MATH 204 C03 – APPLICATION OF LINEAR ALGEBRA TO LINEAR ORDINARY DIFFERENTIAL EQUATIONS

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Outline

We will be concluding our course with an application of linear alegbra to differential equations. In particular, we will show the following theorem:

Theorem. *Let*
$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots a_1 y' + a_0 y = 0$$
with $a_1, \dots, a_n = 0$. *Then the solution set spans an* n*-dimensional subspace of* $\mathcal{C}^\infty(\mathbb{R})$.

Not only will we establish this fact, but we will also show a "standard" basis for the set of solutions, meaning we will find basis elements that coincide with almost any book on ordinary differential equations.

Terms and Previous Theorems

As with previous papers, if $\mathbf{T} : \mathcal{V} \to \mathcal{W}$ is a linear operator, then the kernel of $\mathbf{T}$ is
$$\mathrm{KER}(\mathbf{T}) = \{\mathbf{X} \in \mathcal{V} : \mathbf{T}(\mathbf{X}) = \mathbf{0}\}$$
and the image of $\mathbf{T}$ is
$$\mathrm{IMG}(\mathbf{T}) = \{\mathbf{Y} \in \mathcal{W} : \mathbf{T}(\mathbf{X}) = \mathbf{Y} \text{ for some } \mathbf{X} \in \mathcal{V}\}.$$
If $\mathbf{T} : \mathcal{V} \to \mathcal{V}$ is linear and $p(x) = a_n x^n + \dots a_1 x, +a_0$ is a polynomial with scalar coefficients, we may define (just as in the case of matrices),
$$p(\mathbf{T}) = \sum_{j=0}^{n} a_j \mathbf{T}^j$$
where we always take the convention $\mathbf{T}^0 = \mathbf{I}$, the identity map on $\mathcal{V}$. This is another linear operator from $\mathcal{V}$ to $\mathcal{V}$.

If $\mathbf{T} : \mathcal{V} \to \mathcal{W}$ is a function (not necessarily linear), we will use the following notation
$$\mathbf{T}(\mathcal{V}) = \{\mathbf{T}(\mathbf{X}) : \mathbf{X} \in \mathcal{V}\} \subseteq \mathcal{W}$$
for the **image** of $\mathcal{V}$ under $\mathbf{T}$. Also, if we have subspaces $\mathcal{U}_1, \mathcal{U}_2 \subseteq \mathcal{V}$, then
$$\mathcal{U}_1 + \mathcal{U}_2 = \{\mathbf{X}_1 + \mathbf{X}_2 : \mathbf{X}_i \in \mathcal{U}_i, i = 1, 2\} \subseteq \mathcal{V}$$
is the **span** of the subspaces $\mathcal{U}_1$ and $\mathcal{U}_2$.

Exercise 1. If $\mathcal{U}_1, \mathcal{U}_2$ are subspaces of vector space $\mathcal{V}$, show that $\mathcal{U}_1 + \mathcal{U}_2$ is as well.

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We recall from a previous handout that a linear map $\mathbf{T} : \mathcal{V} \to \mathcal{V}$ is a **projection** if $\mathbf{T}^2 = \mathbf{T}$. We recall the following fact

Lemma 1. *Let* $\mathbf{T} : \mathcal{V} \to \mathcal{V}$ *be a projection, then*

$$\mathcal{V} = \mathrm{KER}(\mathbf{T}) \oplus \mathrm{IMG}(\mathbf{T}).$$

Equivalently, the span of $\mathrm{KER}(\mathbf{T})$ *and* $\mathrm{IMG}(\mathbf{T})$ *is all of* $\mathcal{V}$*, and* $\mathrm{KER}(\mathbf{T})$ *and* $\mathrm{IMG}(\mathbf{T})$ *are independent spaces.*

Let $\mathcal{C}^\infty(\mathbb{R})$ be the real vector space of all smooth functions $f : \mathbb{R} \to \mathbb{R}$. In what follows, we will be forced to consider the complex vector space $\mathcal{C}^\infty(\mathbb{C})$, all smooth functions $f : \mathbb{R} \to \mathbb{C}$. Without delving too deeply into a book on complex variables, we may just consider this as the space

$$\mathcal{C}^\infty(\mathbb{C}) = \{f + \imath g : f, g \in \mathcal{C}^\infty(\mathbb{R})\}.$$

Note that integration and derivatives work as one may suspect, namely if $h \in \mathcal{C}^\infty(\mathbb{C})$ and $h = f + \imath g$, $f, g \in \mathcal{C}^\infty(\mathbb{R})$, then

$$h'(x) = f'(x) + \imath g'(x) \text{ and } \int h\ dx = \int f\ dx + \imath \int g\ dx.$$

Remark. For those who have taken or will take a course in complex variables, $\mathcal{C}^\infty(\mathbb{C})$ is **not** the set of analytic functions on the complex plane.

We define $\mathbf{D} : \mathcal{C}^\infty(\mathbb{C}) \to \mathcal{C}^\infty(\mathbb{C})$ to be the **differential operator**, meaning for every $h \in \mathcal{C}^\infty(\mathbb{C})$,

$$\mathbf{D}(h)(x) = h'(x)$$

We state without proof that $\mathbf{D}$ is linear. We will also refer to the differential operator on $\mathcal{C}^\infty(\mathbb{R})$ by $\mathbf{D}$ as well, as the definition in the previous equation maps $f \in \mathcal{C}^\infty(\mathbb{R})$ to $f' \in \mathcal{C}^\infty(\mathbb{R})$.

Finding Solutions to Linear ODEs

Let us consider a differential equation of the form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

and consider its characteristic polynomial $p(x) = a_n x^n + \cdots + a_1 x + a_0$. Noting that $\mathbf{D}^k(y) = y^{(k)}$ for each k, where $\mathbf{D}$ is the differential operator, we may rewrite our ODE as

$$\begin{aligned}
a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y &= 0 \\
a_n \mathbf{D}^n(y) + a_{n-1}\mathbf{D}^{n-1}(y) + \ldots a_1 \mathbf{D}^1(y) + a_0 \mathbf{D}^0(y) &= 0 \\
(a_n \mathbf{D}^n + a_{n-1}\mathbf{D}^{n-1} + \ldots a_1 \mathbf{D}^1 + a_0 \mathbf{D}^0)(y) &= 0 \\
p(\mathbf{D})(y) &= 0
\end{aligned}$$

We may then conclude y is a solution to our ODE if and only if it is in $\mathrm{KER}(p(\mathbf{D}))$. So we already have reduced solving a differential equation to finding the kernel of a linear function!

The remainder of this section builds to a general solution for any linear ordinary differential homogeneous equation. Our first result uses the Fundamental Theorem of Calculus.

Lemma 2. *Let $\mathbf{D} : \mathcal{C}^\infty(\mathbb{C}) \to \mathcal{C}^\infty(\mathbb{C})$ be the differential operator. Then for each $k > 0$,*

$$\begin{aligned} \mathrm{KER}(\mathbf{D}^k) = \mathcal{P}_{k-1} &= \{a_0 + \cdots + a_{k-1}x^{k-1} : a_0, \ldots, a_{k-1} \in \mathbb{C}\} \\ &= \mathrm{SPAN}_{\mathbb{C}}\{1, x, \ldots, x^{k-1}\} \end{aligned}$$

Proof. We prove this by induction. The solution set for $\mathbf{D}(y) = y' = 0$ is the set of constant functions. This is indeed $\mathcal{P}_0$.

Now assume that the solution set of $\mathbf{D}^k(y) = y^{(k)} = 0$ is $\mathcal{P}_{k-1}$. Any y that satisfies $\mathbf{D}^{k+1}(y) = 0$ must satisfy

$$\mathbf{D}(y) = z \in \mathrm{KER}[\mathbf{D}^k] = \mathcal{P}_{k-1}.$$

Then by the Fundamental Theorem of Calculus,

$$y = \int z\ dx + c \in \mathcal{P}_k.$$

This proves that $\mathrm{KER}(\mathbf{D}^{k+1}) \subseteq \mathcal{P}_k$. We may directly verify that if $y \in \mathcal{P}_k$, then $y^{(k+1)} = 0$, showing our other inclusion. $\square$

Lemma 3. *Let $\mathbf{D} : \mathcal{C}^\infty(\mathbb{C}) \to \mathcal{C}^\infty(\mathbb{C})$ be the differential operator. Then for any $\lambda \in \mathbb{C}$ and $k > 0$,*

$$\begin{aligned} \mathrm{KER}[(\mathbf{D} - \lambda\mathbf{I})^k] &= \left\{ \sum_{j=0}^{k-1} a_j x^j e^{\lambda x} : a_0, \ldots, a_{k-1} \in \mathbb{C} \right\} \\ &= \mathrm{SPAN}_{\mathbb{C}}\{e^{\lambda x}, xe^{\lambda x}, \ldots, x^{k-1}e^{\lambda x}\} \end{aligned}$$

Proof. Let $\mathbf{E} = \mathbf{D} - \lambda\mathbf{I}$ be a linear operator from $\mathcal{V}$ to $\mathcal{V}$. In other words for each smooth f,

$$\mathbf{E}(f)(x) = f'(x) - \lambda f(x).$$

Note that we now must investigate $\mathrm{KER}(\mathbf{E}^k)$ (this is the same the space as in the lemma). We will define another map $\mathbf{M}$ by

$$\mathbf{M}(f)(x) = e^{\lambda x} f(x).$$

This is an invertible linear map, and in fact

$$\mathbf{M}^{-1}(f)(x) = \frac{f(x)}{e^{\lambda x}}.$$

We will now claim that $\mathbf{E}$ and $\mathbf{D}$ are similar. Specifically, for each smooth f,

$$\begin{aligned} \mathbf{M}^{-1} \circ \mathbf{E} \circ \mathbf{M}(f(x)) &= \mathbf{M}^{-1} \circ \mathbf{E}(e^{\lambda x} f(x)) \\ &= \mathbf{M}^{-1}(\lambda e^{\lambda x} f(x) + e^{\lambda x} f'(x) - \lambda e^{\lambda x} f(x)) \\ &= \mathbf{M}^{-1}(e^{\lambda x} f'(x)) \\ &= f'(x) \\ &= \mathbf{D}(f), \end{aligned}$$

or $\mathbf{M}^{-1} \circ \mathbf{E} \circ \mathbf{M} = \mathbf{D}$. It follows that $\mathbf{M}^{-1} \circ \mathbf{E}^k \circ \mathbf{M} = \mathbf{D}^k$.

Exercise 2. Show that $\mathrm{KER}[\mathbf{E}^k] = \mathbf{M}(\mathrm{KER}[\mathbf{D}^k]) = \{\mathbf{M}(f) : f \in \mathrm{KER}(\mathbf{D}^k)\}$.

So as $\{1, \ldots, x^{k-1}\}$ is a basis for the kernel of $\mathbf{D}^k$, $\{e^{\lambda x}, xe^{\lambda x}, \ldots, x^{k-1}e^{\lambda x}\}$ is a basis for the kernel of $\mathbf{E}^k$. $\square$

The following lemma is a technical one. At the discretion of the reader, understanding the statement and result is more important to the rest of the paper than the proof, and therefore the proof may avoided on first reading. We will prove it only for the sake of completeness. This lemma indicates that if $\mathbf{X}$ is a generalized eigenvector of $\mathbf{T}$, then $p(\mathbf{T})\mathbf{X}$ is always a linear combination of the λ-chain of $\mathbf{X}$.

Lemma 4. *Let $\mathbf{T} : \mathcal{V} \to \mathcal{V}$ be linear, $p(x)$ a polynomial and $\mathbf{X}_1, \ldots, \mathbf{X}_k$ be a λ-chain, or*

$$\mathbf{T}(\mathbf{X}_j) = \begin{cases} \lambda \mathbf{X}_j + \mathbf{X}_{j-1}, & j > 1 \\ \lambda \mathbf{X}_1, & j = 1. \end{cases}$$

Then

$$p(T)\mathbf{X}_k = \sum_{j=1}^{k} p^{(k-j)}(\lambda)\mathbf{X}_j = p(\lambda)\mathbf{X}_k + p'(\lambda)\mathbf{X}_{k-1} + \ldots p^{(k-1)}(\lambda)\mathbf{X}_1.$$

Proof. We start by proving an initial claim, that for each $\ell \geq 0$,

$$\mathbf{T}^{\ell}(\mathbf{X}_k) = \sum_{j=1}^{k} m(\lambda, \ell + j - k, \ell)\mathbf{X}_j,$$

where

$$m(\lambda, e, \ell) = \begin{cases} \lambda^e, & e \geq \ell \\ \lambda^e \prod_{r=e+1}^{\ell} r, & 0 \leq e < \ell \\ 0, & e < 0. \end{cases}$$

If $p(x) = a_n x^n + \ldots a_1 x + a_0$, we remark that

$$p^{(k)}(x) = \sum_{j=0}^{n} a_j m(x, j-k, j) = \sum_{j'=0}^{n-k} a_{j'+k} m(x, j', j'+k).$$

This statement may be verified by induction (relate $\frac{d}{dx} m(x, e, \ell)$ to an m function with different terms).

Exercise 3. Prove this initial claim. You may want to use induction on ℓ. In fact, for $\ell = 1$,

$$\begin{aligned} \textstyle\sum_{j=1}^{k} m(\lambda, 1+j-k, 1)\mathbf{X}_j &= \textstyle\sum_{j=k-1}^{k} m(\lambda, 1+j-k, 1)\mathbf{X}_j \\ &= m(\lambda, 0, 1)\mathbf{X}_{k-1} + m(\lambda, 1, 1)\mathbf{X}_k \\ &= \mathbf{X}_{k-1} + \lambda \mathbf{X}_k \\ &= \mathbf{T}(\mathbf{X}_k). \end{aligned}$$

To use induction, assume the claimed identity for ℓ and then use the fact that

$$\mathbf{T}^{\ell+1} = \mathbf{T} \circ \mathbf{T}^{\ell} = \mathbf{T}^{\ell} \circ \mathbf{T}$$

to prove the identity for $\ell + 1$.

We now use this claim. Let $a_0, \ldots, a_n$ be the coefficients of $p(x)$, then

$$\begin{aligned} p(\mathbf{T})\mathbf{X}_k &= \textstyle\sum_{\ell=0}^{n} a_\ell \mathbf{T}^{\ell}(\mathbf{X}_k) \\ &= \textstyle\sum_{\ell=0}^{n} a_\ell \sum_{j=1}^{k} m(\lambda, \ell+j-k, \ell)\mathbf{X}_j \\ &= \textstyle\sum_{j=1}^{k} \left(\sum_{\ell=0}^{n} a_\ell m(\lambda, \ell+j-k, \ell) \sum\right) \mathbf{X}_j \\ &= \textstyle\sum_{j=1}^{k} \left(\sum_{t=-(k-j)}^{n-(k-j)} a_{t+(k-j)} m(\lambda, t, t+(k-j))\right) \mathbf{X}_j \\ &= \textstyle\sum_{j=1}^{k} \left(\sum_{t=0}^{n-(k-j)} a_{t+(k-j)} m(\lambda, t, t+(k-j))\right) \mathbf{X}_j \\ &= \textstyle\sum_{j=1}^{k} p^{(k-j)}(\lambda)\mathbf{X}_j. \end{aligned}$$

We have concluded the proof. Between the third and fourth line, we use the subsitution $t = \ell + j - k$. □

Corollary 5. *If* $\mathbf{X}$ *is a* k^{th} *order* λ*-eigenvector of linear map* $\mathbf{T} : \mathcal{V} \to \mathcal{V}$*, and* $p(x)$ *is a non-zero polynomial with* λ *as root of mulitplity* m *(*$m = 0$ *if* λ *is not a root of* p*), then*

$$\mathbf{X} \in \mathrm{KER}[p(\mathbf{T})] \iff k \leq m.$$

Proof. Consider the λ-chain $\mathbf{X}_k = \mathbf{X}, \mathbf{X}_{k-1}, \ldots, \mathbf{X}_1$. We know from Lemma 4, that

$$p(\mathbf{T})\mathbf{X} = p(\lambda)\mathbf{X} + p'(\lambda)\mathbf{X}_{k-1} + \ldots p^{(k-2)}(\lambda)\mathbf{X}_2 + p^{(k-1)}(\mathbf{T})\mathbf{X}_1.$$

The right hand side is $\mathbf{0}$ if and only if $p(\lambda) = p'(\lambda) = \ldots p^{(k-1)}(\lambda) = 0$. This occurs if and only if λ is a root of multiplicity $m \geq k$. □

Theorem 6. *Suppose* $\mathbf{T} : \mathcal{V} \to \mathcal{V}$ *is a linear operator over complex vector space* $\mathcal{V}$ *and* p *is a polynomial with complex coeffients. Then if*

$$p(x) = \alpha(x - \lambda_1)^{m_1} \cdots (x - \lambda_k)^{m_k}$$

for $\alpha \neq 0$ *and distinct* $\lambda_1, \ldots, \lambda_k$*, then*

$$\mathrm{KER}(p(\mathbf{T})) = \mathrm{KER}[(\mathbf{T} - \lambda_1\mathbf{I})^{m_1}] + \cdots + \mathrm{KER}[(\mathbf{T} - \lambda_k\mathbf{I})^{m_k}].$$

Proof. We will prove this by induction on n, the degree of p.

Exercise 4. If $n = 1$, then the statement holds. Also, if $p(x)$ is a polynomial with only one root of multiplicty m, then the statement holds.

By inductive hypothesis, we now assume that any polynomial $f(x)$ of degree less than n satisfies

$$\mathrm{KER}(f(\mathbf{T})) = \mathrm{KER}[(\mathbf{T} - \lambda'_1\mathbf{I})^{m'_1}] + \cdots + \mathrm{KER}[(\mathbf{T} - \lambda'_k\mathbf{I})^{m'_{k'}}],$$

where $\lambda'_1, \ldots, \lambda'_{k'}$ are the roots of q of multiplicities $m'_1, \ldots, m'_{k'}$, and will prove the claim for $p(x)$ of degree n. Let

$$p(x) = \alpha(x - \lambda_1)^{m_1} \cdots (x - \lambda_k)^{m_k}$$

and define the polynomial $q(x)$ by $p(x) = (x - \lambda_1)^{m_1} q(x)$, or

$$q(x) = \alpha(x - \lambda_2)^{m_2} \cdots (x - \lambda_k)^{m_k}.$$

Because of the previous exercise, we may assume that $k > 1$. We will show two cases, as the first is much easier and instructive while the second follows a very similar argument. In either case, we will construct a projection, which we will call $\mathbf{Q}$, whose image is $\mathrm{KER}[(\mathbf{T} - \lambda_1\mathbf{I})^{m_1}]$ and $\mathrm{KER}(Q) \cap \mathrm{KER}[p(\mathbf{T})] = \mathrm{KER}[q(\mathbf{T})]$. By Lemma 1 and our inductive assumption, this implies that

$$\mathrm{KER}[p(\mathbf{T})] = \mathrm{KER}[(\mathbf{T} - \lambda_1)^{m_1}] + \mathrm{KER}[q(\mathbf{T})] = \sum_{j=1}^{k} \mathrm{KER}[(\mathbf{T} - \lambda_j)^{m_j}]$$

which will conclude our proof.

Suppose first that $m_1 = 1$. Consider $\mathbf{X} \in \mathrm{KER}[p(\mathbf{T})]$, then

$$\mathbf{0} = p(\mathbf{T})\mathbf{X} = (\mathbf{T} - \lambda_1\mathbf{I})(q(\mathbf{T})(\mathbf{X}))$$

implies that $q(\mathbf{T})\mathbf{X} \in \mathrm{KER}(\mathbf{T} - \lambda_1)$ or $\mathrm{IMG}(q(\mathbf{T}))$ is contained in the λ-eigenspace. Remember that $q(x)$ does not have λ_1 as a root. By Lemma 4, if we consider any λ-eigenvector $\mathbf{Y}$, then

$$q(\mathbf{T})\mathbf{Y} = q(\lambda_1)\mathbf{Y},$$

where $q(\lambda_1) \neq 0$. This in turn implies that the λ_1-eigenspace is contained in $\mathrm{IMG}(q(\mathbf{T}))$, which indicates equality. Define

$$\mathbf{Q} = \frac{1}{q(\lambda_1)} q(\mathbf{T}).$$

Exercise 5. Verify that $\mathbf{Q}$ is a projection with the same image and kernel as $q(\mathbf{T})$.

We now consider the more general case, $m_1 > 1$. Define the following polynomials for $0 \leq j < m_1$,

$$q_j(x) = (x - \lambda_1)^j q(x).$$

If $\mathbf{X}$ is a generalized λ_1-eigenvector of order ℓ, then again by Lemma 4,

$$q_j(\mathbf{T})\mathbf{X} = q(\mathbf{T})(\mathbf{T} - \lambda_1 \mathbf{I})^j \mathbf{X}$$

will be either $\mathbf{0}$ if $\ell \leq j$ or a sum of the elements in the λ-chain of $\mathbf{X}$ of order at most $\ell - j$. Also, the coefficient for the $\ell - j$ order λ-eigenvector (what we would call the leading term) is always $q(\lambda_1) \neq 0$.

Exercise 6. Show that we may, for correctly chosen coefficients $c_0, \ldots, c_{m_1 - 1}$, define

$$\mathbf{Q} = c_0 q_0(\mathbf{T}) + \cdots + c_{m_k - 1} q_{m_k - 1}(\mathbf{T})$$

that is a projection on the generalized λ_1-eigenspace. *Hint:* Start by considering $\mathbf{X}_1$, an eigenvector of order 1 (a “true” eigenvector). Note that in this case $\mathbf{Q}(\mathbf{X}) = c_0 q(\mathbf{T})\mathbf{X}_1$ as before. So $c_0 = \frac{1}{q(\lambda_1)}$. Now consider an eigenvector $\mathbf{X}_2$ of order 2. How many polynomials will give non-zero vectors? What should be assigned to the coefficients (other than c_0, which have already selected) so that $\mathbf{Q}\mathbf{X}_2 = \mathbf{X}_2$? How do we progress for orders $3, 4, \ldots, m_{k-1}$?

Our constructed projection $\mathbf{Q}$ has image $\mathrm{KER}[(\mathbf{T} - \lambda_1 \mathbf{I})^{m_1}]$. Each polynomial $q_j(x)$ has $q(x)$ as a factor and is a polynomial of degree less than n. The polynomial, which we will call $s(x)$, that defines $\mathbf{Q}$, also has $q(x)$ as a factor and is of degree less than n, as it is a linear combination of the q_j's. In particular $\mathrm{KER}[q(\mathbf{T})] \subseteq \mathrm{KER}[s(\mathbf{T}) = \mathbf{Q}]$ so $\mathrm{KER}[q(\mathbf{T})] \subseteq \mathrm{KER}[p(\mathbf{T})] \cap \mathrm{KER}[\mathbf{Q}]$. The polynomial $s(x)$ has either more roots than $q(x)$ or has some roots of higher multiplicity. However, λ_1 is not a root of $s(x)$, as

$$s(\mathbf{T})\mathbf{Y}_1 = \mathbf{Q}\mathbf{Y}_1 = \mathbf{Y}_1 \neq 0$$

for any λ_1-eigenvalue $\mathbf{Y}_1$. So we may use Corollary 5 to determine that for any root μ of $s(x)$ and $\mathbf{Y}$ a μ-eigenvector of $\mathbf{T}$ of order higher than the mulitplicity of μ as a root of $q(x)$, then $p(\mathbf{T})\mathbf{Y} \neq \mathbf{0}$. This allows us to conclude that

$$\mathrm{KER}[\mathbf{Q}] \cap \mathrm{KER}[p(\mathbf{T})] = \mathrm{KER}[q(\mathbf{T})]$$

as desired. □

Corollary 7. *Consider the linear first order differential equation*

$$a_0 y + a_1 y' + a_2 y'' + \cdots + a_n y^{(n)} = 0$$

and let

$$p(x) = \sum_{j=0}^{k} a_j x^j$$

be its characteristic polynomial with roots $\lambda_1, \dots, \lambda_k$ with respective multiplicities $m_1, \dots, m_k$. The solution set of the ODE is

$$\mathrm{SPAN}\left(\bigcup_{j=1}^{k}\bigcup_{\ell=0}^{m_j-1}\{x^\ell e^{\lambda_j x}\}\right).$$

In particular, the solution set is an n dimensional subspace of $\mathcal{C}^\infty(\mathbb{C})$.

Complex Extensions of Real Vector Spaces

Definition. Let $\mathcal{V}$ be a real vector space. We will call

$$\mathcal{V}_\mathbb{C} = \{\mathbf{X} + \imath\mathbf{Y} : \mathbf{X}, \mathbf{Y} \in \mathcal{V}\}$$

the **complex extension** of $\mathcal{V}$.

Exercise 7. $\mathcal{V}_\mathbb{C}$ is a complex vector space. Show that $\dim_\mathbb{C} \mathcal{V}_\mathbb{C} = \dim_\mathbb{R} \mathcal{V}$ and $\dim_\mathbb{R} \mathcal{V}_\mathbb{C} = 2\dim_\mathbb{R} \mathcal{V}$ when $\mathcal{V}$ is finite dimensional.

Definition. Let $\mathbf{T}$ be a linear operator on real vector space $\mathcal{V}$. Then we define $\mathbf{T}_\mathbb{C} : \mathcal{V}_\mathbb{C} \to \mathcal{V}_\mathbb{C}$, the **complex extension** of $\mathbf{T}$ to $\mathcal{V}_\mathbb{C}$ as

$$\mathbf{T}_\mathbb{C}(\mathbf{X} + \imath\mathbf{Y}) = \mathbf{T}(\mathbf{X}) + \imath(\mathbf{Y})$$

where $\mathbf{Z} = \mathbf{X} + \imath\mathbf{Y} \in \mathcal{V}_\mathbb{C}$.

Exercise 8. Show that $\mathbf{T}_\mathbb{C}$ is a linear operator on $\mathcal{V}_\mathbb{C}$ and

$$\mathrm{KER}(\mathbf{T}_\mathbb{C}) = \{\mathbf{X} + \imath\mathbf{Y} : \mathbf{X}, \mathbf{Y} \in \mathrm{KER}(\mathbf{T})\}$$

Definition. Let $\mathcal{V}$ be a real vector space and $\mathcal{V}_\mathbb{C}$ its complex extension. Then we define the following functions. Let $\mathbf{Z} = \mathbf{X} + \imath\mathbf{Y} \in \mathcal{V}_\mathbb{C}$ with $\mathbf{X}, \mathbf{Y} \in \mathcal{V}$.

(1) $\bar{\cdot} : \mathcal{V}_\mathbb{C} \to \mathcal{V}_\mathbb{C}$ is the **complex conjugate** map, given by
$$\overline{\mathbf{Z}} = \overline{\mathbf{X} + \imath\mathbf{Y}} = \mathbf{X} - \imath\mathbf{Y}$$

(2) $\mathfrak{Re} : \mathcal{V}_\mathbb{C} \to \mathcal{V}$ is the **real part** map, given by
$$\mathfrak{Re}(\mathbf{Z}) = \mathfrak{Re}(\mathbf{X} + \imath\mathbf{Y}) = \mathbf{X}.$$

(3) $\mathfrak{Im} : \mathcal{V}_\mathbb{C} \to \mathcal{V}$ is the **imaginary part** map, given by
$$\mathfrak{Im}(\mathbf{Z}) = \mathfrak{Im}(\mathbf{X} + \imath\mathbf{Y}) = \mathbf{Y}.$$

Exercise 9. Show that the three given maps are not linear when treating $\mathcal{V}_\mathbb{C}$ as a complex vector space. However, if $\mathcal{V}_\mathbb{C}$ is treated as a real vector space (it is twice the dimension of $\mathcal{V}$ over $\mathbb{R}$), then they are linear. Also, verify that

$$\mathfrak{Re}(\mathbf{Z}) = \frac{1}{2}(\mathbf{Z} + \overline{\mathbf{Z}}) \text{ and } \mathfrak{Im}(\mathbf{Z}) = \frac{1}{2\imath}(\mathbf{Z} - \overline{\mathbf{Z}})$$

for each $\mathbf{Z} \in \mathcal{V}_\mathbb{C}$.

Lemma 8. *Let $\mathbf{T} : \mathcal{V} \to \mathcal{V}$ be linear on $\mathcal{V}$, a real vector space. Then for every $\mathbf{Z} \in \mathcal{V}_\mathbb{C}$,*

$$\mathfrak{Re}(\mathbf{T}_\mathbb{C}(\mathbf{Z})) = \mathbf{T}(\mathfrak{Re}(\mathbf{Z})).$$

(This is usually expressed as $\mathfrak{Re} \circ \mathbf{T}_\mathbb{C} = \mathbf{T} \circ \mathfrak{Re}$.) Likewise, $\mathfrak{Im} \circ \mathbf{T}_\mathbb{C} = \mathbf{T} \circ \mathfrak{Im}$.

Proof. Exercise. □

Exercise 10. Let $\mathcal{V}$ be a real vector space and $\mathcal{V}_{\mathbb{C}}$ its complex extension. Show that for any subspace $\mathcal{W} \subseteq \mathcal{V}_{\mathbb{C}}$, the sets $\mathfrak{Re}(\mathcal{W}), \mathfrak{Im}(\mathcal{W}) \subset \mathcal{V}$ are equal and either (both) is a real subspace of $\mathcal{V}$.

Lemma 9. *Let $\mathcal{V}$ be a real vector space and $\mathcal{V}_{\mathbb{C}}$ its complex extension. Let $\mathcal{W} \subseteq \mathcal{V}_{\mathbb{C}}$ be subspace with $\dim_{\mathbb{C}} \mathcal{W} = n < \infty$. Then the following are equivalent:*

(1) *$\mathcal{W} = \overline{\mathcal{W}}$, or $\forall \mathbf{X}$, $\mathbf{X} \in \mathcal{W}$ if and only if $\overline{\mathbf{X}} \in \mathcal{W}$.*
(2) *There exists a basis $\{\mathbf{X}_1, \dots, \mathbf{X}_n\}$ of $\mathcal{W}$ such that $\mathbf{X}_i \in \mathcal{V}$ for $1 \leq i \leq n$.*
(3) *Let $\mathcal{W}' = \mathfrak{Re}(\mathcal{W}) \subseteq \mathcal{V}$. Then $\mathcal{W} = \mathcal{W}' + \imath\mathcal{W}'$.*

Moreover, if $\mathcal{W}$ satisfies these conditions, then the subspace $\mathcal{W}'$ has the basis from (2) as its basis (the span is taken over $\mathbb{R}$ rather than $\mathbb{C}$).

Proof. We will leave the following to the reader.

Exercise 11. Show that (2) implies (1) and (3) implies (1).

We will show that (1) implies (2). Start with any basis

$$\{\mathbf{Z}_1, \dots, \mathbf{Z}_n\}$$

of $\mathcal{W}$. Then $\mathfrak{Re}(\mathbf{Z}_i) = \frac{1}{2}(\mathbf{Z}_i + \overline{\mathbf{Z}_i})$ and $\mathfrak{Im}(\mathbf{Z}_i) = \frac{1}{2\imath}(\mathbf{Z}_i - \overline{\mathbf{Z}_i})$ belong to both $\mathcal{V}$ and $\mathcal{W}$, and therefore

$$\{\mathfrak{Re}(\mathbf{Z}_1), \mathfrak{Im}(\mathbf{Z}_1), \dots, \mathfrak{Re}(\mathbf{Z}_n), \mathfrak{Im}(\mathbf{Z}_n)\}$$

contains $2n$ vectors in $\mathcal{V}$, but still spans $\mathcal{W}$ (when considering $\mathrm{SPAN}_{\mathbb{C}}$). We may therefore take a subset of these elements to form a basis of $\mathcal{W}$.

We now show that (2) implies (3). Let $\{\mathbf{X}_1, \dots, \mathbf{X}_n\}$ be a basis of $\mathcal{W}$ with $\mathbf{X}_i \in \mathcal{V}$. Then for any vector $\mathbf{Z} \in \mathcal{W}$, there exists unique $a_1 + \imath b_1, \dots, a_n + \imath b_n \in \mathbb{C}$ such that

$$\mathbf{Z} = (a_1 + \imath b_1)\mathbf{X}_1 + \cdots + (a_n + \imath b_n)\mathbf{X}_n = (a_1\mathbf{X}_1 + \cdots + a_n\mathbf{X}_n) + \imath(b_1\mathbf{X}_1 + \cdots + b_n\mathbf{X}_n).$$

This allows us to conclude that $\mathcal{W} \subseteq \mathcal{W}' + \imath\mathcal{W}'$. Moreover, this tells us that our final claim holds, namely, our basis $\{\mathbf{X}_1, \dots, \mathbf{X}_n\}$ is also a basis for $\mathcal{W}'$.

To show the other inclusion, select any $\mathbf{X}, \mathbf{Y} \in \mathcal{V}$, then

$$\mathbf{X} = a_1\mathbf{X}_1 + \cdots + a_n\mathbf{X}_n \text{ and } \mathbf{Y} = b_1\mathbf{X}_1 + \cdots + b_n\mathbf{X}_n$$

for unique real values $a_1, \dots, a_n, b_1, \dots, b_n$. By letting $c_i = a_i + \imath b_i$, we may find $\mathbf{Z} = \sum_{i=1}^n c_i\mathbf{X}_i$ such that $\mathbf{Z} \in \mathcal{W}$ and $\mathbf{Z} = \mathbf{X} + \imath\mathbf{Y}$. Therefore $\mathcal{W}' + \imath\mathcal{W}' \subseteq \mathcal{W}$. ☐

Lemma 10. *If $\mathbf{T} : \mathcal{V} \to \mathcal{V}$ is linear on real a vector space $\mathcal{V}$, then $\mathrm{KER}(\mathbf{T}_{\mathbb{C}})$ is closed under conjugation and*

$$\mathrm{KER}(\mathbf{T}) = \mathfrak{Re}(\mathrm{KER}(\mathbf{T}_{\mathbb{C}})) = \mathfrak{Im}(\mathrm{KER}(\mathbf{T}_{\mathbb{C}})).$$

Proof. Let $\mathcal{W} = \mathrm{KER}(\mathbf{T}_{\mathbb{C}})$ and $\mathcal{W}' = \mathfrak{Re}(\mathcal{W}) = \mathfrak{Im}(\mathcal{W})$.

We will first prove that $\overline{\mathcal{W}} = \mathcal{W}$. It suffices to show that if $\mathbf{Z} \in \mathcal{W}$, then $\overline{\mathbf{Z}} \in \mathcal{W}$. By Exercise 8, if $\mathbf{Z} \in \mathcal{W}$, then $\mathbf{Z} = \mathbf{X} + \imath\mathbf{Y}$ where $\mathbf{T}(\mathbf{X}) = \mathbf{T}(\mathbf{Y}) = \mathbf{0}$. But then

$$\mathbf{T}_{\mathbb{C}}(\overline{\mathbf{Z}}) = \mathbf{T}_{\mathbb{C}}(\mathbf{X} - \imath\mathbf{Y}) = \mathbf{T}(\mathbf{X}) - \imath\mathbf{T}(\mathbf{Y}) = \mathbf{0} - \imath\mathbf{0} = \mathbf{0}$$

or $\overline{\mathbf{Z}} \in \mathcal{W}$ as well.

We now show that $\mathrm{KER}(\mathbf{T}) = \mathcal{W}'$. Let $\mathbf{Z} \in \mathcal{W}$. Then by Exercise 8, $\mathbf{Z} = \mathbf{X} + \imath\mathbf{Y}$ for $\mathbf{X}, \mathbf{Y} \in \mathrm{KER}(\mathbf{T})$. In particular $\mathfrak{Re}(\mathbf{Z}) = \mathbf{X} \in \mathrm{KER}(\mathbf{T})$. This implies that $\mathfrak{Re}(\mathrm{KER}(\mathbf{T}_{\mathbb{C}})) \subseteq \mathrm{KER}(\mathbf{T})$. Now assumse that $\mathbf{X} \in \mathrm{KER}(\mathbf{T})$. Let $\mathbf{Z} = \mathbf{X} + \imath\mathbf{0}$ and note that $\mathfrak{Re}(\mathbf{Z}) = \mathbf{X}$ and $\mathbf{T}_{\mathbb{C}}(\mathbf{Z}) = \mathbf{0}$. We conclude that $\mathrm{KER}(\mathbf{T}) \subseteq \mathfrak{Re}(\mathrm{KER}(\mathbf{T}_{\mathbb{C}}))$. ☐

Finding Solutions in the Space $\mathcal{C}^\infty(\mathbb{R})$

We now consider the differential operator, but this time acting only on $\mathcal{C}^\infty(\mathbb{R})$ rather than on the complex vector space. We also will be only considering linear ODEs such that all of the coefficients are real. It follows from Corollary 7 that we may define our n-dimensional solution set but in $\mathcal{C}^\infty(\mathbb{C})$. If all of the roots of the characteristic polynomial are real, then the solution set given works and defines our n-dimensional space in $\mathcal{C}^\infty(\mathbb{R})$. If there are complex roots, then by Lemmas 9 and 10, we must take a common real basis for our solution set.

Corollary 11. *Consider the linear first order differential equation*

$$a_0y + a_1y' + a_2y'' + \cdots + a_ny^{(n)} = 0$$

and let

$$p(x) = \sum_{j=0}^{k} a_jx^j$$

be its characteristic polynomial with real coefficients. Let $\lambda_1, \ldots, \lambda_\ell$ be the real roots of p with respective multiplicities $m_1, \ldots, m_\ell$ and let $\mu_1, \overline{\mu_1}, \ldots, \mu_k, \overline{\mu_k}$ be the pairs of complex conjugate roots with multiplicities $m'_1, \ldots, m'_k$. Then the solution set is

$$\mathrm{SPAN}_{\mathbb{R}}\left(\left(\bigcup_{j=1}^{\ell}\bigcup_{r=0}^{m_j-1}\{x^re^{\lambda_jx}\}\right)\cup\left(\bigcup_{j=1}^{k}\bigcup_{r=0}^{m'_k-1}\{x^re^{\alpha_jx}\cos(\beta x), x^re^{\alpha_jx}\sin(\beta x)\}\right)\right)$$

where $\mu_j = \alpha_j + \imath\beta_j$, $\alpha_j, \beta_j \in \mathbb{R}$, $\beta_j \neq 0$.

Proof. By Corollary 7, we may find an n-(complex)dimensional subspace of $\mathcal{C}^\infty(\mathbb{C})$ that is the set of solutions to $p(\mathbf{D})y = 0$. This is of the form

$$\mathrm{SPAN}_{\mathbb{C}}\left(\left(\bigcup_{j=1}^{\ell}\bigcup_{r=0}^{m_j-1}\{x^re^{\lambda_jx}\}\right)\cup\left(\bigcup_{j=1}^{k}\bigcup_{r=0}^{m'_j-1}\{x^re^{\mu_jx}, x^re^{\overline{\mu_j}x}\}\right)\right).$$

By Lemma 10, the solution set is the real part of the complex solution set. By Lemma 9, we may find a basis in $\mathcal{C}^\infty(\mathbb{R})$ that spans the complex solution set (by taking complex linear combinations) and the real solution set (by taking real linear combinations instead). For each real root λ_j, the associated basis elements already belong to $\mathcal{C}^\infty(\mathbb{R})$. For each $1 \leq j \leq k$ and $1 \leq r \leq m'_j - 1$, note that

$$x^re^{\mu_jx} = x^re^{\alpha_jx+\imath\beta_jx} = x^re^{\alpha_jx}e^{\imath\beta_jx} = x^re^{\alpha_jx}\cos(\beta_jx) + \imath x^re^{\alpha_jx}\sin(\beta_jx)$$

and likewise $x^re^{\overline{\mu_j}x} = x^re^{\alpha_jx}\cos(\beta_jx) + \imath x^re^{\alpha_jx}\sin(\beta_jx)$.

Exercise 12. Show that

$$\mathrm{SPAN}_{\mathbb{C}}\{x^re^{\alpha_jx}\cos(\beta_jx), x^re^{\alpha_jx}\sin(\beta_jx)\} = \mathrm{SPAN}_{\mathbb{C}}\{x^re^{\mu_jx}, x^re^{\overline{\mu_j}x}\}$$

We conclude that we may take the basis stated as our $\mathcal{C}^\infty(\mathbb{R})$ basis by replacing our complex basis elements $x^re^{\mu_jx}, x^re^{\overline{\mu_j}x}$ with elements as listed in the above exercise. $\square$

Examples

Finding Solutions to Linear ODEs. We begin with a few examples to illustrate the proof of Theorem 6

Example. Suppose $\mathbf{T} : \mathcal{V} \to \mathcal{V}$ satisfies
$$\mathbf{T}^2 = \mathbf{I}.$$
Let $p(x) = x^2 - 1 = (x+1)(x-1)$, then the original equation becomes
$$p(\mathbf{T}) = \mathbf{T}^2 - \mathbf{I} = (\mathbf{T}+\mathbf{I})(\mathbf{T}-\mathbf{I}) = \mathbf{0}.$$
So in this case $\mathrm{KER}[p(\mathbf{T})] = \mathcal{V}$. We would then define $q(x) = x - 1$. As the proof suggests, we see that
$$q(\mathbf{T})\mathbf{X} = (\mathbf{T}-\mathbf{I})\mathbf{X} = 0 \iff \mathbf{TX} = \mathbf{X}$$
if and only if $\mathbf{X}$ is a 1-eigenvector. Likewise, let $\mathbf{Y} = q(\mathbf{T})\mathbf{X}$ for $\mathbf{X} \in \mathcal{V}$. Then
$$\mathbf{TY} = \mathbf{T}(\mathbf{T}-\mathbf{I})\mathbf{X} = \mathbf{T}^2\mathbf{X} - \mathbf{TX} = (\mathbf{I}-\mathbf{T})\mathbf{X} = -\mathbf{Y}$$
or that the image of $q(\mathbf{T})$ is the (-1)-eigenspace. We need to find $\mathbf{Q}$, a projection with the same image and kernel as $q(\mathbf{T})$. Let $\mathbf{Y}$ be a (-1)-eigenvector of $\mathbf{T}$. Then
$$q(\mathbf{T})\mathbf{Y} = \mathbf{TY} - \mathbf{IY} = -2\mathbf{Y}.$$
So let $\mathbf{Q} = -\frac{1}{2}q(\mathbf{T})$. We then may say that in deed (by Lemma 1) that
$$\mathcal{V} = \mathrm{KER}(\mathbf{T}-\mathbf{I}) \oplus \mathrm{KER}(\mathbf{T}+\mathbf{I}),$$
or equivalenty that every $\mathbf{X} \in \mathcal{V}$ may be uniquely expressed as $\mathbf{X} = \mathbf{X}_1 + \mathbf{X}_{-1}$ where $\mathbf{X}_{\pm 1}$ is in the (± 1)-eigenspace.

This result has a number of applications to specific vector spaces in this course:

- $\mathcal{V} = M(n,n)$. $T : \mathcal{V} \to \mathcal{V}$ is defined by $\mathbf{T}(\mathbf{A}) = \mathbf{A}^t$. We conclude that every matrix $\mathbf{A}$ may be uniquely written as
$$\mathbf{A} = \mathbf{B} + \mathbf{C}$$
where $\mathbf{B}^t = \mathbf{B}$ and $\mathbf{C}^t = -\mathbf{C}$.
- $\mathcal{V} = M_C(n,n)$ (treated as a real vector space). Then if $\mathbf{T} : \mathcal{V} \to \mathcal{V}$ is defined by $\mathbf{T}(\mathbf{A}) = \overline{\mathbf{A}}$, then any matrix $\mathbf{A}$ may be uniquely expressed as
$$\mathbf{A} = \mathbf{X} + \imath\mathbf{Y}$$
where $\mathbf{X}, \mathbf{Y} \in M(n,n)$.
- $\mathcal{V} = \mathcal{C}^\infty(\mathbb{R})$. $\mathbf{T} : \mathcal{V} \to \mathcal{V}$ defined by $\mathbf{T}(f(x)) = f(-x)$. Any function $f \in \mathcal{C}^\infty(\mathbb{R})$ may be uniquely expressed as
$$f = g + h$$
where f is even and g is odd.

We now go through our proof with an eigenvalue of mulitplicity 2.

Example. Let $p(x) = x^3 - 2x^2 + x$, $\mathbf{T} : \mathcal{V} \to \mathcal{V}$ be linear. Let $q(x) = x$ so that $p(x) = (x-1)^2 q(x)$. Let $\mathbf{X} \in \mathcal{K} = \mathrm{KER}[p(\mathbf{T})]$, then
$$\mathbf{0} = p(\mathbf{T})\mathbf{X} = (\mathbf{T}-\mathbf{I})^2 q(\mathbf{T})\mathbf{X}$$
implies that the image of $q(\mathbf{T})$ restricted to $\mathcal{K}$ belongs to $\mathrm{KER}[(\mathbf{T}-\mathbf{I})^2]$. Also, by definition $\mathrm{KER}[q(\mathbf{T})] = \mathrm{KER}[\mathbf{T}]$. The proof suggests considering two polynomials:
$$q_0(x) = q(x) \text{ and } q_1(x) = (x-1)q(x).$$

Also, we must consider 1-eigenvectors of orders 1 and 2. If $\mathbf{Y}_1$ is an order 1 eigenvector, we see that

$$\begin{aligned} q_0(\mathbf{T})\mathbf{Y}_1 &= q(1)\mathbf{Y}_1 &= \mathbf{Y}_1, \\ q_1(\mathbf{T})\mathbf{Y}_1 &= q_1(1)\mathbf{Y}_1 &= \mathbf{0}. \end{aligned}$$

If we consider $\mathbf{Y}_2$, an order 2 eigenvector, and $\mathbf{Y}_1$ as the order 1 eigenvector in its chain, then

$$\begin{aligned} q_0(\mathbf{T})\mathbf{Y}_2 &= q(1)\mathbf{Y}_2 + q'(1)\mathbf{Y}_1 &= \mathbf{Y}_2 + \mathbf{Y}_1, \\ q_1(\mathbf{T})\mathbf{Y}_2 &= q_1(1)\mathbf{Y}_2 + q_1'(1)\mathbf{Y}_1 &= 2\mathbf{Y}_1. \end{aligned}$$

If we let $s(x) = q_0(x) - \frac{1}{2}q_1(x) = \frac{3}{2}x - \frac{1}{2}x^2$ and $\mathbf{Q} = s(\mathbf{T})$, then for any 1-eigenvector (of either order) $\mathbf{Y}$, $\mathbf{QY} = \mathbf{Y}$. Now $s(x)$ is a polynomial with roots 0 and 3 (both of mulitplicity 1), so the kernel of $\mathbf{Q}$ is the span of 1-eigenvectors and 3-eigenvectors. However, only 1-eigenvectors belong in the kernel of $p(\mathbf{T})$, as 3 is not a root of p. Therefore, the only elements that are in the kernels of both $\mathbf{Q}$ and $p(\mathbf{T})$ are 1-eigenvectors, precisely the kernel of $q(\mathbf{T})$. We then may conclude, as $\mathbf{Q}$ is a projection, that

$$\mathrm{KER}[p(\mathbf{T})] = \mathrm{KER}[\mathbf{T}] \oplus \mathrm{KER}[(\mathbf{T} - \mathbf{I})^2].$$

The following example makes clear a key point, what Theorem 6 does NOT say.

Example. Let $\mathbf{A} \in M_C(3,3)$ with Jordan canonical form

$$\mathbf{A} \sim \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

We already know from this that $\mathrm{NULL}(\mathbf{A} - 3\mathbf{I})$ is one dimensional, $\mathrm{NULL}(\mathbf{A} - 2\mathbf{I})$ is one dimensional and $\mathrm{NULL}[(\mathbf{A} - 2\mathbf{I})^2]$ is two dimensional. The Cayley-Hamilton theorem tells us that indeed

$$p_{\mathbf{A}}(\mathbf{A}) = 12\mathbf{I} - 16\mathbf{A} + 7\mathbf{A}^2 - \mathbf{A}^3 = \mathbf{0}.$$

However, if we pick a very large polynomial, like $q(x) = (x-1)^3(x-2)^2(x-3)(x-10)^{100}$, then

$$q(\mathbf{A}) = \mathbf{0}$$

still holds. Theorem 6 tells us that in this case

$$\mathbb{C}^3 = \mathrm{NULL}[(\mathbf{A} - \mathbf{I})^3] + \mathrm{NULL}[(\mathbf{A} - 2\mathbf{I})^2] + \mathrm{NULL}[\mathbf{A} - 3\mathbf{I}] + \mathrm{NULL}[(\mathbf{A} - 10\mathbf{I})^{100}].$$

This statement is techinically still true, as every eigenspace is trivial (just the zero vector) for all values other than 2 and 3. Specifically, this theorem does NOT guarantee that every eigenspace has full dimension, or even positive dimension!

Complex Extensions of Real Vector Spaces. We give a few basic examples of complex extensions (that we have alreay seen in this course).

Example. $\mathbb{R}^n$ is a real n-dimensional vector space. $\mathbb{C}^n = \mathbb{R}^n + \imath\mathbb{R}^n$ is its complex extention. It has complex dimension n.

Example. $M(n,n)$ is a real vector space of dimension n^2.

$$M_C(n,n) = M(n,n) + \imath M(n,n)$$

is its complex extension. If $\{\mathbf{E}_{\ell,j}\}_{\ell,j=1}^{n}$ is the standard basis for $M(n,n)$, then it also a complex standard basis for $M_C(n,n)$ as well. If we wanted to treat $M_C(n,n)$ as a real vector space, then our basis would have to be

$$\{\mathbf{E}_{\ell,j}, \imath\mathbf{E}_{\ell,j}\}_{\ell,j=1}^{n}.$$

So $\dim_{\mathbb{C}} M_C(n,n) = n^2$ while $\dim_{\mathbb{R}} M_C(n,n) = 2n^2$.

Example. $\mathbf{A} \in M(n,n)$ naturally defines a linear operation on $\mathbb{R}^n$. The complex extension of $\mathbb{R}^n$ is $\mathbb{C}^n$. The complex extension of $\mathbf{A}$ to this space is again multiplication by $\mathbf{A}$, as if $\mathbf{Z} = \mathbf{X} + \imath\mathbf{Y}$, $\mathbf{X}, \mathbf{Y} \in \mathbb{R}^n$, then

$$\mathbf{AZ} = \mathbf{AX} + \imath\mathbf{AY}.$$

We now show some examples related to Lemma 9.

Example. Let $\mathcal{W} = \mathrm{SPAN}_{\mathbb{C}}\left\{\begin{bmatrix}\imath\\1\end{bmatrix}\right\}$ be a one dimensional subspace of $\mathbb{C}^2$. Note that for any $z = a + \imath b \in \mathbb{C}$,

$$z\begin{bmatrix}\imath\\1\end{bmatrix} = (a+\imath b)\begin{bmatrix}\imath\\1\end{bmatrix} = \begin{bmatrix}-b\\a\end{bmatrix} + \imath\begin{bmatrix}a\\b\end{bmatrix}$$

So we may redefine $\mathcal{V}$ as

$$\mathcal{W} = \left\{\begin{bmatrix}-b\\a\end{bmatrix} + \imath\begin{bmatrix}a\\b\end{bmatrix} : a, b \in \mathbb{R}\right\}.$$

We may directly see that

$$\mathcal{W}' = \mathfrak{Re}(\mathcal{W}) = \mathfrak{Im}(\mathcal{W}) = \mathbb{R}^2.$$

However, $\mathcal{W} \neq \mathcal{W}' + \imath\mathcal{W}' = \mathbb{C}^2$. We may verify that other equivalent conditions fail, such as

$$\overline{\begin{bmatrix}\imath\\1\end{bmatrix}} = \begin{bmatrix}-\imath\\1\end{bmatrix} \notin \mathcal{W}.$$

So we have an example that fails this lemma.

Example. Let $\mathcal{W} = \mathrm{SPAN}_{\mathbb{C}}\left\{\begin{bmatrix}1\\\imath\\1\end{bmatrix}, \begin{bmatrix}\imath\\1\\\imath\end{bmatrix}\right\} \subseteq \mathbb{C}^3$. Our basis is not closed under conjugation, but we will see that the equivalent conditions of Lemma 9 hold for $\mathcal{W}$. In particular, let $\mathbf{Z}_1, \mathbf{Z}_2$ be our listed basis vectors. Then

$$\overline{\mathbf{Z}_1} = -\imath\mathbf{Z}_2 \text{ and } \overline{\mathbf{Z}_2} = -\imath\mathbf{Z}_1.$$

So for any vector $\mathbf{Z} = c_1\mathbf{Z}_1 + c_2\mathbf{Z}_2 \in \mathcal{W}$, $\overline{\mathbf{Z}} = -\imath\overline{c_1}\mathbf{Z}_2 - \imath\overline{c_2}\mathbf{Z}_1 \in \mathcal{W}$ as well. To fulfill the second condition, consider

$$\{\mathfrak{Re}(\mathbf{Z}_1), \mathfrak{Im}(\mathbf{Z}_1), \mathfrak{Re}(\mathbf{Z}_2), \mathfrak{Im}(\mathbf{Z}_2)\} = \left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}\right\} = \left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}.$$

This is a basis for $\mathcal{W}$ with elements in $\mathbb{R}^3$. We then may find that $\mathcal{W}' = \mathfrak{Re}(\mathcal{W})$ is just the span of this basis, but over $\mathbb{R}$ and that $\mathcal{W} = \mathcal{W}' + \imath\mathcal{W}'$.

Finding Solutions in the Space $\mathcal{C}^\infty(\mathbb{R})$. We end with solutions to ODE problems.

Example.
$$y'' - 6y' + 8y = 0$$
has characteristic polynomial $p(x) = x^2 - 6x + 8 = (x-2)(x-4)$. The general solution is then
$$y = c_1 e^{2x} + c_2 e^{4x}.$$

Example.
$$y'''(x) - 3y''(x) + 3y'(x) - y(x) = 0$$
has characteristic polynomial $p(x) = (x-1)^3$. The general solution is therefore
$$y(x) = c_1 e^x + c_2 x e^x + c_3 x^2 e^x.$$

Example.
$$y''''(x) + 2y''(x) + y = 0$$
has polynomial $p(x) = x^4 + 2x^2 + 1 = (x^2+1)^2$. The general solution is
$$y(x) = c_1 \cos(x) + c_2 \sin(x) + c_3 x \cos(x) + c_4 x \sin(x).$$

Example.
$$y^{(7)} - 2y^{(6)} - 6y^{(5)} + 16y^{(4)} - 32y'' + 32y' = 0$$
has polynomial $p(x) = x(x-2)^2(x+2)^2(x^2-2x+2)$ and general solution
$$y = c_1 + c_2 e^{2x} + c_3 x e^{2x} + c_4 e^{-2x} + c_5 x e^{-2x} + c_6 e^x \cos(x) + c_7 e^x \sin(x).$$