfrak{h} \) and \( m = \left\lfloor \frac{n}{2} \right\rfloor \).

**Proof.** By Lemma A7 applied to \( F \), for each class \( a \in \mathfrak{C} \) we have

\[
\tau_a(\Gamma^a \backslash G_a) = \text{vol}_C((\Gamma^a \cap S) \backslash C_a) \text{vol}_3(\Gamma^a \backslash \mathfrak{S}),
\]

where \( S = \{ g \in G_a \mid gz = z \text{ for every } z \in \mathfrak{S} \} \). By Lemma A1, \( S = \{ (\pm1) \}_{v \in a} \) so this becomes

\[
(5.6) \quad \tau_a(\Gamma^a \backslash G_a) \text{vol}_C(C_a)^{-1} = [\Gamma^a \cap \{ \pm1 \}]^{-1} \text{vol}_3(\Gamma^a \backslash \mathfrak{S}).
\]

This together with Theorem 2.1 and (2.1) gives

\[
(5.7) \quad \text{Mass}(\Lambda, \varphi) = 2|D_k|^2 \frac{(n-1)}{4} \text{vol}_C(C_a)^{-1} \prod_{v \in \mathfrak{h}} \beta_v(\Lambda, \psi)^{-1}
\]

\[
= 2|D_k|^2 \frac{(n-1)}{4} \prod_{v \in \mathfrak{a}} \text{vol}_C(C_v)^{-1} \prod_{v \in \mathfrak{h}} \beta_v(\Lambda, \psi)^{-1}.
\]

From here, we complete the proof in 3 parts:

**Part 1:** First we prove the case where \( \varphi_v \) is positive definite at all \( v \in \mathfrak{a} \). In this case \( C_v = G_v \) for all \( v \in \mathfrak{a} \), so by (5.7) we have

\[
\text{Mass}(\Lambda, \varphi) = 2|D_k|^2 \frac{(n-1)}{4} \prod_{v \in \mathfrak{a}} \beta_v(\psi)^{-1} \prod_{v \in \mathfrak{h}} \beta_v(\Lambda, \psi)^{-1}.
\]

By (2.9), \( \phi_v = \psi_v \psi \sigma_v \) and \( |\det(\sigma_v)|_v = \left( \frac{|\det(\psi)|_v}{|\det(\psi)|_v} \right)^{1/2} \). Combining this with Lemma 3.2 we have

\[
2|D_k|^2 \frac{(n-1)}{4} \prod_{v \in \mathfrak{a}} \left( |\det(\psi)|_v^{-(n+1)/2} |\det(\phi_v)|_v^{n+1} \beta_v(\phi_v)^{-1} \right) \prod_{v \in \mathfrak{h}} \left( |\det(\psi)|_v^{-(n+1)/2} |\det(\phi_v)|_v^{n+1} \beta_v(\Lambda_v, \phi_v)^{-1} \right),
\]
which by the product formula and det(φ_v) = ±1 for all v ∈ a, gives
\[
2|D_k|^{\frac{n(n-1)}{4}} \prod_{v \in a} \beta_v(\phi_v)^{-1} \prod_{v \in h} \left( |\det(\phi_v)|^{\frac{n-1}{2}} \beta_v(\Lambda_v, \phi_v)^{-1} \right).
\]

Substituting (3.3) and (4.1.1), using (3.4), and noticing \( \prod_{v|2} 2^{e_v} = 2^n \), we get
\[
2|D_k|^{\frac{n(n-1)}{4}} \left( 2^{-(n+1)d} \prod_{j=1}^{n} \Gamma(j/2) \right)^{\frac{d}{2}} \left( \prod_{i=1}^{\frac{n-1}{2}} \zeta_k(2i) \prod_{v|\epsilon} \lambda_v \right) \left\{ \begin{array}{ll}
1 & \text{if } 2 \nmid n, \\
L_k(\frac{e}{2}, \chi) & \text{if } 2|n.
\end{array} \right.
\]

By rearranging terms we obtain
\[
2|D_k|^{\frac{n(n-1)}{4}} \left( 2^{-(n-1)d} \right) \left( \prod_{j=1}^{\frac{n-1}{2}} \Gamma(j/2) \right)^{\frac{d}{2}} \left( \prod_{i=1}^{\frac{n-1}{2}} \zeta_k(2i) \right) \left\{ \begin{array}{ll}
\bar{\Lambda} : A \frac{n-1}{2} \prod_{v|\epsilon} \lambda_v & \text{if } 2 \nmid n, \\
L_k(\frac{e}{2}, \chi) & \text{if } 2|n.
\end{array} \right.
\]

\[
= 2|D_k|^{\frac{n(n-1)}{4}} \left( 2^{-(n-1)d} \right) \left[ \prod_{j=1}^{\frac{n-1}{2}} \left( \frac{(2j-1)!(2\pi)^{2j}}{(2\pi)^{2j}} \right)^{\frac{d}{2}} \zeta_k(2j) \right] \left\{ \begin{array}{ll}
\bar{\Lambda} : A \frac{n-1}{2} \prod_{v|\epsilon} \lambda_v & \text{if } 2 \nmid n, \\
[2^{n-1}(\frac{e}{2} - 1)(2\pi)^{-\frac{e}{2}}]^d L_k(\frac{e}{2}, \chi) & \text{if } 2|n.
\end{array} \right.
\]

\[
= 2|D_k|^{\frac{n(n-1)}{4}} \left[ \prod_{j=1}^{\frac{n-1}{2}} \left( \frac{(2j-1)!}{(2\pi)^{2j}} \right)^{\frac{d}{2}} \zeta_k(2j) \right] \left\{ \begin{array}{ll}
\bar{\Lambda} : A \frac{n-1}{2} \prod_{v|\epsilon} \lambda_v & \text{if } 2 \nmid n, \\
[2^{-\frac{e}{2}}(\frac{e}{2} - 1)(2\pi)^{-\frac{e}{2}}]^d L_k(\frac{e}{2}, \chi) & \text{if } 2|n,
\end{array} \right.
\]

\[
= 2|D_k|^{\frac{(n-1)^2}{4}} \left[ \prod_{j=1}^{\frac{n-1}{2}} D_k^\frac{e}{2} \left( \frac{(2j-1)!(2\pi)^{2j}}{(2\pi)^{2j}} \right)^{\frac{d}{2}} \zeta_k(2j) \right] \left\{ \begin{array}{ll}
\bar{\Lambda} : A \frac{n-1}{2} \prod_{v|\epsilon} \lambda_v & \text{if } 2 \nmid n, \\
D_k^\frac{e}{2} \left( \frac{(e - 1)(2\pi)^{-\frac{e}{2}}}{(2\pi)^{-\frac{e}{2}}} \right)^d L_k(\frac{e}{2}, \chi) & \text{if } 2|n.
\end{array} \right.
\]

**Part 2:** Now suppose that all v ∈ a are real, but perhaps φ_v is indefinite at some v. Take
\[
b_v^e = 2^{\frac{e}{2}} \pi^{\frac{(n-1)^2}{4}} \Gamma_v(r_v/2) \Gamma_v(n/2)^{-1}
\]
as above where $r_v$ is defined by the normalization of $\varphi_v$ in (2.9). For each indefinite $v$, we add an additional factor of $b_v^\varphi$ from the formula in part 1, which is seen by observing

$$\text{vol}_C(C_v)^{-1} = \left(2\pi \frac{-n(n+1)}{4} \prod_{j=1}^{n} \Gamma(j/2) \right) b_v^\varphi$$

and that $b_v^\varphi = 1$ if $v$ is definite. Combined with the previous formula this proves the case where all $v \in a$ are real.

**Part 3:** Finally, consider arbitrary $v \in a$. We define $r_v = 0$ for $v$ complex, and so for such $v$ we have $b_v^\varphi = 1$. Since each complex place replaces two real places in the totally real formula, we again have a correction factor. The relevant calculation to check for $v$ complex is

$$\text{vol}_C(C_v)^{-1} = \left(2\pi \frac{-n(n+1)}{4} \prod_{j=1}^{n} \Gamma(j/2) \right)^2 \left(2^{-\frac{(n-1)(n-2)}{2}} \prod_{j=1}^{n} \Gamma(j/2)^{-1} \right) b_v^\varphi.$$  

This together with Part 2 proves the theorem. \qed

One interesting application of Theorem 5.1 is to the case of a maximal indefinite quadratic form $(\Lambda, \varphi)$ in $n \geq 3$ variables. In this case our formula explicitly computes the volume of the quotient $\Gamma^a \setminus \mathbb{3}$.

**Corollary 5.2.** Let $(\Lambda, \varphi)$ be a maximal indefinite quadratic form in $n \geq 3$ variables and let $D$ the subgroup of $G_\Lambda$ stabilizing $\Lambda$. Then

$$\text{vol}(\Gamma^a \setminus \mathbb{3}) = \varepsilon [k_\Lambda^x : k^x \sigma(D)] \text{Mass}(\Lambda, \varphi)$$

where $\sigma$ is the spinor norm map $G_\Lambda^x \rightarrow k^x/(k^x)^2$ (see [Shi, (2.1.1)]) and $\varepsilon$ is either 1 or 2 depending on whether $\dim(V)$ is odd or even. If $k$ has class number one, then

$$\text{vol}(\Gamma^a \setminus \mathbb{3}) = \varepsilon \text{Mass}(\Lambda, \varphi).$$

**Proof.** Since $n \geq 3$ and $\varphi$ is indefinite, the classes and the spinor genera in the genus of $\Lambda$ coincide. From this and [Shi, Lemma 2.3(4)] we know that the number of classes is $[k_\Lambda^x : k^x \sigma(D)]$. We also know that $\nu(\Gamma^a)$ is independent of the class $a$ by [Shi, Thrm 5.10(1)]. Finally, $-1 \in \Gamma^a$ exactly when $\det(-1_n) = 1$ which happens exactly when $2\dim(V)$. This proves the first assertion.

For the second part, from [Shi, Lemma 2.5] we know that $k_\Lambda^x/k^x \sigma(D)$ is a quotient of the ideal class group of $k$. Thus if the class number of $k$ is one, then $[k_\Lambda^x : k^x \sigma(D)] = 1$. \qed

§6 Mass formula for $a$-maximal lattices

In this section we use the local similitude groups $\widetilde{G}_p^\varphi$ to show that the mass of the genus of $a$-maximal lattices is the same for many ideals $a$. We do this by noticing that $\text{Mass}(\Lambda, \varphi)$ depends only on the volume of non-archimedian stabilizer $D_h$ of $(\Lambda, \varphi)$, and then showing that the action of $\widetilde{G}_p^\varphi$ preserves these volumes.
Let $\widetilde{G}_p^\varphi = \{ \tilde{g} \in GL_n(k_p) \mid \xi(\tilde{g}) \varphi \tilde{g} = \xi(g) \varphi g \text{ for some } \xi(g) \in k_p^* \}$ be the local group of similitudes of $\varphi$, and let $\Xi_p(\varphi) = \{ \xi(\tilde{g}) \mid \tilde{g} \in G_p^\varphi \}$ be the set of similitude multipliers of $G_p^\varphi$. We also recall the local decomposition

$$(V_p, \varphi_p) = (H_{2r}, \eta_{2r}) \bigoplus (W_p, \theta_p)$$

where $(H_{2r}, \eta_{2r}) \cong \bigoplus_{i=1}^r (F_p^2, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix})$ and $(W_p, \theta_p)$ is anisotropic of dimension $t_p$. By [OM, 63:19, p170] we know that $t_p \leq 4$. One can easily compute $\Xi_p(\eta_{2r}) = F_p^\varphi$, and so $\Xi_p(\varphi) = \Xi_p(\eta_{2r}) \cap \Xi_p(\theta_p) = \Xi_p(\theta_p)$.

**Lemma 6.1.** Suppose $\Lambda$ and $\Lambda'$ are two lattices in the quadratic space $(V, \varphi)$ over $k$ with stabilizers $D, D' \subset G_\Lambda^\varphi$ respectively. Then

$$\frac{\text{Mass}(\Lambda, \varphi)}{\text{Mass}(\Lambda', \varphi)} = \frac{\text{vol}(D_h^\alpha)}{\text{vol}(D_h)}$$

where the local volumes are defined by $\text{vol}(D_v) = \int_{D_v} \omega_G$.

**Proof.** This follows by combining (5.7) with the remarks after (2.8). □

**Lemma 6.2.** Suppose $D_p$ is an open compact subgroup of $G_p^\varphi$ and $\alpha \in G_p^\varphi$, then $\text{vol}(D_p) = \text{vol}(\alpha^{-1}D_p \alpha)$.

**Proof.** This is equivalent to showing that the volume form $\omega_G$ on $G_v$ is invariant under conjugation by $\alpha$. To see this holds, following the procedure of §2 we can realize $G$ as a fibre of the map $\mathcal{F} : \widetilde{G}_v \to k_v^*$ given by $\mathcal{F}(\tilde{g}) = \xi(\tilde{g})$, which gives $\omega_G = \omega_G \wedge \mathcal{F}(\frac{d\xi}{\xi})$. Since $\widetilde{G}_v$ is unimodular and $\frac{d\xi}{\xi}$ is clearly invariant under conjugation, we see that $\omega_G$ is also invariant. □

**Theorem 6.3.** Suppose $(\Lambda', \varphi)$ is a non-degenerate $\alpha$-maximal lattice of dimension $n \geq 3$ defined over a number field $k$. Then

$$\text{Mass}(\Lambda', \varphi) = \text{Mass}(\Lambda, \varphi)$$

where $(\Lambda, \varphi)$ is a maximal lattice, and $a_p$ satisfies the following conditions:

- If $n$ is odd, then $a_p$ is a square,
- If $n$ is even and $t_p = 2$, then $a_p$ is a norm from $K_p = k_p \left( \sqrt{(-1)^{n/2}d_p} \right)$.

This mass is explicitly given in Theorem 5.1.

**Proof.** By Lemmas 6.1 and 6.2, it suffices to show for all primes $p$ that $\Lambda'_p = \tilde{g}\Lambda_p$ for some $\tilde{g} \in G_p^\varphi$, and by [Shi2, Lemma 5.9, p33] we know this is true for any two $\alpha$-maximal lattices. By comparing their values under $\varphi$ we see that

$$\Lambda_p \text{ is } O_p\text{-maximal } \iff \tilde{g}\Lambda_p \text{ is } \xi(\tilde{g})O_p\text{-maximal},$$

so the proof reduces to characterizing the set $\text{ord}_p(\Xi_p(\theta_p))$. We do this by using the local models for $(W_p, \theta_p)$ in [Shi, §3.2].
If \( t_p \) is odd then we can never find a similitude of odd valuation, since if \( \text{ord}_p(\xi(\tilde{g})) \) is odd then taking determinants gives \( \text{ord}_p(\det(\tilde{g})^2) = \text{ord}_p(\xi(\tilde{g}))^{2} \) which is odd. Conversely, if \( \pi_p \) is a uniformizer in \( k_p \) then we can construct \( \pi_p^2 \) in \( \Xi(\Theta_p) \) by using \( \tilde{g} = \text{diag}[\pi_p, \ldots, \pi_p] \).

If \( t_p = 0 \), then \( (V_p, \varphi_p) \) is a direct sum of hyperbolic planes and \( \Xi_p(\theta_p) = k_p^{\times} \).

If \( t_p = 2 \), then \( (W_p, \theta_p) \cong (K_p, cN_{K_p/(K_p))} \) where \( K_p = k_p(\sqrt{-\det(\varphi)}) \) and \( c \in k^{\times} \). Therefore \( K_p^{\times} \subseteq \tilde{G}_p^{a} \) and so \( \Xi_p(\theta_p) \cong N_{K_p/(K_p)}(K_p^{\times}) \).

If \( t_p = 4 \), then \( (W_p, \theta_p) \cong (B_p, N_{B_p/k_p}(x)) \) where \( B_p \) is a division quaternion algebra over \( k_p \). Since \( B^{\times} \subseteq \tilde{G}_p^{a} \) and \( N_{B_p/k_p}(B_p^{\times}) = k_p^{\times} \), we have \( \Xi_p(\theta_p) = k_p^{\times} \). \( \Box \)

Remark. In terms of the invariants \((n_p, d_p, c_p)\) for the local quadratic space \((V_p, \varphi_p)\), the condition \( t_v = 1 \) is equivalent to \( c_p = \left(\frac{(-1)^{n/2}(-1)^{n/2}d_p}{p}\right) \) when \( n \) is odd, and \( t_v = 2 \) is equivalent to \([K_p : k_p] = 2\) when \( n \) is even and \( K_p \) is as above.

Appendix

It will be convenient to know a few lemmas about matrices of differentials. If we take \( x = (x_{ij}) \) to be a matrix of functions, then we define the matrix \( dx \) to be the matrix \((dx_{ij})\) of differentials of \( x \).

**Lemma A1.** Let \( \mathfrak{g}_v \) be a symmetric space of the type described in \$4.2 \) or \$4.3. Then \( \{g \in G_v \mid g z = z \text{ for every } z \in \mathfrak{g}_v\} = \{\pm 1_n\} \).

**Proof.** This is the analogous statement of [Shi2, Prop. 6.4(5)] for orthogonal groups, and has the same proof with obvious modifications. \( \Box \)

**Lemma A2.** Let \( dx \) and \( dx' \) be two \( r \times t \) matrices of linearly independent differentials, and suppose \( dx' = a(dx) \) for some \( r \times r \) constant matrix \( a \). Then

\[
\bigwedge_{i,k} dx'_{ik} = \det(a)^{t} \bigwedge_{i,k} dx_{ik}.
\]

Similarly, if \( dx' = (dx)a' \) for some \( t \times t \) constant matrix \( a' \), then

\[
\bigwedge_{i,k} dx'_{ik} = \det(a')^{r} \bigwedge_{i,k} dx_{ik}.
\]

**Proof.** This is well known, and follows from the action of \( a \) (resp. \( a' \)) on a column (resp. row) vector. \( \Box \)

**Lemma A3.** Let \( dx \) and \( dx' \) be two \( n \times n \) matrices of linearly independent differentials and suppose \( dx' = a(dx) \) for some \( n \times n \) constant matrix \( a \). Then

\[
\bigwedge_{i \leq k} dx'_{ik} = \prod_{l=1}^{n} \det_{l}(a) \bigwedge_{i \leq k} dx_{ik} + \sum \left( \text{terms containing at least one factor } dx_{ik} \text{ with } i > k \right).
\]

**Proof.** It will be enough to analyze the columns \( k \geq k_0 \), proving inductively that for each \( 1 \leq k_0 \leq n \) we have

\[
(A3.1) \quad \bigwedge_{i \leq k \geq k_0} dx'_{ik} = \prod_{l=k_0}^{n-1} \det_{l}(a) \bigwedge_{i \leq k \geq k_0} dx_{ik} + \Omega,
\]
where \( \Omega \) is a sum of terms each containing at least one factor \( dx_{ik} \) with \( i > k \).

If \( k_0 = n \) then
\[
\bigwedge_{i \leq k_0} dx'_{ik_0} = \bigwedge_{i \leq k_0} \sum_j a_{ij} dx_{jk_0} \\
= \bigwedge_{i \leq k_0} \sum_{\sigma \in S_n} a_{i\sigma(i)} dx'_{\sigma(i)k_0} \\
= \det(a) \bigwedge_{i \leq k_0} dx_{ik_0}
\]
since the only non-zero terms in the wedge product come from permutations of the row index \( i \).

Proceeding inductively, we consider the row \( k_0 \) and assume (A3.1) holds for all \( k > k_0 \). Then
\[
(A3.2) \quad \bigwedge_{i \leq k} dx'_{ik} = \bigwedge_{i \leq k_0} dx'_{ik_0} \wedge \bigwedge_{i \leq k, k \geq k_0+1} dx'_{ik} \\
= \left( \bigwedge_{i \leq k_0} \sum_j a_{ij} dx_{jk_0} \right) \wedge \left( \prod_{l=k_0+1}^{n-1} \det(a) \bigwedge_{i \leq l, k \geq k_0+1} dx_{ik} + \Omega \right).
\]
The terms \( dx_{jk_0} \) of \( \bigwedge_{i \leq k_0} \sum_j a_{ij} dx_{jk_0} \) with \( j > k_0 \) cannot contribute to the term \( \bigwedge_{i \leq k, k \geq k_0+1} dx_{ik} \) since the entries of \( dx \) are linearly independent. Therefore the only terms which contribute to it are the \( dx_{jk_0} \) with \( j \leq k_0 \) and these can be written as the following sum over permutations on the row index \( i \):
\[
\bigwedge_{i \leq k_0} \sum_{j \leq k_0} a_{ij} dx_{jk_0} = \bigwedge_{i \leq k_0} \sum_{\sigma \in S_{k_0}} a_{i\sigma(i)} dx'_{\sigma(i)k_0} \\
= \det_{k_0}(a) \bigwedge_{i \leq k_0} dx_{ik_0}.
\]
Combining this with (A3.2), we prove (A3.1). Our lemma then follows from (A3.1) by taking \( k_0 = 1 \). \( \square \)

**Lemma A4.** Let \( dx \) and \( dx' \) be two skew-symmetric \( n \times n \) matrices of differentials whose upper triangular coordinates are linearly independent, and suppose \( dx' = 'a(dx)a \) for some \( n \times n \) constant matrix \( a \). Then
\[
\bigwedge_{i > k} dx'_{ik} = \det(a)^{n-1} \bigwedge_{i > k} dx_{ik}.
\]

**Proof.** This is proved in the same way as Lemma 3.2, the only difference being that the computation for scalar matrices here gives \( \det(a)^{n-1} \). \( \square \)
Lemma A5. Let $dx$ and $dx'$ be two skew-symmetric $n \times n$ matrices of differentials whose upper triangular coordinates are linearly independent, and suppose $dx' = (dx)a$ for some $n \times n$ constant matrix $a$. Then

$$\bigwedge_{i > k} dx'_{ik} = \prod_{l=1}^{n-1} \det_l(a) \bigwedge_{i > k} dx_{ik}.$$ 

Proof. We prove by induction that

(A5.1) $$\bigwedge_{k < i} dx'_{ik} = \prod_{l=i_0}^{n-1} \det_l(a) \bigwedge_{k < i} dx_{ik}$$

for all $1 \leq i_0 \leq n$.

In the case $i_0 = n$, the non-zero terms of $\bigwedge_{k < n} \sum_j dx_{nj} a_{jk}$ come from choosing one term $dx_{nj} a_{jk}$ for each $k$ with no repetition among the $j$ indices. Thus the $j$ index is a permutation of the $k$ index, and we have

$$\bigwedge_{k < n} \sum_{\sigma \in S_{n-1}} dx_{n \sigma(k)} a_{\sigma(k)k} = \det_{n-1}(a) \bigwedge_{k < n} dx_{nk}.$$ 

Now suppose $i_0 < n$. By induction we have

$$\bigwedge_{k < i} dx_{ik} = \left( \bigwedge_{k < i_0} dx_{i_0 j} a_{jk} \right) \wedge \left( \bigwedge_{k < i} \sum_{j \geq i_0 + 1} dx_{ij} a_{jk} \right)$$

$$= \left( \bigwedge_{k < i_0} dx_{i_0 j} a_{jk} \right) \wedge \left( \prod_{l=i_0+1}^{n-1} \det_l(a) \bigwedge_{k < l} dx_{ik} \right).$$

By skew-symmetry of $dx$, we see that all of the terms in $\bigwedge_{k < i} \sum_j dx_{i_0 j} a_{jk}$ with $j \geq i_0$ would give zero when wedged together with $\bigwedge_{k < i, i \geq i_0 + 1} dx_{ik}$. Thus the only terms that contribute have the form

$$\sum_{\sigma \in S_{i_0}} dx_{i_0 \sigma(k)} a_{jk} = \det_{i_0-1}(a) \bigwedge_{k < i_0} dx_{i_0 k},$$

which together with the above proves (A5.1). Our result follows from (A5.1) by taking $i_0 = 1$. \qed

We now state two basic lemmas about volume forms on manifolds.

Lemma A6. Let $F : X \to Y$ be a map of $C^\infty$-manifolds of dimensions $n$ and $m$ respectively, with $\text{rank}(F) = m$. Suppose that $X$ is a group acting on $Y$ and the map $F$ commutes with this action. Choose $p \in Y$ and let $C = F^{-1}(p)$ be the fibre over $p$. Given $X$-invariant volume forms $\omega_X$ and $\omega_Y$ on $X$ and $Y$ respectively, we can define a unique volume form $\omega_C$ on $C$ by choosing $\omega \in \left( \bigwedge^{n-m} \right)^*(X)$ such that

(A6.1) $$\omega \wedge F^*(\omega_Y) = \omega_X.$$
and taking $\omega_C$ to be the restriction $\omega|_C$ of $\omega$ to $C$. Further, $\omega_C$ is $C$-invariant and when computing $\omega_C$ it suffices to take forms on $X$ with coefficients in the fibre $C$ over $p$.

**Proof.** In this situation, the forms on $X$ are determined by their definition on any neighborhood, so it is sufficient to check locally on $X$.

Choose a point $q \in F^{-1}(p) \subset X$. Taking $y_1, \cdots, y_m$ to be a set of coordinates on $Y$ in some neighborhood of $p$, we can pull these back to give coordinates $x_1, \cdots, x_m$ on some neighborhood of $q$ in $X$. Since $F^{-1}(p)$ is a regular submanifold of $X$, we can extend these to give a complete set of coordinates $x_1, \cdots, x_n$ on a possibly smaller neighborhood of $q$. In these coordinates we have

\begin{equation}
\omega_X = f(x) \prod_{i=1}^{n} dx_i,
\end{equation}

\begin{equation}
F^*(\omega_Y) = f_1(x) \prod_{i=1}^{m} dx_i.
\end{equation}

From this we see that any $\omega$ on $X$ satisfying (A6.1) must have the form

\begin{equation}
\omega = \frac{f(x)}{f_1(x)} \prod_{i=1}^{m} dx_i + \sum \left( \text{terms containing at least one factor from } \{dx_1, \cdots, dx_m\} \right).
\end{equation}

Such an $\omega$ exists and is a volume form since both $\omega_X$ and $\omega_Y$ are nowhere vanishing. Uniqueness of $\omega_C$ follows since $x_1, \cdots, x_m$ are constant on $C$, so all terms of (A6.4) except the first term vanish on $C$.

To see the $C$-invariance of $\omega_C$, let $c_0 \in C$ act on (A6.1). This gives

\[ c_0^* \wedge F^*(\omega_Y) = \omega_X. \]

But by uniqueness of $\omega_C$ we have the second part of

\[ c_0^*(\omega_C) = c_0^*(\omega)|_C = \omega_C, \]

so $\omega_C$ is $C$-invariant.

The final assertion is easy, and can be checked in the coordinates $x_1, \cdots, x_n$ above. We write $f_1(x) = f_2(x) + f_3(x)$ where $f_3(x)$ has coefficients all of which are zero on $C$, and observe that the $f_3(x)$ term disappears whether we restrict coefficients before or after choosing $\omega$. \qed

**Lemma A7.** Suppose we are in the setting of Lemma A6, and take some Fuchsian subgroup $\Gamma \subseteq X$. We let $\mu_C, \mu_X,$ and $\mu_Y$ denote the measures associated to $\omega_C, \omega_X,$ and $\omega_Y$ respectively. Then

\[ \mu_X(\Gamma \setminus X) = \mu_Y(\Gamma \setminus Y) \mu_C(\Gamma \cap S \setminus C), \]

where $S = \{ x \in X \mid xy = y \text{ for every } y \in Y \}$. 

References


