Fundamental groups of klt varieties

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1. Fundamental groups in Algebraic Geometry

- $X$ normal algebraic variety over $\mathbb{C}$

we can view $X$ as top space with the Euclidean metric and define the topological fundamental group

$$\pi_1(X) = \{ \text{closed loops } y: [0,1] \to X \}$$

Problem: $\pi_1(X)$ need not be algebraic, nor need the associated universal cover

$$\tilde{\pi}_1(X) \to \pi_1(X)$$
be algebraic.

**Solution:** consider finite, étale covers

\[ f_i : X_i \rightarrow X, \quad i \in I \]

\( I \) is a directed set by \( j \preceq i \iff \exists j \in I \) finitely étale

\( f_{ij} \) yield morphisms \( g_{ij} : \text{Aut}_x(X_i) \rightarrow \text{Aut}_x(X_j) \)

\[ G_i \]

as yet a proj system of groups \( \prod (G_i)_{i \in I}, (g_{ij})_{j \leq i} \)

Define the étale fundamental group

\[ \hat{\pi}_1(X) := \lim_{\rightarrow} G_i = \left\{ g \in \prod G_i \mid g_j = g_{ij}(g) \quad \forall j \leq i \right\} \]

Easy to see: \( \hat{\pi}_1(X) \) is the profinite completion of \( \pi_1(X) \)

\[ G = \lim_{\leftarrow} G/N \quad \text{finite index} \]

Attention: in general, there is no associated “universal étale cover” in the category of algebraic varieties
There is a natural group homomorphism

\[ \pi_n(x) \to \pi_n(x) \]

with dense image, but not necessarily injective.

E.g., \( G \) may be very bad but have no non-trivial subgroup of finite index \( \Rightarrow G \) is trivial.

\( G \) may be infinite but rather well-behaved and \( G \) may be nasty:

\[ X = \mathbb{C}^* = \mathbb{R} \times S^1 \quad \Rightarrow \quad \pi_1(X) = \mathbb{Z} \]

\[ \mathbb{Z}^2 = \mathbb{R}^2 \]

but \( \mathbb{Z} = \lim_{\leftarrow} \mathbb{Z}/n\mathbb{Z} \)

\[ = \prod_{p \text{ prime}} \mathbb{Z}_p \]

\[ \mathbb{Z}_p \text{ are the p-adic integers} \]

The best situation is of course when \( \pi_1(X) \) is finite.

2. Fundamental groups of singularities

~ two viewpoints

Global: if \( X \) is smooth, then a cover is étale iff it is étale in codimension 1.
maybe we should allow covers to ramify in codim 2 for singular X?!  

equiv: over X_{reg}  

\[ \text{consider} \ \tilde{\mu}_n(X_{reg}) \]  

local: \((X,x)\), X locally at \(x\) is contractible, so we should at least allow ramification over \(x\)  

\[ \text{consider local fundamental group} \]  

\[ \tilde{\pi}_1(X, x) = \tilde{\pi}_1(X \setminus \{x\}) \]  

maybe better regional fundamental group  

\[ \tilde{\pi}_1^{\text{reg}}(X, x) = \tilde{\pi}_1(X_{\text{reg}}) \]  

For a log pair \((X, \Delta)\), we may allow covers that ramify over \(\Delta\) in a controlled manner, i.e. \(f: (Y, \delta, \Delta_Y) \to (X, 2)\)  

such \(\Delta_Y\) is given by \(f^* (K_X + \Delta) = K_Y + \Delta_Y\) is effective.  

How to decompose \(\Delta = \Delta_1 + \Delta_2\)  

\[ \sum (1 - \frac{1}{m_i}) \Delta_i \]  

standard coefficients
$X = (X, \Delta)$ defines an orbifold structure.

and the orbifold fundamental group

$\pi_1(X, \Delta_x) = \pi_1(X^{\ast}/d^i) / \langle g_i \rangle$

yields what we want.

3. Fundamental groups of klt's and fano's

Thus (classical) Fano manifolds are simply connected.

less classical (Takayama '93): $(X^X) \log \text{Fano} \Rightarrow X \text{ simply conn.}

Thus (8.2020)

1) $(X, \Delta) \text{ log Fano}$, then $\pi_1(X_{\text{reg}}, \Delta|_{X_{\text{reg}}})$ finite.

2) $(X, \Delta, x)$ klt sing, then $\pi_1^{\text{reg}}(X, \Delta, x)$ finite.

Basic Idea: "local to global induction" 

(1) $(\dim X = n - 1) \Rightarrow (2) \dim X = n$
Theorem \((3.2)\) \(\dim X = n \Rightarrow (1) \dim X = n\)

\((4)\) \textit{global to local}

\textit{basic construction: Kollar, component/plt blowups}

going back to Shokurov/Prokhorov in the 90ies

\(\nu\) except divisor of \(X\) which is of Fano type.

\underline{Lemma} (Existence of plt blowups, Xu 14)

Let \(p \in (X)\) be a klt point. There exists a \(\mathbb{Q}\)-divisor \(D\) on \(X\) and if \(f : Y \rightarrow X\) birational:

\(1\) A prime divisor \(E\) on \(Y\) with \(\text{center}_X(E) = p\);

\(2\) \((- (k \sigma f^{-1}(E)) + E)\) and \(-E\) ample over \(X\);

\(3\) \((X,\delta + E)\) blt on \(X\) \(\not\exists \beta, \text{mld}(p, X, \delta + E) = 0\) and \(E\) is unique over \(p\) without divisor \(-1\)

\textit{Proof:}
Choose general ample $L$ on $X$ passing through $p$ with small coeff. $s_{*}\Delta$ is lc at $p$ and $lt$ on $X \times E_p$

- log reg $g: \mathcal{Z} \to (X, \Delta + L)$ or $E_g(g)$ supports a relative ample $A$

- If we take $0 < \delta < 2$ s.t. $\text{EG}^{*}L + FA \cdot L'$ is a general ample $\mathbb{Q}$-div, then there exists $0 < \delta$ s.t. in the formula

$$g^{*} (K_{X \times \Delta + (\delta + 3)} L) \equiv K_{2} + g_{*}^{-1}(\Delta + L + L' + E\sum a_{i}E_{i})$$

there is a unique $a_{i} = a_{1} = 1$ and $a_{i} < 1$ for $i \geq 2$ and $\text{can}_{X}(E_{i}) > 0$.

- Now we consider the del pair

$$\left( \mathcal{Z}, g_{*}^{-1}(\Delta + L) + E_{1} + L' + \sum_{i \geq 2} E_{i} \right)$$

$$= (\Delta_{Z})$$

having $K_{2} + \Delta_{Z} \equiv \sum (1 - a_{i}) E_{i}.$

Now run a $(K_{2} + D_{2})$-MMP over $X$ with scaling of $L'$, by [BC+HMJ], it terminates with a good minimal model $h: W \to X$

- $h: \mathcal{Z} \dashrightarrow W$ contracts precisely the $E_{i} (i \geq 2)$
so \( E_W = \phi_k^*(E) \) is the div part of \( E_X(E) \). 

on \( W \)

\[
K_W + h^{-\cdot} (\Delta + tL) + E_W + \phi_k^* L' \equiv 0
\]

\( =: \Delta_W \)

\( (W, \Delta_W) \) is plt and for \( \epsilon \) sufficiently small, 
\( K_W + \Delta_W + \epsilon \phi_k^* L' \) is nef over \( X \).

Let \( f: Y \to X \) be the log canonical model of 
\( (W, \Delta_W + \epsilon \phi_k^* L') \).

\( \phi_k^* A = \lambda E_W \) for some \( \lambda > 0 \), so we have that

\[-\epsilon S_1 E_W = \epsilon \phi_k^* A \equiv_{X, \Delta_W} K_W + \Delta_W + \epsilon \phi_k^* L'
\]
is nef over \( X \).

Now define \( H := H + (\log \phi)^* L' \)

\( W \to Y \) is small and \((Y, E_Y + f_*^{-1}(H + t))\) plt since \( (W, \Delta_W) \) is.

\( -E_Y \) is ample and

\[
(KE_Y + f_*^{-1}(\Delta + \epsilon)) \equiv_{X, \Delta_W} (1 + \epsilon (E_X, x, \epsilon)) E
\]
is ample as well.

Now Writing
\[ (K_y + f^*\Delta + E)|_E = K_E + \text{Diff}_E f^*\Delta \]

Then the pair \((\Sigma, \Gamma)\) is log Fano.

**Proof of (A):** global to local (for \(\Sigma\))

\(\text{pc}(X, \Delta)\) left sing, \(f: Y \to X\) plt blowup as above

\[ \ldots \to (X_2, \rho_2) \xrightarrow{\phi_2} (X_1, \rho_1) \xrightarrow{\phi_1} (X, \rho) \]

seq of finite morphisms etc. etc. over \(X\) is \(\mathbb{P}\)

want to show: \(\exists i \in \mathbb{N}\), s.t. \(\phi_i\) trivial \(\Rightarrow i = i_0\)

Let \(Y_i\) be the normalized main component of \(X_i \times X_i\); we have a commutative diagram:

\[
\begin{array}{ccc}
E_{i+1} & \xrightarrow{\chi_{i+1}} & Y_i \times E_i \\
\downarrow \phi_i & & \downarrow \phi_i \\
\text{pr} \times X & \xrightarrow{\phi_i} & X_0 \times X
\end{array}
\]

\(E_i = \text{pullback of } \Delta \text{ and } \phi_i\) defined by \(f^*\Delta + E_i = K_X \cdot \phi_i^{-1} E_i + E_i \)

\(\phi_i = \eta_i\) for \(E_i\) is \(\xi_i\).
Thus \( \psi_i/E_i \cdot (E_{in}, R_{in}) \Rightarrow (E_i, R_i) \)

is log étale in codim 1.

by **Assumption**: \( E_{i+1} \), the \( \psi_{i+1}/E_{i+1} \) is trivial for \( i \geq i \), so \( \psi_i \) is totally ramified over \( E_i \).

Let \( \gamma \) be a loop around a general point of \( E_j \), cutting \( \psi_j \) to a surface though this general point, by Mumford’s 61 work on surface singularities, we get that \( \gamma \) is torsion in

\[ \pi_1(Y_j/E_j) = \pi_1(X_j, \{p_j\}) \]

so for \( k \geq j \) \( \phi_j : X_j \rightarrow X_j-k \) is trivial

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**(3) local to global**

**Basic idea**: express \( \pi_1(Y_{reg}\setminus D) \) as an orbifold fundamental group supported on log res of \( f : X \rightarrow Y \)

using the assumption that...
$H_0$ is finite for $x$, $y$.

How? $f$ is continuous!

Let $y_1$ be a loop around a general point of $E_i \in \text{Ex}(f)$

so $y_i$ is of finite order $m_i$ in $\pi_1^{\text{reg}}(X, p)

\text{Hence } \pi_1(Y_{\text{reg}}, D) = \pi_1 \left( X \setminus \text{Ex}(f), f^{-1}_{\ast} D'' \right)

\langle \langle y_i \rangle \rangle

= \pi_1 \left( X, f^{-1}_{\ast} D'' + \sum (1 - \frac{1}{m_i}) E_i \right)

so the following prop will prove part (B):

**Proposition**

Let $(Y, D)$ be log Fano with log reg nef $f : X \to Y$. Then for any choice of $m_i \in \mathbb{N}_{\geq 1}$,

$\pi_1 \left( X, f^{-1}_{\ast} D'' + \sum (1 - \frac{1}{m_i}) E_i \right)$
is finite.

**Proof:** notation as above, \( a_i := \text{disc}(E_i, Y_i) \)
\[ c_i := a_i - \frac{1}{m_i} \]
\[ L := -f^*(K_{Y+D}) - \sum c_i E_i + \sum (\Gamma_i c_i - c_i) E_i + f^* D' \]
\[ -1 < c_i < 0 \]
Consider the orbifold \( Z = (X, \mathcal{A}) \)
\[ K_Z = K_X + \mathcal{A} \] is the orbifold canonical divisor
\[ K_Z + L = \sum_{0 < c_i} \Gamma_i c_i E_i \] is an effective orbifold divisor on \( X \)

**Claim:** produce many sections on this divisor.

As a method of Taubes' GG generalized to orbifolds.

By Claudon 10, the \( \mathcal{P} \)-reduction as Shafarevich map is available for orbifolds, i.e. we have a commutative diagram:

\[ \tilde{\mathcal{X}} \xrightarrow{\tilde{\pi}} \mathcal{P} (\mathcal{X}) \]
\[ \sqrt[\pi_1(X)]{\tilde{\mathcal{X}}} \]
with:

- $\overline{\gamma}$ are top locally trivial fibrations on open subsets of $X, \overline{X}$
- $\gamma$ parameterizes maximal suborbifolds $Y \subset X$ with $\text{im}(\overline{\pi}(\gamma) \to \overline{\pi}(X))$ finite
- $\tilde{\gamma}$ parameterizes maximal compact suborbifolds of $\overline{X}$

Now let $F$ be a generic fiber of $\overline{\gamma}$

- take $\mathcal{E} \in \mathcal{H}^0(X, K_X + L)$
- pullback $\tilde{\mathcal{E}} := (\pi|_{F \times U})^* \left( \pi_1 | F \times U \right) \cdot (\gamma \circ \pi)|^* \mathcal{E}$
- $\tilde{\mathcal{E}}$ is a smooth $\tilde{\mathcal{T}} = u^* \mathcal{L}$-valued $(k, 0)$-form on $\tilde{X}$
Now we want to use Nadel vanishing for orbifolds to produce \( v \in H^0(\mathcal{X}, K_{\mathcal{X} + \mathbb{L}}) \).

Idea: \( \omega = \bar{\vartheta}(\omega) = \pi^* \omega \cdot \bar{\vartheta}(\pi \circ \pi^*) \rho \)

is square integrable and \( \bar{\vartheta} \)-closed.

Nadel \( \Rightarrow \) \( \mathcal{X} \) co with \( \bar{\vartheta}(\omega) = \omega \), square integrable,

\[ \gamma = \mathbb{L} - \omega \] non-trivial, square-integrable and holomorphic, because \( \bar{\vartheta}(\omega) = 0 \).

At this point, we can use the standard theory of Gromov, which gives that

\[ p(v \otimes 2\mathbb{L}) = \sum_{g \in \pi_1(K)} g^* v \otimes 2\mathbb{L} \]

(the Poincaré series)

for \( \pi_1(K) \) infinite, \( \mathcal{E} \) regular for two different partitions \( K = \sum K_i = \sum K_i^j \)

\[ \bigotimes p(v \otimes 2\mathbb{L}) \text{ and } \bigotimes p(v \otimes 2\mathbb{L}^j) \]
are linearly independent sections of

\[ H^0(X, (K_X + L)^{\otimes 2K}) \]

\( \text{effective except } m \in \mathbb{Z} \)

THANKS!