MATH 204 C03 – DIRECT SUMS AND PROJECTIONS

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Outline

We have discussed two notions in class that do not appear in the text: projections and direct sums. This is designed as a supplement to the material put on the board with extra examples. These two concepts are connected, and we express this as Propositions 1-3 in the third section. We conclude by tying in our results to the discussion of *Proj* and *Orth* as defined in the text.

Terms

Let $\mathcal{V}$ and $\mathcal{W}$ be a vector spaces and $T : \mathcal{V} \to \mathcal{W}$ a linear transformation. Then

$$\begin{array}{lcl} \mathrm{IM}(T) & \leftrightarrow & \text{Image of } T \\ \mathrm{KER}(T) & \leftrightarrow & \text{Kernel of } T \end{array}$$

The span of a collection $S \subset \mathcal{V}$ is SPAN(S).

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1. Projections

We always will assume that $\mathcal{V}$ is a vector space.

Definition. A linear map $P : \mathcal{V} \to \mathcal{V}$ is a **projection** if

$$P^2 = P \circ P = P$$

or equivalently

$$P(\mathbf{Y}) = \mathbf{Y}, \text{ for all } \mathbf{Y} \in \mathrm{IM}(P)$$

Remark. If P is a projection, then $P^n = P \circ \cdots \circ P = P$ for any $n > 1$ by induction.

Example. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by the matrix

$$T(\mathbf{X}) = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}}_{\mathbf{A}} \mathbf{X}.$$

Is T a projection? Well T^2 is defined by $\mathbf{A}^2$, and

$$\begin{aligned}
\mathbf{AA} &= \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \\
&= \begin{bmatrix} 1+0+0 & 0+0+0 & 0+0+0 \\ \frac{1}{2}+\frac{1}{4}-\frac{1}{4} & 0+\frac{1}{4}+\frac{1}{4} & 0-\frac{1}{4}-\frac{1}{4} \\ \frac{1}{2}-\frac{1}{4}+\frac{1}{4} & 0-\frac{1}{4}-\frac{1}{4} & 0+\frac{1}{4}+\frac{1}{4} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \\
&= \mathbf{A}
\end{aligned}$$

So T^2 is defined by $\mathbf{A}$, which tells us that $T^2 = T$.

Exercise. What is $\mathrm{IM}(T)$ and $\mathrm{KER}(T)$?

Example. The function $T : \mathcal{C}(\mathbb{R}) \to \mathcal{C}(\mathbb{R})$ ($\mathcal{C}(\mathbb{R})$ is the set of real valued, continuous functions) defined by

$$T(f) = \int_0^x f(t)dt$$

is not a projection. Sure it is linear, but consider the function $g(x) = x$.

$$T(g) = \int_0^x t dt = \frac{x^2}{2}$$

while

$$T^2(g) = T(\frac{x^2}{2}) = \int_0^x \frac{t^2}{2} dt = \frac{x^3}{6} \neq T(g).$$

We conclude that $T \neq T^2$ and so T is NOT a projection.

We have discussed a few general points about projections, as follows:

T/F: If $P, Q : \mathcal{V} \to \mathcal{V}$ are projections and $\mathrm{IM}(P) = \mathrm{IM}(Q)$, then $P = Q$.

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FALSE: For a simple counterexample, consider the two linear transformations defined by

$$\mathbf{P} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } \mathbf{Q} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

Each define projections as $\mathbf{P}^2 = \mathbf{P}$ and $\mathbf{Q}^2 = \mathbf{Q}$. The image of each is the x-axis in $\mathbb{R}^2$. But they are not the same transformation, as

$$\mathbf{P}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \mathbf{Q}\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

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T/F: If $P, Q : \mathcal{V} \to \mathcal{V}$ are projections such that $\mathrm{IM}(P) = \mathrm{IM}(Q)$ AND $\mathrm{KER}(P) = \mathrm{KER}(Q)$, then $P = Q$.

*****

TRUE: Call $\mathcal{W}_0$ the mutual kernel and $\mathcal{W}_1$ the mutual image. Consider any $\mathbf{X} \in \mathcal{V}$ and let

$$\mathbf{Y} = P(\mathbf{X}) \text{ and } \mathbf{Y}' = Q(\mathbf{X}).$$

We see that $Q(\mathbf{X} - \mathbf{Y}') = Q(\mathbf{X}) - Q(\mathbf{Y}') = \mathbf{Y}' - \mathbf{Y}' = \mathbf{0}$, so

$$\mathbf{X} = \mathbf{Y}' + \mathbf{Z}'$$

where $\mathbf{Y}' \in \mathcal{W}_1$ and $\mathbf{Z}' \in \mathcal{W}_0$. So it follows that

$$\begin{aligned} \mathbf{Y} = P(\mathbf{X}) &= P(\mathbf{Y}' + \mathbf{Z}') = P(\mathbf{Y}') + P(\mathbf{Z}') \\ &= \mathbf{Y}' + \mathbf{0} = \mathbf{Y}'. \end{aligned}$$

Therefore, for every $\mathbf{X} \in \mathcal{V}$, $P(\mathbf{X}) = Q(\mathbf{X})$.

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T/F: Let $P : \mathcal{V} \to \mathcal{V}$ be a projection. There exists a unique projection $Q : \mathcal{V} \to \mathcal{V}$ such that

$$\mathrm{IM}(P) = \mathrm{KER}(Q) \text{ and } \mathrm{KER}(P) = \mathrm{IM}(Q).$$

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TRUE: If such a Q exists, it is unique by our previous T/F. So we simply need to find Q. Let

$$Q = I_{\mathcal{V}} - P$$

where $I_{\mathcal{V}}$ is the identity function on $\mathcal{V}$.

Exercise. Finish this argument by showing that:
– $Q \circ Q = Q$ (Q is a projection).
– $\mathrm{KER}(Q) = \mathrm{IM}(P)$.
– $\mathrm{IM}(Q) = \mathrm{KER}(P)$.
(*No single argument should have a long proof!*).

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T/F: Let $P, Q : \mathcal{V} \to \mathcal{V}$ be projections. Then $P + Q : \mathcal{V} \to \mathcal{V}$ is a projection.

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FALSE: We give two counterexamples, the second less trivial than the first. Let

$$P = Q = I_{\mathcal{V}}$$

As long as $\mathcal{V}$ doesn't consist of just one element (the zero element), then for any $\mathbf{X} \in \mathcal{V}$, $\mathbf{X} \neq \mathbf{0}$,

$$(P+Q)(\mathbf{X}) = P(\mathbf{X}) + Q(\mathbf{X}) = \mathbf{X} + \mathbf{X} = 2\mathbf{X} \neq \mathbf{X}.$$

Because $(P+Q)$ is not the identity on its image, it is not a projection.

For our second counter example, consider $\mathcal{V} = \mathbb{R}^2$ and

$$\mathbf{P} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } \mathbf{Q} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

These are both projections.

Exercise. Confirm this, and determine their images and kernels.

However,

$$\mathbf{P} + \mathbf{Q} = \frac{1}{2}\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \text{ and } (\mathbf{P} + \mathbf{Q})^2 = \frac{1}{2}\begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$$

so $\mathbf{P} + \mathbf{Q} \neq (\mathbf{P} + \mathbf{Q})^2$.

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T/F: Let $P, Q : \mathcal{V} \to \mathcal{V}$ be projections. Then $P + Q : \mathcal{V} \to \mathcal{V}$ is a projection if and only if

$$\mathrm{IM}(P) \subset \mathrm{KER}(Q) \text{ and } \mathrm{IM}(Q) \subset \mathrm{KER}(P).$$

(*Note that the above conditions imply that* $\mathrm{IM}(P) \cap \mathrm{IM}(Q) = \{\mathbf{0}\}$).

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TRUE: First assume that

$$\mathrm{IM}(P) \subset \mathrm{KER}(Q) \text{ and } \mathrm{IM}(Q) \subset \mathrm{KER}(P)$$

holds. Let $\mathbf{Z} \in \mathrm{IM}(P+Q)$, then $\mathbf{Z} = \mathbf{X} + \mathbf{Y}$ for some $\mathbf{X} \in \mathrm{IM}(P)$ and $\mathbf{Y} \in \mathrm{IM}(Q)$. So

$$\begin{aligned} (P+Q)(\mathbf{Z}) &= P(\mathbf{Z}) + Q(\mathbf{Z}) \\ &= P(\mathbf{X}) + P(\mathbf{Y}) + Q(\mathbf{X}) + Q(\mathbf{Y}) \\ &= \mathbf{X} + \mathbf{0} + \mathbf{0} + \mathbf{Y} \\ &= \mathbf{Z}. \end{aligned}$$

Because $P + Q$ is a projection on its image, it is a projection.

Now suppose that there exists $\mathbf{X} \in \mathrm{IM}(P)$ that does not belong to $\mathrm{KER}(Q)$ and $P + Q$ is a projection. Let $\mathbf{Y} = Q(\mathbf{X})$, and note that this is non-zero. Then

$$\begin{aligned} \mathbf{X} + \mathbf{Y} &= (P+Q)(\mathbf{X}) \\ &= (P+Q)^2(\mathbf{X}) \\ &= (P+Q)(\mathbf{X} + \mathbf{Y}) \\ &= P(\mathbf{X}) + P(\mathbf{Y}) + Q(\mathbf{X}) + Q(\mathbf{Y}) \\ &= \mathbf{X} + P(\mathbf{Y}) + 2\mathbf{Y} \\ &\text{or} \\ P(\mathbf{Y}) &= -\mathbf{Y}. \end{aligned}$$

But because $\mathbf{Y} \neq \mathbf{0}$, we have a vector $\mathbf{Y} \in \mathrm{IM}(P)$ but $P(\mathbf{Y}) \neq \mathbf{Y}$, which contradicts that P is a projection.

We may repeat the same argument by switching the roles of P and Q. We may then conclude that if

$$\mathrm{IM}(P) \subset \mathrm{KER}(Q) \text{ and } \mathrm{IM}(Q) \subset \mathrm{KER}(P)$$

fails, $P + Q$ can not be a projection.

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Remark. Our text refers to a class of transformations as projections. They are technically correct, as all of their maps are projections. However, their maps are **orthogonal projections**, they are projections P such that $\mathrm{IM}(P) \perp \mathrm{KER}(P)$ which means

$$\mathbf{X} \cdot \mathbf{Y} = 0, \text{ for all } \mathbf{X} \in \mathrm{IM}(P) \text{ and } \mathbf{Y} \in \mathrm{KER}(P).$$

In the language of the book, $Proj_{\mathcal{W}}$ is the unique projection with image $\mathcal{W}$ and kernel $\mathcal{W}^{\perp}$.

2. Direct Sums

We begin by giving a bsic definition

Definition. Let $\mathcal{W}_1, \mathcal{W}_2 \subset \mathcal{V}$ be subspaces of vector space $\mathcal{V}$. We say that $\mathcal{V}$ is the **direct sum** of $\mathcal{W}_1$ and $\mathcal{W}_2$, or

$$\mathcal{V} = \mathcal{W}_1 \oplus \mathcal{W}_2$$

if the following two conditions holds:

(1) $\mathcal{W}_1 + \mathcal{W}_2 := \{\mathbf{X} + \mathbf{Y} | \mathbf{X} \in \mathcal{W}_1, \mathbf{Y} \in \mathcal{W}_2\} = \mathcal{V}$.
"The span of $\mathcal{W}_1$ and $\mathcal{W}_2$ is all of $\mathcal{V}$."

(2) $\mathcal{W}_1 \cap \mathcal{W}_2 = \{\mathbf{0}\}$.
"$\mathcal{W}_1$ and $\mathcal{W}_2$ are independent."

Remark. We may state an equivalent definition as follows:
$\mathcal{V} = \mathcal{W}_1 \oplus \mathcal{W}_2$ if every $\mathbf{X} \in \mathcal{V}$ may be uniquely written as

$$\mathbf{X} = \mathbf{X}_1 + \mathbf{X}_2$$

where $\mathbf{X}_i \in \mathcal{W}_i$, $i \in \{1, 2\}$. We leave the proof to the reader, but we note the following:

- Every $\mathbf{X} \in \mathcal{V}$ may be written as $\mathbf{X} = \mathbf{X}_1 + \mathbf{X}_2$ if and only if $\mathcal{V} = \mathcal{W}_1 + \mathcal{W}_2$. This is the definition of $\mathcal{W}_1 + \mathcal{W}_2$.
- If $\mathcal{W}_1$ and $\mathcal{W}_2$ are not independent, then how many ways can we express any element in $\mathcal{W}_1 \cap \mathcal{W}_2$?
- If $\mathbf{X} = \mathbf{X}_1 + \mathbf{X}_2 = \mathbf{X}_1' + \mathbf{X}_2'$ where $\mathbf{X}_i' \neq \mathbf{X}_i$, can we use the fact that both sums equal $\mathbf{X}$ to find a common non-zero element in $\mathcal{W}_1 \cap \mathcal{W}_2$?

Example. $\mathbb{R}^2 = \mathbb{R}_1 \oplus R_2$, where $\mathbb{R}_1$ is the x-axis and $\mathbb{R}_2$ is the y-axis. There are many more choices. Any two lines that are not parallel and pass through the origin define two subspaces $\mathcal{W}_1$ and $\mathcal{W}_2$ such that their span is $\mathbb{R}^2$ and, because their intersection is the origin, are independent. So any $\mathbf{X} \in \mathbb{R}^2$ may be uniquely expressed as the sum of two points, one on the line $\mathcal{W}_1$ and the other on line $\mathcal{W}_2$.

Example. Say $\mathcal{V}$ is a vector space with basis $\mathcal{B} = \{\mathbf{B}_1, \ldots, \mathbf{B}_n\}$. Let $\mathcal{W}_1 = \text{SPAN}(\mathbf{B}_1, \ldots, \mathbf{B}_k)$ and $\mathcal{W}_2 = \text{SPAN}(\mathbf{B}_{k+1}, \ldots, \mathbf{B}_n)$ for some $1 \leq k < n$. Then

$$\mathcal{V} = \mathcal{W}_1 \oplus \mathcal{W}_2.$$

The following properties were addressed in class, for $\mathcal{V}$ a vector space and $\mathcal{W}_i$ representing subspaces.:

- If everything is finite dimensional and $\mathcal{V} = \mathcal{W}_1 + \mathcal{W}_2$, then

$$\dim(\mathcal{V}) \leq \dim(\mathcal{W}_1) + \dim(\mathcal{W}_2).$$

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Let $\mathcal{B} = \{\mathbf{Y}_1, \ldots, \mathbf{Y}_n\}$ and $\mathcal{B}' = \{\mathbf{Z}_1, \ldots, \mathbf{Z}_m\}$ be bases for $\mathcal{W}_1$ and $\mathcal{W}_2$ respectively. Then any $\mathcal{X} \in \mathcal{V}$ may be expressed as $\mathbf{Y} \in \mathcal{W}_1$ and $\mathbf{Z} \in \mathcal{W}_2$. But these are expressed as a linear combination of their basis vectors so

$$\mathbf{X} = a_1\mathbf{Y}_1 + \ldots a_n\mathbf{Y}_n + b_1\mathbf{Z}_1 + \cdots + b_m\mathbf{Z}_m.$$

We conclude that

$$\mathcal{S} = \{\mathbf{Y}_1, \ldots, \mathbf{Y}_n, \mathbf{Z}_1, \ldots, \mathbf{Z}_m\}$$

spans $\mathcal{V}$. A basis for $\mathcal{V}$ can be therefore expressed as a subset of $\mathcal{S}' \subseteq \mathcal{S}$ of size k, so

$$\dim(\mathcal{V}) = k \leq n + m = \dim(\mathcal{W}_1) + \dim(\mathcal{W}_2).$$

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- If we have finite dimensions and instead $\mathcal{V} = \mathcal{W}_1 \oplus \mathcal{W}_2$, then
$$\dim(\mathcal{V}) = \dim(\mathcal{W}_1) + \dim(\mathcal{W}_2).$$

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Exercise. Prove this equality by first noting that $\mathcal{V} = \mathcal{W}_1 + \mathcal{W}_2$, so the above inequality holds. Given that $\mathcal{W}_1 \cap \mathcal{W}_2 = \{0\}$, show that $\mathcal{S}$ in the previous proof is a basis for $\mathcal{V}$.

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- $\mathcal{W}_1 \oplus \mathcal{W}_2 = \mathcal{W}_2 \oplus \mathcal{W}_1$.

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This is just definitional given that addition and intersections commute:

$$\mathcal{W}_1 \cap \mathcal{W}_2 = \mathcal{W}_2 \cap \mathcal{W}_1$$

and

$$\mathbf{Y} + \mathbf{Z} = \mathbf{Z} + \mathbf{Y}, \text{ for all } \mathbf{Y} \in \mathcal{W}_1, \mathbf{Z} \in \mathcal{W}_2.$$

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- $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 = (\mathcal{W}_1 \oplus \mathcal{W}_2) \oplus \mathcal{W}_3$.
All we are saying here is that if $\mathcal{U} = \mathcal{W}_1 \oplus \mathcal{W}_2$ (any vector in $\mathcal{U}$ may be uniquely expressed as a sum of vectors in $\mathcal{W}_1$ and $\mathcal{W}_2$), then
$$\mathcal{U} \oplus \mathcal{W}_3 = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3.$$
This says that any $\mathbf{X} \in \mathcal{U} \oplus \mathcal{W}_3$ may be uniquely written as $\mathbf{X} = \mathbf{Y} + \mathbf{Z}$ for some $\mathbf{Y} \in \mathcal{U}$ and $\mathbf{Z} \in \mathcal{W}_3$. Because $\mathbf{Y} = \mathbf{Y}_1 + \mathbf{Y}_2$ for some unique $\mathbf{Y}_1 \in \mathcal{W}_1$ and $\mathbf{Y}_2 \in \mathcal{W}_2$,
$$\mathbf{X} = \mathbf{Y}_1 + \mathbf{Y}_2 + \mathbf{Z}$$
and these choices are all unique.

3. The Relationship Between the Two

We finish by making some useful remarks about the relationship between direct sums and projections. Namely a direct sum $\mathcal{W}_1 \oplus \mathcal{W}_2$ exists if and only if a projection exists with image $\mathcal{W}_1$ and kernel $\mathcal{W}_2$. We prove this in the first two propositions below. We conclude with the relationship between projections P, Q and $P+Q$ (if this is indeed a projection!).

Proposition 1. *If $P : \mathcal{V} \to \mathcal{V}$ is a projection, then*
$$\mathcal{V} = \mathrm{IM}(P) \oplus \mathrm{KER}(P).$$

Proof. We first will show that $\mathcal{V} = \mathrm{IM}(P) + \mathrm{KER}(P)$. The first inclusion, $\mathrm{IM}(P) + \mathrm{KER}(P) \subseteq \mathcal{V}$, is clear as each set on the right is a subspace of $\mathcal{V}$, so their sum will be a subset as well. We then need to show that $\mathcal{V} \subseteq \mathrm{IM}(P) + \mathrm{KER}(P)$. Let $\mathbf{X} \in \mathcal{V}$. Let $\mathbf{Y} = P(\mathbf{X}) \in \mathrm{IM}(P)$ and $\mathbf{Z} = \mathbf{X} - \mathbf{Y}$. Then $\mathbf{X} = \mathbf{Y} + \mathbf{Z}$ and
$$P(\mathbf{Z}) = P(\mathbf{X} - \mathbf{Y}) = P(\mathbf{X}) - P(\mathbf{Y}) = \mathbf{Y} - \mathbf{Y} = \mathbf{0}$$
or $\mathbf{Z} \in \mathrm{KER}(P)$. So $\mathbf{X} \in \mathrm{IM}(P) + \mathrm{KER}(P)$. This finishes the proof of equality.

We now show that $\mathrm{IM}(P) \cap \mathrm{KER}(P) = \{\mathbf{0}\}$. $\mathbf{0}$ is contained in the intersection, so we show that any vector in this set must be $\mathbf{0}$ as well. Let $\mathbf{X} \in \mathrm{IM}(P) \cap \mathrm{KER}(P)$. Then
$$\mathbf{X} \underset{\mathbf{X} \in \mathrm{IM}(P)}{=} P(\mathbf{X}) \underset{\mathbf{X} \in \mathrm{KER}(P)}{=} \mathbf{0}.$$
□

Proposition 2. *If $\mathcal{V}$ is a vector space and $\mathcal{W}_1, \mathcal{W}_2 \subseteq \mathcal{V}$ are subspaces such that*
$$\mathcal{V} = \mathcal{W}_1 \oplus \mathcal{W}_2,$$
then there exists a unique projection $P : \mathcal{V} \to \mathcal{V}$ such that $\mathrm{IM}(P) = \mathcal{W}_1$ and $\mathrm{KER}(P) = \mathcal{W}_2$.

Proof. Any $\mathbf{X} \in \mathcal{V}$ has a unique expression $\mathbf{X} = \mathbf{Y} + \mathbf{Z}$, where $\mathbf{Y} \in \mathcal{W}_1$ and $\mathbf{Z} \in \mathcal{W}_2$. So the transformation $P : \mathcal{V} \to \mathcal{V}$ defined by
$$P(\mathbf{X}) = \mathbf{Y}$$
is well defined. We now show that P is linear. The zero element is uniquely expressed as $\mathbf{0} = \mathbf{0} + \mathbf{0}$, so
$$P(\mathbf{0}) = \mathbf{0}.$$
If $\mathbf{X}, \mathbf{X}' \in \mathcal{V}$, they have unique expressions
$$\mathbf{X} = \mathbf{Y} + \mathbf{Z} \text{ and } \mathbf{X}' = \mathbf{Y}' + \mathbf{Z}'$$
where $\mathbf{Y}, \mathbf{Y}' \in \mathcal{W}_1$ and $\mathbf{Z}, \mathbf{Z}' \in \mathcal{W}_2$. $\mathbf{X} + \mathbf{X}'$ may be expressed (just by addition) as
$$\mathbf{X} + \mathbf{X}' = \mathbf{Y} + \mathbf{Y}' + \mathbf{Z} + \mathbf{Z}'.$$
This must be **the** expression for $\mathbf{X} + \mathbf{X}'$ by uniqueness. So
$$P(\mathbf{X} + \mathbf{X}') = \mathbf{Y} + \mathbf{Y}' = P(\mathbf{X}) + P(\mathbf{X}').$$
By the same reasoning $\mathbf{X} = \mathbf{Y} + \mathbf{Z}$ implies that $c\mathbf{X} = c\mathbf{Y} + c\mathbf{Z}$ for any scalar c, so
$$P(c\mathbf{X}) = c\mathbf{Y} = cP(\mathbf{X}).$$
Now we need show that

- P is a projection.
- P has image $\mathcal{W}_1$ and kernel $\mathcal{W}_2$.

If $P(\mathbf{X}) = \mathbf{Y}$, $\mathbf{Y}$ has the unique sum $\mathbf{Y} = \mathbf{Y} + \mathbf{0}$, so

$$P^2(\mathbf{X}) = P(P(\mathbf{X})) = P(\mathbf{Y}) = \mathbf{Y} = P(\mathbf{X}).$$

Therefore $P^2 = P$ so it is a projection.

By our choice of P, $\mathrm{IM}(P) \subseteq \mathcal{W}_1$. We now show that $\mathcal{W}_1 \subseteq \mathrm{IM}(P)$. As before, if $\mathbf{Y} \in \mathcal{W}_1$, its unique sum is $\mathbf{Y} = \mathbf{Y} + \mathbf{0}$, so

$$P(\mathbf{Y}) = \mathbf{Y} \in \mathrm{IM}(P).$$

Exercise. Show that $\mathrm{KER}(P) = \mathcal{W}_2$.

- Show that if $\mathbf{Z} \in \mathcal{W}_2$, then $P(\mathbf{Z}) = \mathbf{0}$.
- If $P(\mathbf{X}) = \mathbf{0}$, write the unique sum $\mathbf{X} = \mathbf{Y} + \mathbf{Z}$. What can you say about $\mathbf{Y}$? What does this say about $\mathbf{X}$?

□

Our final claim concerns sum of projections.

Proposition 3. *Suppose $P, Q : \mathcal{V} \to \mathcal{V}$ are projections. Assume that $P + Q$ is a projection as well. Then*

$$\mathrm{IM}(P + Q) = \mathrm{IM}(P) \oplus \mathrm{IM}(Q)$$

and

$$\mathrm{KER}(P + Q) = \mathrm{KER}(P) \cap \mathrm{KER}(Q).$$

This implies that

$$\mathcal{V} = \mathrm{IM}(P) \oplus \mathrm{IM}(Q) \oplus \mathrm{KER}(P) \cap \mathrm{KER}(Q)$$

as well.

Proof. We assume that P, Q and $P + Q$ are projections. It follows immediately that

$$\mathrm{IM}(P + Q) = \mathrm{IM}(P) + \mathrm{IM}(Q).$$

Because $P + Q$ is a projection, we proved at the end of the section on projections that (among other things) $\mathrm{IM}(P) \subseteq \mathrm{KER}(Q)$. We see that

$$\{0\} \subseteq \mathrm{IM}(P) \cap \mathrm{IM}(Q) \subseteq \mathrm{KER}(Q) \cap \mathrm{IM}(Q) = \{0\},$$

so $\mathrm{IM}(P + Q) = \mathrm{IM}(P) \oplus \mathrm{IM}(Q)$.

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If $\mathbf{Z} \in \mathrm{KER}(P) \cap \mathrm{KER}(Q)$, then

$$(P + Q)(\mathbf{Z}) = P(\mathbf{Z}) + Q(\mathbf{Z}) = \mathbf{0} + \mathbf{0} = \mathbf{0}.$$

So $\mathrm{KER}(P) \cap \mathrm{KER}(Q) \subseteq \mathrm{KER}(P + Q)$

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Now assume $\mathbf{Z} \in \mathrm{KER}(P + Q)$. Then

$$\mathbf{0} = (P + Q)(\mathbf{Z}) = P(\mathbf{Z}) + Q(\mathbf{Z}).$$

This may happen if and only if

$$P(\mathbf{Z}) = -Q(\mathbf{Z}).$$

This result is an element of $\mathrm{IM}(P)$ by the left hand side and an element of $\mathrm{IM}(Q)$ by the right hand side. Because $\mathrm{IM}(P) \cap \mathrm{IM}(Q) = \{\mathbf{0}\}$, each side must be $\mathbf{0}$, so $\mathbf{Z} \in \mathrm{KER}(P) \cap \mathrm{KER}(Q)$. □

4. Orthogonal Projections

We end with a note about $Proj_{\mathbf{Y}}$ and $Proj_{\mathcal{W}}$ as listed in the text. Here $\mathcal{V} = \mathbb{R}^n$ and we have a notion of orthogonality. If $\mathbf{Y} \neq \mathbf{0}$, then $Proj_{\mathbf{Y}}$ defined as

$$Proj_{\mathbf{Y}}(\mathbf{X}) = \frac{\mathbf{X} \cdot \mathbf{Y}}{\mathbf{Y} \cdot \mathbf{Y}} \mathbf{Y}.$$

We discussed in class the following:

Exercise. Show that $Proj_{\mathbf{Y}}$ is a projection and has image $\mathrm{SPAN}(\mathbf{Y})$ and kernel $\mathbf{Y}^{\perp} = (\mathrm{SPAN}(\mathbf{Y}))^{\perp}$.

What about $Proj_{\mathcal{W}}$? Well, if $\{\mathbf{Y}_1, \dots, \mathbf{Y}_k\}$ is an orthonormal basis for subspace $\mathcal{W}$, then consider each projection $P_{\mathbf{Y}_i} = Proj_{\mathbf{Y}_i}$.

Exercise. Show that $P_{\mathcal{W}} = P_{\mathbf{Y}_1} + \cdots + P_{\mathbf{Y}_k}$ is a projection.
(How do the images and kernels of each $P_{\mathbf{Y}_i}$ relate to each other?)

We know from Proposition 3 that

$$\mathrm{IM}(P_{\mathcal{W}}) = \mathrm{IM}(P_{\mathbf{Y}_1}) \oplus \cdots \oplus \mathrm{IM}(P_{\mathbf{Y}_k}) = \mathcal{W}.$$

Also,

$$\mathrm{KER}(P_{\mathcal{W}}) = \mathrm{KER}(P_{\mathbf{Y}_1}) \cap \cdots \cap \mathrm{KER}(P_{\mathbf{Y}_k}).$$

Exercise. Show that the right hand side is $\mathcal{W}^{\perp}$.

So this is indeed the orthogonal projection on $\mathcal{W}$. What the text calls $Orth_{\mathcal{W}}$ is the unique projection with image $\mathcal{W}^{\perp}$ and kernel $\mathcal{W}$. Given our results in Section 1, we may simply point out that $Othr_{\mathcal{W}}$ can be nothing else than

$$Orth_{\mathcal{W}} = I - Proj_{\mathcal{W}}$$

where I is the identity function on $\mathbb{R}^n$.